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Quadratic Leibniz conformal superalgebras *

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Abstract: In this paper, we study a class of Leibniz conformal superalgebras called quadratic Leibniz conformal superalgebras. An equivalent characterization of a Leibniz conformal superalgebra $R=\mathbb{C}[\partial]V$ thought three algebraic operations on V are given. By this characterization, several constructions of quadratic Leibniz conformal superalgebras are presented. Moreover, one-dimensional central extensions of quadratic Leibniz conformal superalgebras are considered using some bilinear forms on V. In particular, we also study one-dimensional Leibniz central extensions of quadratic Lie conformal superalgebras.

Key words: Leibniz conformal superalgebra, Lie conformal superalgebra, Gel'fand-Dorfman bialgebra, Leibniz central extension

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1 Introduction

Throughout this paper, we denote by $\mathbb{C}$ the set of complex numbers, $\mathbb{Z}$ the set of integer numbers and $\mathbb{Z}_+$ the set of non-negative integer numbers. All vector spaces are over $\mathbb{C}$ and tensors over $\mathbb{C}$ are denoted by $\otimes$. Moreover, if V is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space with a drect sum $V=V_{\bar{0}}\oplus V_{\bar{1}}$, the space of polynomials of λ with coefficients in V is denoted by $V[\lambda]$.

Lie conformal superalgebra, introduced by Kac in [22], gives an axiomatic description of the singular part of the operator product expansion of chiral fields in conformal field theory. It is an useful tool to study vertex superalgebras (see [22]) and has many applications in the theory of infinite-dimensional Lie superalgebras. Moreover, Lie conformal superalgebras have close connections to Hamiltonian formalism in the theory of nonlinear evolution equations (see [3]). In fact, Lie conformal superalgebra is a Lie pseudo-algebra which can be seen a Lie algebra in a pseudo-tensor category (see [2]). Structure theory and representation theory of finite Lie conformal superalgebras which are finitely generated as $\mathbb{C}[\partial]$-modules are well developed (see [4,13,14,16], [6]- [9], [17,18] and so on).

Conformal superalgebras are quite intriguing subjects in the purely algebraic viewpoint. One can define the conformal analogue of a variety of "usual" superalgebras such as Lie conformal superalgebras, associative conformal superalgebras, etc. The theory of conformal superalgebras sheds new light on the problem of classification of infinite-dimensional algebras of the corresponding "classical" variety.

Leibniz superalgebras are the non-super-commutative analogs of Lie superalgebras (see [23]). Leibniz algebra, first introduced by Bloh in [1], and reintroduced by Loday in [24], arose naturally

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during their studying on periodicity phenomena in algebraic K-theory. The concept of Leibniz superalgebra was first introduced in [15]. The name of left (right) Leibniz superalgebra comes from that the left (right) multiplication is a superderivation. We call the right Leibniz superalgebra as Leibniz superalgebra in this paper.

The topic of this paper is about Leibniz conformal superalgebra. Left Leibniz conformal algebra was introduced in [4] and its cohomology theory was investigated in [4] and [28]. In addition, Leibniz pseudoalgebra was studied in [27]. But, there are few examples of (left) Leibniz conformal algebras which are not Lie conformal algebras. In this paper, our aim is to provide an efficient way to construct Leibniz conformal superalgebras which are not Lie conformal superalgebras. By the correspondence of Leibniz conformal superalgebras and (infinite-dimensional) Leibniz superalgebras, it is also useful to construct infinite-dimensional Leibniz superalgebras which are not Lie superalgebras. Therefore, it is of signification and interesting. For Lie conformal superalgebras, as we know, most of known examples are quadratic Lie conformal superalgebras which were studied in [19, 26]. It was essentially stated in [19] that a "quadratic" Lie conformal superalgebra is equivalent to a bialgebra structure. One is a Lie superalgebra structure , the other is a Novikov superalgebra structure and they satisfy a compatibility condition. This bialgebra structure is called a super Gel'fand-Dorfman bialgebra by Xu in [26]. In fact, a quadratic Lie conformal algebra corresponds to a Hamiltonian pair in [19], which plays fundamental roles in completely integrable systems. Moreover, several constructions of super Gel'fand-Dorfman bialgebras were presented by Xu in [26]. Therefore, it is an useful method to construct Lie conformal superalgebras by using super Gel'fand-Dorfman bialgebras. Note that Lie conformal superalgebras are Leibniz conformal superalgebras. Motivated by this, in this paper, we investigate what are the superalgebra structures underlying quadratic Leibniz conformal superalgebras. Since super Gel'fand-Dorfman bialgebras correspond to Lie conformal superalgebras, if we find other algebra structures which are not super Gel'fand-Dorfman bialgebras underlying Leibniz conformal superalgebras, we can construct Leibniz conformal superalgebras which are not Lie conformal superalgebras. We show that except super Gel'fand-Dorfman bialgebra, there is also a bialgebra structure called ass-Nov-Leibniz superalgebra underlying quadratic Leibniz conformal superalgebras. Thought ass-Nov-Leibniz superalgebras, we can construct many Leibniz conformal superalgebras which are not Lie conformal superalgebras. Moreover, one-dimensional central extensions of quadratic Leibniz conformal superalgebra $R=\mathbb{C}[\partial]V$ are considered using some bilinear forms on V. In particular, we also study one-dimensional Leibniz central extensions of quadratic Lie conformal superalgebras.

This paper is organized as follows. In Section 2, some basic definitions and some facts about Lie conformal superalgebras and Leibniz conformal superalgebras are recalled. We also introduce the definitions of quadratic Lie and Leibniz conformal superalgebras. In Section 3, an equivalent characterization of quadratic Leibniz conformal superalgebra is given. Thought this equivalent characterization, we find that a bialgebra structure called ass-Nov-Leibniz superalgebra can be used to construct Leibniz conformal superalgebras which are not Lie conformal superalgebras. Moreover, several constructions and examples are given. In Section 4, we investigate one-dimensional central extensions of quadratic Leibniz conformal superalgebras $R=\mathbb{C}[\partial]V$ corresponding to ass-

Nov-Leibniz superalgebras using some bilinear forms on V. In particular, one-dimensional Leibniz central extensions of quadratic Lie conformal superalgebras are also considered.

2 Preliminaries

In this section, we will introduce some basic definitions and some facts about Lie conformal superalgebras and Leibniz conformal superalgebras.

Definition 2.1. *A* Lie superalgebra $\mathfrak{g}$ *is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}}$ with an operation $[\cdot,\cdot]$ satisfying $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subset\mathfrak{g}_{\alpha+\beta}$ and the following axioms:*

$$[a,b]=-(-1)^{\alpha\beta}[b,a],\ \ [a,[b,c]]=[[a,b],c]+(-1)^{\alpha\beta}[b,[a,c]],$$

for $a\in\mathfrak{g}_\alpha$, $b\in\mathfrak{g}_\beta$ and $c\in\mathfrak{g}$.

When $\mathfrak{g}_{\bar{1}}=0$, $\mathfrak{g}$ is a Lie algebra.

Definition 2.2. *A* Leibniz superalgebra *L is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $L=L_{\bar{0}}\oplus L_{\bar{1}}$ with an operation $[\cdot,\cdot]$ satisfying $[L_\alpha,L_\beta]\subset L_{\alpha+\beta}$ and the following axioms:*

$$[a,[b,c]]=[[a,b],c]-(-1)^{\beta\gamma}[[a,c],b], \tag{2.1}$$

for $a\in L$, $b\in L_\beta$ and $c\in L_\gamma$.

When $L_{\bar{1}}=0$, L is called a Leibniz algebra*.*

Remark 2.3. *(2.1) is called* right Leibniz identity*. Similarly, there is also a* left Leibniz identity*:*

$$[a,[b,c]]=[[a,b],c]+(-1)^{\alpha\beta}[b,[a,c]], \tag{2.2}$$

for $a\in L_\alpha$, $b\in L_\beta$ and $c\in L$. If a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $L=L_{\bar{0}}\oplus L_{\bar{1}}$ with an operation $[\cdot,\cdot]$ satisfying $[L_\alpha,L_\beta]\subset L_{\alpha+\beta}$ and the left Leibniz identity, L is called a left Leibniz superalgebra*.*

Note that, any Leibniz superalgebra $(L,[\cdot,\cdot])$ can be endowed with a left Leibniz superalgebra structure $(L,[\cdot,\cdot]')$ via $[a,b]'=-(-1)^{\alpha\beta}[b,a]$ for any $a\in L_\alpha$, $b\in L_\beta$ and vice versa.

Obviously, all Lie superalgebras are Leibniz superalgebras. Then, we consider the conformal versions of these algebras.

Definition 2.4. A *Lie conformal superalgebra R* is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R=R_{\bar{0}}\oplus R_{\bar{1}}$ endowed with a $\mathbb{C}$-bilinear map

$$R\times R\to R[\lambda],\qquad a\times b\mapsto [a\,{}_\lambda\,b],$$

satisfying $[R_{\alpha\lambda} R_\beta] \subset R_{\alpha+\beta}[\lambda]$ and the following three axioms ($a \in R_\alpha, b \in R_\beta, c \in R$):

$$Conformal\ \ sesquilinearity: \quad [\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b];$$
$$Skew\ \ symmetry: \quad [a_\lambda b] = -(-1)^{\alpha\beta}[b_{-\lambda-\partial} a];$$
$$Jacobi\ \ identity: \quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{\alpha\beta}[b_\mu [a_\lambda c]].$$

When $R_{\bar{1}} = 0$, R is called a *Lie conformal algebra*.

Definition 2.5. A *Leibniz conformal superalgebra* R is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R = R_{\bar{0}} \oplus R_{\bar{1}}$ endowed with a $\mathbb{C}$-bilinear map

$$R \times R \to R[\lambda], \qquad a \times b \mapsto [a_\lambda b],$$

satisfying $[R_{\alpha\lambda} R_\beta] \subset R_{\alpha+\beta}[\lambda]$ and the following axioms:

$$Conformal\ \ sesquilinearity: \quad [\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b];$$
$$(right)\ \ Leibniz\ \ identity: \quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] - (-1)^{\beta\gamma}[[a_\lambda c]_{-\mu-\partial} b],$$

where $a \in R$, $b \in R_\beta$, $c \in R_\gamma$.

When $R_{\bar{1}} = 0$, R is called a *Leibniz conformal algebra*.

A Lie or Leibniz conformal superalgebra R is called *finite*, if it is finitely generated as a $\mathbb{C}[\partial]$-module; otherwise, it is said to be *infinite*.

Remark 2.6. *Similar to the classical case, there is also the definition of left Leibniz conformal superalgebra. A* left Leibniz conformal superalgebra *R is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module $R = R_{\bar{0}} \oplus R_{\bar{1}}$ endowed with a $\mathbb{C}$-bilinear map $R \times R \to R[\lambda]$, $a \times b \mapsto [a_\lambda b]$, satisfying $[R_{\alpha\lambda} R_\beta] \subset R_{\alpha+\beta}[\lambda]$, conformal sesquilinearity and the left Leibniz identity*

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{\alpha\beta}[b_\mu [a_\lambda c]],$$

where $a \in R_\alpha$, $b \in R_\beta$ and $c \in R$. When $R_{\bar{1}} = 0$, R is the left Leibniz conformal algebra.

Similarly, it is easy to check that any Leibniz conformal superalgebra $(R, [\cdot_\lambda \cdot])$ can be endowed with a left Leibniz conformal superalgebra structure $(R, [\cdot_\lambda \cdot]')$ via $[a_\lambda b]' = -(-1)^{\alpha\beta}[b_{-\lambda-\partial} a]$ for any $a \in R_\alpha$ and $b \in R_\beta$ and vice versa. Since the two algebra structures can be obtained from each other, in this paper, we only study Leibniz conformal superalgebras.

Remark 2.7. *Obviously, all Lie conformal superalgebras are Leibniz conformal superalgebras. In the next section, we will provide an efficient way to construct a class of Leibniz conformal superalgebras which are not Lie conformal superalgebras.*

Example 2.8. *Let L be a Leibniz superalgebra. Then* $CurL = \mathbb{C}[\partial]\otimes L$ *where* $(CurL)_\alpha = \mathbb{C}[\partial]\otimes L_\alpha$ *for* $\alpha \in \{\bar{0},\bar{1}\}$ *can be endowed with a Lebiniz conformal superalgebra structure as follows*

$$[a_\lambda b] = [a,b], \qquad a,\ b \in L. \tag{2.3}$$

CurL is called the current Leibniz superalgebra *associated with L.*

Suppose R is a Lie (or Leibniz) conformal superalgebra. Write $[a_\lambda b] = \sum_{i\in\mathbb{Z}_+}\frac{\lambda^i}{i!}a_{(i)}b$, where $a_{(i)}b \in R$ for every i. There is a natural Lie (or Leibniz) superalgebra associated with it. Let Coeff(R) be the quotient of the $\mathbb{Z}/2\mathbb{Z}$-graded vector space with the basis a_n $(a\in R, n\in\mathbb{Z})$ by the subspace spanned over $\mathbb{C}$ by elements:

$$(\alpha a)_n - \alpha a_n,\ \ (a+b)_n - a_n - b_n,\ \ (\partial a)_n + n a_{n-1}, \quad \text{where}\ \ a,\ \ b\in R,\ \ \alpha\in\mathbb{C},\ \ n\in\mathbb{Z}.$$

Here, if $a\in R_\alpha$, $a_n \in (\mathrm{Coeff}(R))_\alpha$. The operation on Coeff$R$ is defined as follows:

$$[a_m, b_n] = \sum_{j\in\mathbb{Z}_+}\begin{pmatrix} m \\ j\end{pmatrix}(a_{(j)}b)_{m+n-j}. \tag{2.4}$$

Then, Coeff(R) is a Lie (or Leibniz) superalgebra. Of course, if R is not torsion as a $\mathbb{C}[\partial]$-module, Coeff(R) is infinite-dimensional.

Definition 2.9. Suppose that $R=\mathbb{C}[\partial]V$ is a free $\mathbb{C}[\partial]$-module over a $\mathbb{Z}/2\mathbb{Z}$-graded vector space V, where $R_\alpha = \mathbb{C}[\partial]V_\alpha$ for any $\alpha\in\{\bar{0},\bar{1}\}$. The Lie (or Leibniz) conformal supperalgebra R is called *quadratic* if for any a, $b\in V$, the corresponding λ-bracket is of the following form:

$$[a_\lambda b] = \partial u + \lambda v + w, \qquad u,\ v,\ w \in V. \tag{2.5}$$

Definition 2.10. *A* Novikov superalgebra $(A,\circ)$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded vector space* $A = A_{\bar{0}}\oplus A_{\bar{1}}$ *with an operation "*$\circ$*" satisfying* $A_\alpha\circ A_\beta \subset A_{\alpha+\beta}$ *and the following axioms: for* $a\in A_\alpha$, $b\in A_\beta$, $c\in A_\gamma$,

$$(a\circ b)\circ c = (-1)^{\beta\gamma}(a\circ c)\circ b,$$
$$(a\circ b)\circ c - a\circ(b\circ c) = (-1)^{\alpha\beta}[(b\circ a)\circ c - b\circ(a\circ c)].$$

When $A_{\bar{1}} = 0$, *we call A a* Novikov algebra.

Remark 2.11. *Novikov algebra was essentially stated in [19] that it corresponds to a certain Hamiltonian operator. Such an algebraic structure also appeared in [5] from the point of view of Poisson structures of hydrodynamic type. The name "Novikov algebra" was given by Osborn in [25].*

Definition 2.12. *(see [26]) A* super Gel'fand-Dorfman bialgebra $(\mathscr{A},[\cdot,\cdot],\circ)$ *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded vector space* $\mathscr{A}=\mathscr{A}_{\bar{0}}\oplus\mathscr{A}_{\bar{1}}$ *with two algebraic operations* $[\cdot,\cdot]$ *and* $\circ$ *such that* $(\mathscr{A},[\cdot,\cdot])$ *forms a Lie superalgebra,* $(\mathscr{A},\circ)$ *forms a Novikov superalgebra and the following compatibility condition holds:*

$$[a\circ b,c]+[a,b]\circ c-a\circ[b,c]-(-1)^{\beta\gamma}[a\circ c,b] \tag{2.6}$$

$$-(-1)^{\beta\gamma}[a,c]\circ b=0,$$

for $a\in\mathscr{A}$*,* $b\in\mathscr{A}_{\beta}$*, and* $c\in\mathscr{A}_{\gamma}$*. When* $\mathscr{A}_{\bar{1}}=0$*, it is called a* Gel'fand-Dorfman bialgebra.

Theorem 2.13. *(see [26]) A Lie conformal superalgebra* $R=\mathbb{C}[\partial]V$ *is quadratic if and only if* $(V,[\cdot,\cdot],\circ)$ *is a super Gel'fand-Dorfman bialgebra. The correspondence is given as follows*

$$[a_{\lambda}b]=\partial(b\circ a)+\lambda(b*a)+[b,a], \tag{2.7}$$

where $b*a=b\circ a+(-1)^{\alpha\beta}a\circ b$ *for any* $a\in V_{\alpha}$ *and* $b\in V_{\beta}$*.*

3 Characterization of quadratic Leibniz conformal superalgebras

In this section, we give an equivalent characterization of quadratic Leibniz conformal superalgebra and present several constructions of quadratic Leibniz conformal superalgebras.

Theorem 3.1. *Let* V *be a* $\mathbb{Z}/2\mathbb{Z}$*-graded vector space. A quadratic Leibniz conformal superalgebra* $R=\mathbb{C}[\partial]V$ *is equivalent to the quadruple* $(V,\circ,*,[\cdot,\cdot])$ *where* $\circ$*,* $*$ *are two operations on* V *with* $V_{\alpha}\circ V_{\beta}\subset V_{\alpha+\beta}$ *and* $V_{\alpha}*V_{\beta}\subset V_{\alpha+\beta}$*,* $[\cdot,\cdot]$ *is a left Leibniz superalgebra operation, and they satisfy the following conditions:*

$$(x\circ y)\circ z+(x\circ y)*z=-x\circ(y\circ z)+x\circ(y*z)-(-1)^{\alpha\beta}(y\circ(x*z)-y*(x*z)), \tag{3.1}$$

$$(x\circ y)\circ z=-(-1)^{\alpha\beta}(-y*(x\circ z)+y\circ(x\circ z)), \tag{3.2}$$

$$(x*y)\circ z=-x\circ(y\circ z)+(-1)^{\alpha\beta}(2y*(x\circ z)-y\circ(x\circ z)), \tag{3.3}$$

$$(x\circ y)*z=-x*(y\circ z)+x*(y*z), \tag{3.4}$$

$$(x*y)*z=-2x*(y\circ z)+x*(y*z)+(-1)^{\alpha\beta}y*(x*z), \tag{3.5}$$

$$-x*(y\circ z)+(-1)^{\alpha\beta}y*(x\circ z)=0, \tag{3.6}$$

$$[x,y]*z+[x\circ y,z]=-[x,y\circ z]+[x,y*z]+x*[y,z]-(-1)^{\alpha\beta}[y,x*z], \tag{3.7}$$

$$[x,y]\circ z+[x\circ y,z]=x\circ[y,z]-(-1)^{\alpha\beta}([y,x\circ z]+y\circ[x,z]-y*[x,z]), \tag{3.8}$$

$$[x*y,z]=-[x,y\circ z]+x*[y,z]-(-1)^{\alpha\beta}([y,x\circ z]-y*[x,z]), \tag{3.9}$$

for any $x\in V_{\alpha}$*,* $y\in V_{\beta}$ *and* $z\in V$*.*

Proof. Suppose that R is a quadratic Leibniz conformal superalgebra. Then, by its definition, we set

$$[x_\lambda y]=\partial(y\circ x)+\lambda(y*x)+[y,x],\quad \text{where } x,\, y\in V, \tag{3.10}$$

where $\circ$, $*$ and $[\cdot,\cdot]$ are three $\mathbb{C}$-bilinear maps from $V\times V\to V$. Then, it is easy to see that $[R_{\alpha\lambda}R_\beta]\subset R_{\alpha+\beta}[\lambda]$ if and only if $V_\alpha\circ V_\beta\subset V_{\alpha+\beta}$, $V_\alpha * V_\beta\subset V_{\alpha+\beta}$, and $[V_\alpha,V_\beta]\subset V_{\alpha+\beta}$.

Next, we consider the Leibniz identity. For any $x\in V$, $y\in V_\beta$ and $z\in V_\gamma$, we get

$$\begin{aligned}
[x_\lambda\,[y_\mu\,z]] &= [x_\lambda(\partial\,(z\circ y)+\mu(z*y)+[z,y])]\\
&= (\lambda+\partial)[x_\lambda\,(z\circ y)]+\mu[x_\lambda\,(z*y)]+[x_\lambda\,[z,y]]\\
&= (\lambda+\partial)(\partial(z\circ y)\circ x+\lambda(z\circ y)*x+[z\circ y,x])\\
&\quad +\mu(\partial(z*y)\circ x+\lambda(z*y)*x+[z*y,x])\\
&\quad +\partial[z,y]\circ x+\lambda[z,y]*x+[[z,y],x]\\
&= \lambda\partial((z\circ y)\circ x+(z\circ y)*x)+\lambda^2(z\circ y)*x\\
&\quad +\partial^2(z\circ y)\circ x+\partial([z,y]\circ x+[z\circ y,x])\\
&\quad +\mu\partial(z*y)\circ x+\mu\lambda(z*y)*x+\mu[z*y,x]\\
&\quad +\lambda([z,y]*x+[z\circ y,x])+[[z,y],x].
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
[[x_\lambda\,y]_{\lambda+\mu}\,z]] &= [(\partial(y\circ x)+\lambda(y*x)+[y,x])_{\lambda+\mu}\,z]\\
&= (-\lambda-\mu)[(y\circ x)_{\lambda+\mu}\,z]+\lambda[(y*x)_{\lambda+\mu}\,z]+[[y,x]_{\lambda+\mu}\,z]\\
&= (-\lambda-\mu)(\partial z\circ(y\circ x)+(\lambda+\mu)z*(y\circ x)+[z,y\circ x])\\
&\quad +\lambda(\partial z\circ(y*x)+(\lambda+\mu)z*(y*x)+[z,y*x])\\
&\quad +\partial z\circ[y,x]+(\lambda+\mu)z*[y,x]+[z,[y,x]]\\
&= -\lambda^2(z*(y\circ x)-z*(y*x))-\mu^2 z*(y\circ x)-\lambda\mu(2z*(y\circ x)-z*(y*x))\\
&\quad +\lambda\partial(-z\circ(y\circ x)+z\circ(y*x))-\mu\partial z\circ(y\circ x)+\mu(-[z,y\circ x]+z*[y,x])\\
&\quad +\partial z\circ[y,x]+\lambda(-[z,y\circ x]+[z,y*x]+z*[y,x])+[z,[y,x]].
\end{aligned}$$

On the other hand, one can have

$$\begin{aligned}
[[x_\lambda z]_{-\mu-\partial} y] &= [(\partial(z\circ x)+\lambda(z*x)+[z,x])_{-\mu-\partial} y] \\
&= (\mu+\partial)[(z\circ x)_{-\mu-\partial} y]+\lambda[(z*x)_{-\mu-\partial} y]+[[z,x]_{-\mu-\partial} y] \\
&= (\mu+\partial)(\partial y\circ(z\circ x)-(\mu+\partial)y*(z\circ x)+[y,z\circ x]) \\
&\quad +\lambda(\partial y\circ(z*x)-(\mu+\partial)y*(z*x)+[y,z*x]) \\
&\quad +\partial y\circ[z,x]-(\mu+\partial)y*[z,x]+[y,[z,x]] \\
&= -\mu^2 y*(z\circ x)+\partial^2(-y*(z\circ x)+y\circ(z\circ x))-\mu\partial(2y*(z\circ x)-y\circ(z\circ x)) \\
&\quad +\lambda\partial(y\circ(z*x)-y*(z*x))-\lambda\mu y*(z*x) \\
&\quad +\partial([y,z\circ x]+y\circ[z,x]-y*[z,x])+\lambda[y,z*x] \\
&\quad +\mu([y,z\circ x]-y*[z,x])+[y,[z,x]]
\end{aligned}$$

Then, by the Leibniz identity, comparing the coefficients of $\lambda\partial$, ∂^2, $\mu\partial$, λ^2, $\lambda\mu$, μ^2, λ, ∂, μ and $\lambda^0\mu^0\partial^0$, and replacing the positions of z and x in these coefficients, we get $(3.1)-(3.9)$ and $[\cdot,\cdot]$ is a left Leibniz super-algebraic operation.

Now, the proof is finished. □

Corollary 3.2. *If for any $x, y\in V$, $x*y=2x\circ y$ in the quadruple $(V,\circ,*,[\cdot,\cdot])$, then the conditions (3.1)-(3.9) are equivalent to the following equalities:*

$$(x\circ y)\circ z = x\circ(y\circ z), \tag{3.11}$$

$$x\circ(y\circ z) = (-1)^{\alpha\beta} y\circ(x\circ z), \tag{3.12}$$

$$2x\circ[y,z] = [x\circ y,z]+(-1)^{\alpha\beta}[y,x\circ z], \tag{3.13}$$

$$2[x,y]\circ z = [x,y\circ z]-(-1)^{\alpha\beta}[y,x\circ z], \tag{3.14}$$

$$[x,y]\circ z = -x\circ[y,z]+(-1)^{\alpha\beta} y\circ[x,z], \tag{3.15}$$

for $x\in V_\alpha$, $y\in V_\beta$ and $z\in V$.

Proof. Since for any $x, y\in V$, $x*y=2x\circ y$, from (3.4) and (3.6), we can directly obtain (3.11) and (3.12) respectively. Then, according to (3.11) and (3.12), it is easy to check that (3.1), (3.2), (3.3) and (3.5) hold. Then, we only need to prove that (3.7)-(3.9) are equivalent to (3.13)-(3.15). With the assumption, (3.7), (3.8) and (3.9) become

$$2[x,y]\circ z+[x\circ y,z] = [x,y\circ z]+2x\circ[y,z]-(-1)^{\alpha\beta}2[y,x\circ z], \tag{3.16}$$

$$[x,y]\circ z+[x\circ y,z] = x\circ[y,z]-(-1)^{\alpha\beta}([y,x\circ z]-y\circ[x,z]), \tag{3.17}$$

$$2[x\circ y,z] = -[x,y\circ z]+2x\circ[y,z]-(-1)^{\alpha\beta}([y,x\circ z]-2y\circ[x,z]). \tag{3.18}$$

By (3.17), we get

$$2[x\circ y,z] = -2[x,y]\circ z+2x\circ[y,z]-(-1)^{\alpha\beta}2([y,x\circ z]-y\circ[x,z]). \tag{3.19}$$

Then, according to (3.18) and (3.19), we obtain (3.14). By (3.16) and (3.17), we have

$$[x,y]\circ z = x\circ [y,z] + [x,y\circ z] - (-1)^{\alpha\beta} y\circ [x,z] - (-1)^{\alpha\beta}[y,x\circ z]. \tag{3.20}$$

Then, (3.15) follows from (3.20) and (3.14). Finally, by (3.15) and (3.17), we get (3.13). Therefore, (3.7)-(3.9) are equivalent to (3.13)-(3.15). This completes the proof.

□

Remark 3.3. *In this case, by Theorem 2.13 and Theorem 3.1, if $(V,\circ)$ is not super-commutative, then the corresponding Leibniz conformal superalgebra is not a Lie conformal superalgebra. This provides an useful method to construct Leibniz conformal superalgebras which are not Lie conformal superalgebras.*

Corollary 3.4. *If for any $x\in V_\alpha$, $y\in V_\beta$, $x*y = x\circ y + (-1)^{\alpha\beta} y\circ x$ in the quadruple $(V,\circ,*,[\cdot,\cdot])$, then the conditions (3.1)-(3.9) are equivalent to that $(V,\circ)$ is a Novikov superalgebra and the following equalities hold:*

$$[x,y]\circ z + [x\circ y,z] = x\circ [y,z] - (-1)^{\alpha\beta}[y,x\circ z] + (-1)^{\beta\gamma}[x,z]\circ y, \tag{3.21}$$

$$([x,y] + (-1)^{\alpha\beta}[y,x])\circ z = 0, \tag{3.22}$$

$$[x\circ y,z] = -(-1)^{\gamma(\alpha+\beta)}[z,x\circ y], \tag{3.23}$$

for any $x\in V_\alpha$, $y\in V_\beta$ and $z\in V_\gamma$.

Proof. According to Theorem 2.13, (3.1)-(3.6) are equivalent to that $(V,\circ)$ is a Novikov superalgebra. Then, we only need to consider (3.7)-(3.9).

By the assumption, (3.8) is equal to (3.21) and (3.7), (3.9) become

$$\begin{aligned}[x,y]\circ z + (-1)^{\gamma(\alpha+\beta)} z\circ [x,y] + [x\circ y,z] &= (-1)^{\beta\gamma}[x,z\circ y] + x\circ [y,z] \\ &+(-1)^{\alpha(\beta+\gamma)}[y,z]\circ x - (-1)^{\alpha\beta}[y,x\circ z] - (-1)^{\alpha(\beta+\gamma)}[y,z\circ x],\end{aligned} \tag{3.24}$$

$$\begin{aligned}[x\circ y,z] + (-1)^{\alpha\beta}[y\circ x,z] &= -[x,y\circ z] + x\circ [y,z] + (-1)^{\alpha(\beta+\gamma)}[y,z]\circ x \\ &-(-1)^{\alpha\beta}[y,x\circ z] + (-1)^{\alpha\beta} y\circ [x,z] + (-1)^{\beta\gamma}[x,z]\circ y.\end{aligned} \tag{3.25}$$

By (3.21), it follows from (3.25) that

$$-[x,y]\circ z + (-1)^{\alpha\beta}[y\circ x,z] = -[x,y\circ z] + (-1)^{\alpha(\beta+\gamma)}[y,z]\circ x + (-1)^{\alpha\beta} y\circ [x,z]. \tag{3.26}$$

Changing the places of x, y in (3.26), we get

$$-[y,x]\circ z + (-1)^{\alpha\beta}[x\circ y,z] = -[y,x\circ z] + (-1)^{\beta(\alpha+\gamma)}[x,z]\circ y + (-1)^{\alpha\beta} x\circ [y,z]. \tag{3.27}$$

Then, by (3.21), (3.27) can be reduced to (3.22). Similarly, by (3.21) and (3.22), (3.24) can be simplified as (3.23).

Therefore, in this case, (3.1)-(3.9) are equivalent to that $(V,\circ)$ is a Novikov superalgebra, (3.21), (3.22) and (3.23) hold. □

Remark 3.5. *By Theorem 2.13, a quadratic Lie conformal superalgebra is equivalent to a super Gel'fand-Dorfman bialgebra. If in this case, the Leibniz superalgebra is a Lie superalgebra, the corresponding quadratic Leibniz conformal superalgebra is a Lie conformal superalgebras. (3.22) and (3.23) measure how far a quadratic Leibniz conformal superalgebras is a Lie conformal superalgebras in this case.*

Corollary 3.6. *If $*$ is trivial, i.e. for any x, $y \in V$, $x*y=0$ in the quadruple $(V,\circ,*,[\cdot,\cdot])$, then (3.1)-(3.9) are equivalent to the following equalities*

$$(x\circ y)\circ z = x\circ(y\circ z) = 0, \tag{3.28}$$

$$[x\circ y,z] = -[x,y\circ z],\quad [x,y\circ z] = -(-1)^{\alpha\beta}[y,x\circ z], \tag{3.29}$$

$$[x,y]\circ z + [x\circ y,z] = x\circ[y,z] - (-1)^{\alpha\beta}[y,x\circ z] - (-1)^{\alpha\beta} y\circ[x,z], \tag{3.30}$$

for any $x\in V_\alpha$, $y\in V_\beta$ and $z\in V_\gamma$. Moreover, letting $[x,y] = ax\circ y - (-1)^{\alpha\beta} by\circ x$ for any $x\in V_\alpha$, $y\in V_\beta$ and a, $b\in\mathbb{C}$, (3.28)-(3.30) and the fact $[\cdot,\cdot]$ is a Leibniz bracket are equivalent to (3.28).

Proof. It can be easily obtained from Theorem 3.1. ☐

Corollary 3.7. *If $\circ$ is trivial, i.e. for any x, $y\in V$, $x\circ y = 0$ in the quadruple $(V,\circ,*,[\cdot,\cdot])$, then (3.1)-(3.9) are equivalent to the following equalities*

$$(x*y)*z = x*(y*z) = 0, \tag{3.31}$$

$$x*[y,z] = [x*y,z] = 0, \tag{3.32}$$

$$[x,y]*z = [x,y*z] - (-1)^{\alpha\beta}[y,x*z], \tag{3.33}$$

for any $x\in V_\alpha$, $y\in V_\beta$, $z\in V$. In addition, letting $[x,y] = ax*y - (-1)^{\alpha\beta} by*x$ for any $x\in V_\alpha$, $y\in V_\beta$ and a, $b\in\mathbb{C}$, (3.31)-(3.33) and the fact $[\cdot,\cdot]$ is a Leibniz bracket are equivalent to (3.31).

Proof. It can be directly obtained from Theorem 3.1. ☐

Next, we present an example to construct Leibniz conformal superalgebras according to Corollaries 3.6 and 3.7.

Example 3.8. *Let $V=\mathbb{C}a\oplus\mathbb{C}b$ be a 2-dimensional vector space. Define a bilinear operation $\star : V\times V\to V$ by*

$$a\star a = b,\quad a\star b = b\star a = b\star b = 0. \tag{3.34}$$

Then, $(V,\star)$ satisfies the equalities (3.28) and (3.31), if we replace $\circ$ and $*$ by $\star$ respectively.

Therefore, according to Theorem 3.1 and Corollary 3.6, we can get the following quadratic Leibniz conformal algebra $R=\mathbb{C}[\partial]V=\mathbb{C}[\partial]a\oplus\mathbb{C}[\partial]b$ with the λ-product given by

$$[a_\lambda a] = \partial b,\quad [a_\lambda b] = [b_\lambda a] = [b_\lambda b] = 0. \tag{3.35}$$

Similarly, by Theorem 3.1 and Corollary 3.7, we also obtain a quadratic Leibniz conformal algebra $R=\mathbb{C}[\partial]V=\mathbb{C}[\partial]a\oplus\mathbb{C}[\partial]b$ with the λ-product

$$[a_\lambda a]=\lambda b,\quad [a_\lambda b]=[b_\lambda a]=[b_\lambda b]=0. \tag{3.36}$$

Remark 3.9. *In fact, the above example can be generalized as follows. Let $R=\mathbb{C}[\partial]V=\mathbb{C}[\partial]a\oplus\mathbb{C}[\partial]b$ with the λ-product as follows:*

$$[a_\lambda a]=f(\lambda,\partial)b,\ \ [a_\lambda b]=[b_\lambda a]=[b_\lambda b]=0. \tag{3.37}$$

Obviously, R is a Leibniz conformal algebra. Moreover, it is easy to see that R is a Lie conformal algebra if and only if $f(\lambda,\partial)=-f(-\lambda-\partial,\partial)$.

Since in the case of Corollary 3.4, $(V,[\cdot,\cdot],\circ)$ is almost a super Gel'fand-Dorfman bialgebra, in the following, we mainly study the case in Corollary 3.2.

Definition 3.10. *Given a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V=V_{\bar{0}}\oplus V_{\bar{1}}$ with two operations $\circ$ and $[\cdot,\cdot]$, where $V_\alpha\circ V_\beta\subset V_{\alpha+\beta}$ and $[V_\alpha,V_\beta]\subset V_{\alpha+\beta}$. We call $(V,\circ)$ an* associative Novikov superalgebra *if $\circ$ satisfies (3.11) and (3.12). $(V,\circ,[\cdot,\cdot])$ is called an* ass-Nov-Leibniz superalgebra *if $(V,[\cdot,\cdot])$ is a left-Leibniz superalgebra and $\circ$, $[\cdot,\cdot]$ satisfy (3.11)-(3.15).*

Remark 3.11. *Note that the opposite superalgebra of $(V,\circ)$ is an associative and Novikov superalgebra. This is the reason that we take this name.*

Remark 3.12. *Obviously, for an associative Novikov superalgebra $(V,\circ)$, $(V,\circ,[\cdot,\cdot])$ is an ass-Nov-Leibniz superalgebra with $[a,b]=0$ for any a, $b\in V$. In this case, we say the quadratic Leibniz conformal superalgebra $R=\mathbb{C}[\partial]V$ corresponds to the associative Novikov superalgebra $(V,\circ)$.*

Remark 3.13. *By Theorem 3.1, the quadratic Leibniz conformal superalgebra corresponding to the ass-Nov-Leibniz superalgebra $(V,\circ,[\cdot,\cdot])$ is $R=\mathbb{C}[\partial]V$ with the λ-bracket given by*

$$[a_\lambda b]=\partial(b\circ a)+2\lambda(b\circ a)+[b,a], \tag{3.38}$$

for any a, $b\in V$.

Therefore, the coefficient algebra of $R=\mathbb{C}[\partial]V$ is an infinite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V\otimes\mathbb{C}[t,t^{-1}]=V_{\bar{0}}\otimes\mathbb{C}[t,t^{-1}]\oplus V_{\bar{1}}\otimes\mathbb{C}[t,t^{-1}]$ with the Leibniz bracket given by

$$[a\otimes t^m,b\otimes t^n]=(m-n)(b\circ a)\otimes t^{m+n-1}+[b,a]\otimes t^{m+n},\quad a,\ \ b\in V,\ \ m,\ \ n\in\mathbb{Z}. \tag{3.39}$$

Next, we give a construction of associative Novikov superalgebras from commutative associative superalgebras.

Proposition 3.14. *Let* $(A = A_{\bar{0}} \oplus A_{\bar{1}}, \cdot)$ *be a commutative associative superalgebra and* $P : A \to A$ *is an even operator, i.e.* $P(A_\alpha) \subset A_\alpha$ *satisfying*

$$P(P(x) \cdot y) = P(x) \cdot P(y), \ \ x, \ \ y \in A. \tag{3.40}$$

Define

$$x \circ y = P(x) \cdot y, \quad x, \ \ y \in A. \tag{3.41}$$

Then, $(A, \circ)$ *is an associative Novikov superalgebra.*

Proof. For any $x \in A_\alpha$, $y \in A_\beta$ and $z \in A$, we obtain

$$(x \circ y) \circ z = P(P(x) \cdot y) \cdot z,$$

$$x \circ (y \circ z) = P(x) \cdot P(y) \cdot z,$$

and

$$(-1)^{\alpha\beta} y \circ (x \circ z) = (-1)^{\alpha\beta} P(y) \cdot P(x) \cdot z.$$

Since A is a commutative associative superalgebra and P satisfies (3.40), $(x \circ y) \circ z = x \circ (y \circ z) = (-1)^{\alpha\beta} y \circ (x \circ z)$. Therefore, $(A, \circ)$ is an associative Novikov superalgebra. ☐

Remark 3.15. $P : A \to A$ *satisfying (3.40) is called* an averaging operator *on A. More details on averaging operators can be referred to [20].*

Finally, we present an example of ass-Nov-Leibniz algebra.

Example 3.16. *Let* $A = \mathbb{C}L \oplus \mathbb{C}W$ *be a 2-dimensional vector space with an operation defined by*

$$L \circ L = L \circ W = 0, \ \ W \circ L = L, \ \ W \circ W = W. \tag{3.42}$$

It is easy to see that $(A, \circ)$ *is an associative Novikov algebra. Obviously, it is not commutative.*

By a simple computation, we can get that any Leibniz algebra structure $[\cdot, \cdot]$ *on A for* $(A, \circ, [\cdot, \cdot])$ *to be an ass-Nov-Leibniz algebra is as follows:*

$$[L, L] = [L, W] = 0, \quad [W, L] = aL, \ \ [W, W] = bL, \tag{3.43}$$

for some a, $b \in \mathbb{C}$.

Therefore, by Theorem 3.1, we can get a quadratic Leibniz conformal algebra $R = \mathbb{C}[\partial]A$ *corresponding to* $(A, \circ, [\cdot, \cdot])$ *with the* λ*-product as follows:*

$$[L_\lambda L] = [W_\lambda L] = 0, \tag{3.44}$$

$$[L_\lambda W] = (\partial + 2\lambda + a)L, \ \ [W_\lambda W] = (\partial + 2\lambda)W + bL, \tag{3.45}$$

for some a, $b \in \mathbb{C}$. *Denote this Leibniz conformal algebra by* $R_{a,b}$.

By the correspondence of Leibniz conformal algebra and Leibniz algebra, we can also obtain that Coeff$(R_{a,b})$ *is the vector space spanned by* $\{L_n, W_n|_{n \in \mathbb{Z}}\}$ *with the Leibniz product*

$$[L_m, L_n] = [W_n, L_m] = 0, \ \ [L_m, W_n] = (m - n)L_{m+n-1} + aL_{m+n},$$
$$[W_m, W_n] = (m - n)W_{m+n-1} + bL_{m+n}.$$

4 Central extensions of quadratic Leibniz conformal superalgebras

In this section, we will study central extensions of quadratic Leibniz conformal superalgebras.

An extension of a Leibniz conformal superalgebra R by an abelian Leibniz conformal superalgebra C is a short exact sequence of Leibniz conformal superalgebras

$$0 \to C \to \widehat{R} \to R \to 0.$$

$\widehat{R}$ is called an *extension* of R by C in this case. This extension is called *central* if $\partial C = 0$ and $[C_\lambda \widehat{R}] = 0$. Moreover, if R is a Lie conformal superalgebra, we call this extension a *Leibniz central extension* of this Lie conformal superalgebra.

In the following, we investigate the central extensions $\widehat{R}$ of R by a one-dimensional center $\mathbb{C}\mathfrak{c}$, where $\mathfrak{c} \in \widehat{R}_{\bar{0}}$. This implies that $\widehat{R} = R \oplus \mathbb{C}\mathfrak{c}$, and

$$[a_\lambda b]_{\widehat{R}} = [a_\lambda b]_R + \alpha_\lambda(a,b)\mathfrak{c}, \text{for all } \ a,\, b \in R,$$

where $\alpha_\lambda : R \times R \to \mathbb{C}[\lambda]$ is a $\mathbb{C}$-bilinear map. Since $[(\widehat{R}_\alpha)_\lambda \widehat{R}_\beta] \subset \widehat{R}_{\alpha+\beta}[\lambda]$, we get $\alpha_\lambda(a,b) = 0$, where $a \in R_\alpha$, $b \in R_\beta$ and $\alpha + \beta = \bar{1}$. By the axioms of Leibniz conformal superalgebra, α_λ should satisfy the following properties (for all $a \in R$, $b \in R_\beta$, $c \in R_\gamma$) :

$$\alpha_\lambda(\partial a, b) = -\lambda \alpha_\lambda(a,b) = -\alpha_\lambda(a, \partial b), \tag{4.1}$$

$$\alpha_\lambda(a, [b_\mu c]) = \alpha_{\lambda+\mu}([a_\lambda b], c) - (-1)^{\beta\gamma} \alpha_{-\mu}([a_\lambda c], b). \tag{4.2}$$

Theorem 4.1. *Let V be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space and $\widehat{R} = R \oplus \mathbb{C}\mathfrak{c}$ be a central extension of quadratic Leibniz conformal superalgebra $R = \mathbb{C}[\partial]V$ corresponding to the ass-Nov-Leibniz superalgebra $(V, \circ, [\cdot,\cdot])$ by a one-dimensional center $\mathbb{C}\mathfrak{c}$. Set the λ-bracket of $\widehat{R}$ by*

$$\widetilde{[a_\lambda b]} = \partial(b \circ a) + \lambda(b * a) + [b,a] + \alpha_\lambda(a,b)\mathfrak{c}, \tag{4.3}$$

where a, $b \in V$, $b * a = 2b \circ a$ and $\alpha_\lambda(a,b) \in \mathbb{C}[\lambda]$. Assume that $\alpha_\lambda(a,b) = \sum_{i=0}^n \lambda^i \alpha_i(a,b)$ for any a, $b \in V$, where $\alpha_i(\cdot,\cdot) : V \times V \to \mathbb{C}$ are bilinear forms and there exist some a, $b \in V$ such that $\alpha_n(a,b) \neq 0$. Then, we obtain (for any $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$):

(1) If $n > 3$, $\alpha_n(a \circ b, c) = 0$;

(2) If $n \leq 3$, $\alpha_i(x,y) = 0$ if $x \in V_\alpha$, $y \in V_\beta$, $\alpha + \beta = \overline{1}$ for any $i \in \{0,1,2,3\}$, and

$$\alpha_3(a, c \circ b) = \alpha_3(b \circ a, c) = (-1)^{\beta\gamma} \alpha_3(c \circ a, b), \tag{4.4}$$

$$\alpha_2(a, c \circ b) + \alpha_3(a, [c,b]) = \alpha_2(b \circ a, c) + \alpha_3([b,a], c), \tag{4.5}$$

$$2\alpha_2(a, c \circ b) = \alpha_2(b \circ a, c) + 3\alpha_3([b,a], c), \tag{4.6}$$

$$\alpha_2(a, c \circ b) = \alpha_2(b \circ a, c) + (-1)^{\beta\gamma} \alpha_2(c \circ a, b), \tag{4.7}$$

$$\alpha_2(b \circ a, c) + (-1)^{\beta\gamma} \alpha_2(c \circ a, b) = \alpha_3([b,a], c) + (-1)^{\beta\gamma} \alpha_3([c,a], b), \tag{4.8}$$

$$\alpha_1(a, c \circ b) + \alpha_2(a, [c,b]) = \alpha_1(b \circ a, c) + \alpha_2([b,a], c), \tag{4.9}$$

$$\alpha_1(a, c \circ b) = \alpha_2([b,a], c) + (-1)^{\beta\gamma} \alpha_1(c \circ a, b), \tag{4.10}$$

$$\alpha_1(b \circ a, c) - (-1)^{\beta\gamma} \alpha_1(c \circ a, b) = \alpha_2([b,a], c) - (-1)^{\beta\gamma} \alpha_2([c,a], b), \tag{4.11}$$

$$\alpha_0(a, c \circ b) + \alpha_1(a, [c,b]) = \alpha_0(b \circ a, c) + \alpha_1([b,a], c) - (-1)^{\beta\gamma} 2\alpha_0(c \circ a, b), \tag{4.12}$$

$$2\alpha_0(a, c \circ b) = -\alpha_0(b \circ a, c) + \alpha_1([b,a], c) - (-1)^{\beta\gamma} \alpha_0(c \circ a, b) + (-1)^{\beta\gamma} \alpha_1([c,a], b), \tag{4.13}$$

$$\alpha_0(a, [c,b]) = \alpha_0([b,a], c) - (-1)^{\beta\gamma} \alpha_0([c,a], b). \tag{4.14}$$

Proof. By (4.1) and the definition of quadratic Leibniz conformal superalgebra $R = \mathbb{C}[\partial]V$, (4.2) becomes

$$\begin{aligned} \lambda \alpha_\lambda(a, c \circ b) + \mu \alpha_\lambda(a, c * b) + \alpha_\lambda(a, [c,b]) &= (-\lambda - \mu)\alpha_{\lambda+\mu}(b \circ a, c) \\ &+ \lambda \alpha_{\lambda+\mu}(b * a, c) + \alpha_{\lambda+\mu}([b,a], c) - (-1)^{\beta\gamma} \mu \alpha_{-\mu}(c \circ a, b) \\ &- (-1)^{\beta\gamma} \lambda \alpha_{-\mu}(c * a, b) - (-1)^{\beta\gamma} \alpha_{-\mu}([c,a], b). \end{aligned} \tag{4.15}$$

If $n > 3$, by the assumption of $\alpha_\lambda(a,b) = \sum_{i=0}^n \lambda^i \alpha_i(a,b)$ and comparing the coefficients of $\lambda^2 \mu^{n-1}$ in (4.15), we get

$$(n - C_n^2)\alpha_n(b \circ a, c) = 0. \tag{4.16}$$

Therefore, $\alpha_n(b \circ a, c) = 0$.

When $n \leq 3$, taking $\alpha_\lambda(a,b) = \sum_{i=0}^3 \lambda^i \alpha_i(a,b)$ into (4.15) and comparing the coefficients of

λ^4, $\lambda\mu^3$, $\lambda^2\mu^2$, $\lambda^3\mu$, μ^4, λ^3, $\lambda^2\mu$, $\lambda\mu^2$, μ^3, λ^2, $\lambda\mu$, μ^2, λ, μ, $\lambda^0\mu^0$, we obtain

$$\alpha_3(a,c\circ b)=-\alpha_3(b\circ a,c)+\alpha_3(b*a,c), \tag{4.17}$$

$$-4\alpha_3(b\circ a,c)+\alpha_3(b*a,c)+(-1)^{\beta\gamma}\alpha_3(c*a,b)=0, \tag{4.18}$$

$$-2\alpha_3(b\circ a,c)+\alpha_3(b*a,c)=0, \tag{4.19}$$

$$\alpha_3(a,c*b)=-4\alpha_3(b\circ a,c)+3\alpha_3(b*a,c), \tag{4.20}$$

$$-\alpha_3(b\circ a,c)+(-1)^{\beta\gamma}\alpha_3(c\circ a,b)=0, \tag{4.21}$$

$$\alpha_2(a,c\circ b)+\alpha_3(a,[c,b])=-\alpha_2(b\circ a,c)+\alpha_2(b*a,c)+\alpha_3([b,a],c), \tag{4.22}$$

$$\alpha_2(a,c*b)=-3\alpha_2(b\circ a,c)+2\alpha_2(b*a,c)+3\alpha_3([b,a],c), \tag{4.23}$$

$$-3\alpha_2(b\circ a,c)+\alpha_2(b*a,c)+3\alpha_3([b,a],c)-(-1)^{\beta\gamma}\alpha_2(c*a,b)=0, \tag{4.24}$$

$$-\alpha_2(b\circ a,c)+\alpha_3([b,a],c)-(-1)^{\beta\gamma}\alpha_2(c\circ a,b)+(-1)^{\beta\gamma}\alpha_3([c,a],b)=0, \tag{4.25}$$

$$\alpha_1(a,c\circ b)+\alpha_2(a,[c,b])=-\alpha_1(b\circ a,c)+\alpha_1(b*a,c)+\alpha_2([b,a],c), \tag{4.26}$$

$$\alpha_1(a,c*b)=-2\alpha_1(b\circ a,c)+\alpha_1(b*a,c)+2\alpha_2([b,a],c)+(-1)^{\beta\gamma}\alpha_1(c*a,b), \tag{4.27}$$

$$-\alpha_1(b\circ a,c)+\alpha_2([b,a],c)+(-1)^{\beta\gamma}\alpha_1(c\circ a,b)-(-1)^{\beta\gamma}\alpha_2([c,a],b)=0, \tag{4.28}$$

$$\alpha_0(a,c\circ b)+\alpha_1(a,[c,b])=-\alpha_0(b\circ a,c)+\alpha_0(b*a,c)+\alpha_1([b,a],c)-(-1)^{\beta\gamma}\alpha_0(c*a,b), \tag{4.29}$$

$$\alpha_0(a,c*b)=-\alpha_0(b\circ a,c)+\alpha_1([b,a],c)-(-1)^{\beta\gamma}\alpha_0(c\circ a,b)+(-1)^{\beta\gamma}\alpha_1([c,a],b), \tag{4.30}$$

$$\alpha_0(a,[c,b])=\alpha_0([b,a],c)-(-1)^{\beta\gamma}\alpha_0([c,a],b). \tag{4.31}$$

Then, it is easy to check that (4.17)-(4.31) are equivalent to (4.4)-(4.14). This completes the proof. □

Corollary 4.2. *Let $R=\mathbb{C}[\partial]V$ be a finite quadratic Leibniz conformal superalgebra corresponding to the ass-Nov-Leibniz superalgebra $(V,\circ,[\cdot,\cdot])$ where for any $z\in V$, there exist some non-negative integer m and x_i, $y_i\in V$ such that $z=\sum_{i=0}^m x_i\circ y_i$. Set $\widehat{R}=R\oplus\mathbb{C}\mathfrak{c}$ be a central extension of $(R,[\cdot_\lambda\cdot])$ with the λ-product given by (4.3). Then, for any a, $b\in V$, we obtain $\alpha_\lambda(a,b)=\sum_{i=0}^3\lambda^i\alpha_i(a,b)$ where $\alpha_i(\cdot,\cdot):V\times V\to\mathbb{C}$ are bilinear forms with $\alpha_i(x,y)=0$ if $x\in V_\alpha$, $y\in V_\beta$, $\alpha+\beta=\overline{1}$ for any $i\in\{0,1,2,3\}$ and satisfy (4.4)-(4.14).*

Proof. Since $R=\mathbb{C}[\partial]V$ is a finitely generated and free $\mathbb{C}[\partial]$-module, we can assume that $\alpha_\lambda(a,b)=\sum_{i=0}^n\lambda^i\alpha_i(a,b)$ for any a, $b\in V$ and some non-negative integer n. Then, by Theorem 4.1 and the condition that for any $z\in V$, there exist some m and x_i, $y_i\in V$ such that $z=\sum_{i=0}^m x_i\circ y_i$, we get that if $n>3$, for any $z\in V$ and $c\in V$, $\alpha_n(z,c)=\sum_{i=0}^m\alpha_n(x_i\circ y_i,c)=0$. Hence, for any a, $b\in V$, we obtain $\alpha_\lambda(a,b)=\sum_{i=0}^3\lambda^i\alpha_i(a,b)$. Then, by Theorem 4.1, we can obtain this corollary. □

Remark 4.3. *It should be pointed out that there are some natural conditions to make the Novikov algebra $(V,\circ)$ satisfy the assumption in Corollary 4.2. For example, $(V,\circ)$ is simple or $(V,\circ)$ has a left unit or a right unit.*

Corollary 4.4. *Let $R=\mathbb{C}[\partial]V$ be a quadratic Leibniz conformal superalgebra corresponding to the associative Novikov superalgebra $(V,\circ)$ where for any $z\in V$, there exist some non-negative integer m and $x_i, y_i\in V$ such that $z=\sum_{i=0}^m x_i\circ y_i$. Set $\widehat{R}=R\oplus\mathbb{C}\mathfrak{c}$ be a central extension of $(R,[\cdot_\lambda\cdot])$ with the λ-product given as follows:*

$$\widetilde{[a_\lambda b]}=\partial(b\circ a)+2\lambda(b\circ a)+\alpha_\lambda(a,b),\qquad a,\ b\in V. \tag{4.32}$$

Then, for any a, $b\in V$, we obtain $\alpha_\lambda(a,b)=\alpha_0(a,b)+\lambda\alpha_1(a,b)+\lambda^3\alpha_3(a,b)$ and the bilinear forms $\alpha_0(\cdot,\cdot)$, $\alpha_1(\cdot,\cdot)$ and $\alpha_3(\cdot,\cdot)$ on V satisfy $\alpha_i(x,y)=0$ if $x\in V_\alpha$, $y\in V_\beta$, $\alpha+\beta=\overline{1}$ for any $i\in\{1,3\}$ and the following equalities:

$$\alpha_3(a,c\circ b)=\alpha_3(b\circ a,c)=(-1)^{\beta\gamma}\alpha_3(c\circ a,b), \tag{4.33}$$

$$\alpha_1(a,c\circ b)=\alpha_1(b\circ a,c)=(-1)^{\beta\gamma}\alpha_1(c\circ a,b), \tag{4.34}$$

$$\alpha_0(a,c\circ b)=-\alpha_0(b\circ a,c)=-(-1)^{\beta\gamma}\alpha_0(c\circ a,b). \tag{4.35}$$

Proof. For any a, $b\in V$, set $\alpha_\lambda(a,b)=\sum_{i=0}^{n_{a,b}}\lambda^i\alpha_i(a,b)$ where $\alpha_i(\cdot,\cdot)$ are bilinear forms on V and $n_{a,b}$ is a non-negative integer depending on a and b. With the same process as in Theorem 4.1, (4.2) becomes

$$\begin{aligned}\lambda\alpha_\lambda(a,c\circ b)+\mu\alpha_\lambda(a,c*b)&=(-\lambda-\mu)\alpha_{\lambda+\mu}(b\circ a,c)\\ &+\lambda\alpha_{\lambda+\mu}(b*a,c)-(-1)^{\beta\gamma}\mu\alpha_{-\mu}(c\circ a,b)-(-1)^{\beta\gamma}\lambda\alpha_{-\mu}(c*a,b).\end{aligned} \tag{4.36}$$

For fixed a, b, c, there are only finite elements of V appearing in $\alpha_\lambda(\cdot,\cdot)$ in (4.36). Therefore, we may assume the degrees of all $\alpha_\lambda(\cdot,\cdot)$ in (4.36) are smaller than some non-negative integer. So, we set $\alpha_\lambda(a,c\circ b)=\sum_{i=0}^n\lambda^i\alpha_i(a,c\circ b)$, $\cdots$ and $\alpha_{-\mu}(c*a,b)=\sum_{i=0}^n(-\mu)^i\alpha_i(c*a,b)$.

If $n>3$, by comparing the coefficients of $\lambda^2\mu^{n-1}$ in (4.36), we get

$$(n-C_n^2)\alpha_n(b\circ a,c)=0. \tag{4.37}$$

Therefore, $\alpha_n(b\circ a,c)=0$. Repeating this process, we can get $\alpha_m(b\circ a,c)=0$ for all $n\geq m>3$.

By the discussion above, for any a, b, $c\in V$, we get $\alpha_m(b\circ a,c)=0$ for all $m>3$. Since for any $z\in V$, there exist some m and x_i, $y_i\in V$ such that $z=\sum_{i=0}^m x_i\circ y_i$, we get $\alpha_m(z,c)=0$ for all $m>3$. Hence, $\alpha_\lambda(a,b)=\sum_{i=0}^3\alpha_i(a,b)$. Moreover, it is easy to check that (4.33)-(4.35) are equivalent to (4.4)-(4.14) with $[\cdot,\cdot]$ trivial. Then, by Theorem 4.1, we get this corollary. □

Remark 4.5. *Note that this corollary also holds for the quadratic Leibniz conformal superalgebras corresponding to associative Novikov superalgebras which are not finite.*

Corollary 4.6. *Let $(V,\circ,[\cdot,\cdot])$ be an ass-Nov-Leibniz superalgebra and $R=\mathbb{C}[\partial]V$ be the corresponding quadratic Leibniz conformal superalgebra. Set $\alpha_i(\cdot,\cdot)$ $(i=0,1,2,3)$ be bilinear forms on V with $\alpha_i(x,y)=0$ if $x\in V_\alpha$, $y\in V_\beta$, $\alpha+\beta=\overline{1}$ and satisfy (4.4)-(4.14). Define bilinear forms φ_i : $Coeff(R)\times Coeff(R)\to\mathbb{C}$ as follows:*

$$\varphi_0(a\otimes t^m,b\otimes t^n)=\alpha_0(a,b)\delta_{m+n+1,0}, \tag{4.38}$$

$$\varphi_1(a\otimes t^m,b\otimes t^n)=m\alpha_1(a,b)\delta_{m+n,0}, \tag{4.39}$$

$$\varphi_2(a\otimes t^m,b\otimes t^n)=m(m-1)\alpha_2(a,b)\delta_{m+n-1,0}, \tag{4.40}$$

$$\varphi_3(a\otimes t^m,b\otimes t^n)=m(m-1)(m-2)\alpha_3(a,b)\delta_{m+n-2,0}, \tag{4.41}$$

for a, $b\in V$, m, $n\in\mathbb{Z}$. Then, φ_0, φ_1, φ_2, φ_3 are 2-cocycles of Leibniz superalgebra $Coeff(R)$.

Proof. Suppose that $\widehat{R}$ is the central extension defined by (4.3). Then, the coefficient algebra of $\widehat{R}$ is $\mathrm{Coeff}(\widehat{R})=\mathrm{Coeff}(R)\oplus\mathbb{C}\mathfrak{c}_{-1}$ with the following Leibniz bracket:

$$\begin{aligned}[a_m,b_n] \;=\; & [a,b]_{m+n}+m(a\circ b)_{m+n-1}-n(b\circ a)_{m+n-1}+\alpha_0(a,b)\delta_{m+n+1,0}\mathfrak{c}_{-1}\\ & +m\alpha_1(a,b)\delta_{m+n,0}\mathfrak{c}_{-1}+m(m-1)\alpha_2(a,b)\delta_{m+n-1,0}\mathfrak{c}_{-1}\\ & +m(m-1)(m-2)\alpha_3(a,b)\delta_{m+n-2,0}\mathfrak{c}_{-1},\end{aligned}$$

for any a, $b\in V$, and m, $n\in\mathbb{Z}$, and $\mathfrak{c}_{-1}$ is in the center of $\mathrm{Coeff}(\widehat{R})$. Obviously, $\mathrm{Coeff}(\widehat{R})$ is the central extension of $\mathrm{Coeff}(R)$ by a one-dimensional center $\mathbb{C}\mathfrak{c}_{-1}$. Then, we naturally obtain 4 kinds of 2-cocycles of $\mathrm{Coeff}(R)$. ☐

Example 4.7. *Let $R_{0,0}$ be the quadratic Leibniz conformal algebra corresponding to the associative Novikov algebra $(A,\circ)$ given in the Example 3.16. By Corollary 4.4, for investigating the central extensions of $R_{0,0}$ by a one-dimensional center $\mathbb{C}\mathfrak{c}$, we only need to compute the bilinear forms $\alpha_0(\cdot,\cdot)$, $\alpha_1(\cdot,\cdot)$, $\alpha_3(\cdot,\cdot)$ on A satisfying (4.33)-(4.35).*

By a simple computation, it is easy to get that $\alpha_3(L,L)=\alpha_3(W,L)=\alpha_1(L,L)=\alpha_1(W,L)=0$, $\alpha_3(L,W)=\alpha_3(W,W)=\beta$, $\alpha_1(L,W)=\alpha_1(W,W)=\gamma$ and $\alpha_0(L,L)=\alpha_0(L,W)=\alpha_0(W,L)=\alpha_0(W,W)=0$ where α, $\gamma\in\mathbb{C}$.

Therefore, any central extension of $R_{0,0}$ by a one-dimensional center $\mathbb{C}\mathfrak{c}$ is $\widetilde{R_{0,0}}=R_{0,0}\oplus\mathbb{C}\mathfrak{c}$ with $\mathfrak{c}$ the center of $\widetilde{R_{0,0}}$ and the Leibniz bracket as follows:

$$[L_\lambda L]=[W_\lambda L]=0,\quad [L_\lambda W]=(\partial+2\lambda)L+(\gamma\lambda+\beta\lambda^3)\mathfrak{c},$$

$$[W_\lambda W]=(\partial+2\lambda)W+(\gamma\lambda+\beta\lambda^3)L,$$

where β, $\gamma\in\mathbb{C}$.

Then, by Corollary 4.6, we get a central extension Coeff$(R_{0,0})\oplus\mathbb{C}\mathfrak{c}$ of Coeff$(R_{0,0})$ by a one-dimensional center $\mathbb{C}\mathfrak{c}$ with the Leibniz bracket as follows:

$$[L_m,L_n]=[W_n,L_m]=0,$$
$$[L_m,W_n]=(m-n)L_{m+n-1}+(m\gamma\delta_{m+n,0}+m(m-1)(m-2)\beta\delta_{m+n-2,0})\mathfrak{c},$$
$$[W_m,W_n]=(m-n)W_{m+n-1}+(m\gamma\delta_{m+n,0}+m(m-1)(m-2)\beta\delta_{m+n-2,0})\mathfrak{c},$$

where β, $\gamma\in\mathbb{C}$.

Next, we study the Leibniz central extensions of quadratic Lie conformal superalgebras.

Theorem 4.8. *Let V be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space and $\widehat{R}=R\oplus\mathbb{C}\mathfrak{c}$ be a Leibniz central extension of quadratic Lie conformal superalgebra $R=\mathbb{C}[\partial]V$ corresponding to the super Gel'fand-Dorfman bialgebra $(V,\circ,[\cdot,\cdot])$ by a one-dimensional center $\mathbb{C}\mathfrak{c}$. Set the λ-bracket of $\widehat{R}$ by*

$$\widetilde{[a_\lambda b]}=\partial(b\circ a)+\lambda(b*a)+[b,a]+\alpha_\lambda(a,b)\mathfrak{c},\tag{4.42}$$

where $a\in V_\alpha$, $b\in V_\beta$, $b*a=b\circ a+(-1)^{\alpha\beta}a\circ b$ and $\alpha_\lambda(a,b)\in\mathbb{C}[\lambda]$. Assume that $\alpha_\lambda(a,b)=\sum_{i=0}^n\lambda^i\alpha_i(a,b)$ for any a, $b\in V$, where all $\alpha_i(\cdot,\cdot):V\times V\to\mathbb{C}$ are bilinear forms and there exist some a, $b\in V$ such that $\alpha_n(a,b)\neq 0$. Then, we obtain (for any $a\in V_\alpha$, $b\in V_\beta$, $c\in V_\gamma$):
(1) If $n>3$, $\alpha_n(a\circ b,c)=0$;
(2) If $n\leq 3$, $\alpha_i(x,y)=0$ if $x\in V_\alpha$, $y\in V_\beta$, $\alpha+\beta=\overline{1}$ for any $i\in\{0,1,2,3\}$, and

$$\alpha_3(a\circ b,c)=(-1)^{\alpha\beta}\alpha_3(b\circ a,c)=(-1)^{\alpha\beta}\alpha_3(a,c\circ b)=(-1)^{\beta(\alpha+\gamma)}\alpha_3(c\circ a,b),\tag{4.43}$$
$$\alpha_2(a,c\circ b)+\alpha_3(a,[c,b])=(-1)^{\alpha\beta}\alpha_2(a\circ b,c)+\alpha_3([b,a],c),\tag{4.44}$$
$$\alpha_2(a,c*b)=-\alpha_2(b\circ a,c)+(-1)^{\alpha\beta}2\alpha_2(a\circ b,c)+3\alpha_3([b,a],c),\tag{4.45}$$
$$-\alpha_2(b\circ a,c)+\alpha_3([b,a],c)-(-1)^{\beta\gamma}\alpha_2(c\circ a,b)+(-1)^{\beta\gamma}\alpha_3([c,a],b)=0,\tag{4.46}$$
$$\alpha_2(a,c*b)=\alpha_2(b*a,c)+(-1)^{\beta\gamma}\alpha_2(c*a,b),\tag{4.47}$$
$$\alpha_1(a,c\circ b)+\alpha_2(a,[c,b])=(-1)^{\alpha\beta}\alpha_1(a\circ b,c)+\alpha_2([b,a],c),\tag{4.48}$$
$$-\alpha_1(b\circ a,c)+\alpha_2([b,a],c)+(-1)^{\beta\gamma}\alpha_1(c\circ a,b)-(-1)^{\beta\gamma}\alpha_2([c,a],b)=0,\tag{4.49}$$
$$\alpha_1(a,c*b)=(-1)^{\alpha\beta}\alpha_1(a\circ b,c)-\alpha_1(b\circ a,c)+2\alpha_2([b,a],c)+(-1)^{\beta\gamma}\alpha_1(c*a,b),\tag{4.50}$$
$$\alpha_0(a,c\circ b)+\alpha_1(a,[c,b])=(-1)^{\alpha\beta}\alpha_0(a\circ b,c)+\alpha_1([b,a],c)-(-1)^{\beta\gamma}\alpha_0(c*a,b),\tag{4.51}$$
$$\alpha_0(a,c*b)=\alpha_1([b,a],c)-\alpha_0(b\circ a,c)-(-1)^{\beta\gamma}\alpha_0(c\circ a,b)+(-1)^{\beta\gamma}\alpha_1([c,a],b),\tag{4.52}$$
$$\alpha_0(a,[c,b])=\alpha_0([b,a],c)-(-1)^{\beta\gamma}\alpha_0([c,a],b).\tag{4.53}$$

Proof. The proof of (1) is similar to that in Theorem 4.1 and it is easy to see that (4.17)-(4.31) are equivalent to (4.43)-(4.53) when $a*b=a\circ b+(-1)^{\alpha\beta}b\circ a$ for $a\in V_\alpha$, $b\in V_\beta$. This concludes this theorem. ∎

Corollary 4.9. *Let V be a $\mathbb{Z}/2\mathbb{Z}$-graded vector space and $R = \mathbb{C}[\partial]V$ be a finite quadratic Lie conformal superalgebra corresponding to the super Gel'fand-Dorfman bialgebra $(V, \circ, [\cdot,\cdot])$ where for any $z \in V$, there exist some non-negative integer m and x_i, $y_i \in V$ such that $z = \sum_{i=0}^{m} x_i \circ y_i$. Set $\widehat{R} = R \oplus \mathbb{C}\mathfrak{c}$ be a Leibniz central extension of $(R, [\cdot_\lambda \cdot])$ with the λ-product given by (4.42). Then, for any a, $b \in V$, we obtain $\alpha_\lambda(a,b) = \sum_{i=0}^{3} \lambda^i \alpha_i(a,b)$ where $\alpha_i(\cdot,\cdot) : V \times V \to \mathbb{C}$ are bilinear forms with $\alpha_i(x,y) = 0$ if $x \in V_\alpha$, $y \in V_\beta$, $\alpha + \beta = \overline{1}$ for any $i \in \{0,1,2,3\}$ and satisfy (4.43)-(4.53).*

Proof. It can be obtained from Theorem 4.8 with a similar proof with that in Corollary 4.2. □

Corollary 4.10. *Let $R = \mathbb{C}[\partial]V$ be a quadratic Lie conformal superalgebra corresponding to the Novikov superalgebra $(V, \circ)$ where for any $z \in V$, there exist some non-negative integer m and x_i, $y_i \in V$ such that $z = \sum_{i=0}^{m} x_i \circ y_i$. Set $\widehat{R} = R \oplus \mathbb{C}\mathfrak{c}$ be a Leibniz central extension of $(R, [\cdot_\lambda \cdot])$ with the λ-product given as follows:*

$$\widetilde{[a_\lambda b]} = \partial(b \circ a) + \lambda(b * a) + \alpha_\lambda(a,b)\mathfrak{c}, \tag{4.54}$$

where $a * b = a \circ b + (-1)^{\alpha\beta} b \circ a$ for any $a \in V_\alpha$ and $b \in V_\beta$. Then, for any a, $b \in V$, we obtain $\alpha_\lambda(a,b) = \sum_{i=0}^{3} \lambda^i \alpha_i(a,b)$ and the bilinear forms $\alpha_i(\cdot,\cdot)$ satisfy $\alpha_i(x,y) = 0$ if $x \in V_\alpha$, $y \in V_\beta$, $\alpha + \beta = \overline{1}$ for any $i \in \{0,1,2,3\}$ and the following equalities for any $a \in V_\alpha$, $b \in V_\beta$, $c \in V_\gamma$:

$$\alpha_3(a \circ b, c) = (-1)^{\alpha\beta} \alpha_3(b \circ a, c) = (-1)^{\alpha\beta} \alpha_3(a, c \circ b) = (-1)^{\beta(\alpha+\gamma)} \alpha_3(c \circ a, b), \tag{4.55}$$

$$\alpha_2(a, c \circ b) = (-1)^{\alpha\beta} \alpha_2(a \circ b, c) = -(-1)^{\alpha(\beta+\gamma)} \alpha_2(c \circ b, a), \tag{4.56}$$

$$\alpha_2(a, b \circ c) = -(-1)^{\beta\gamma} \alpha_2(b \circ a, c) + (-1)^{\beta(\alpha+\gamma)} \alpha_2(a \circ b, c), \tag{4.57}$$

$$\alpha_1(a, c \circ b) = (-1)^{\alpha\beta} \alpha_1(a \circ b, c) = (-1)^{\alpha(\beta+\gamma)} \alpha_1(c \circ b, a), \tag{4.58}$$

$$\alpha_0(a, c \circ b) + (-1)^{\beta\gamma} \alpha_0(c \circ a, b) = -\alpha_0(b \circ a, c) - (-1)^{\beta\gamma} \alpha_0(a, b \circ c), \tag{4.59}$$

$$\alpha_0(a, c \circ b) + (-1)^{\beta\gamma} \alpha_0(c \circ a, b) = (-1)^{\alpha\beta} \alpha_0(b \circ a, c) - (-1)^{(\alpha+\beta)\gamma} \alpha_0(a \circ c, b). \tag{4.60}$$

Proof. It can be obtained from Theorem 4.8 by a similar proof with that in Corollary 4.4. □

Corollary 4.11. *Let $(V, \circ, [\cdot,\cdot])$ be a super Gel'fand-Dorfman bialgebra and $R = \mathbb{C}[\partial]V$ be the corresponding quadratic Lie conformal superalgebra. Set $\alpha_i(\cdot,\cdot)$ $(i = 0,1,2,3)$ be bilinear forms on V with $\alpha_i(x,y) = 0$ if $x \in V_\alpha$, $y \in V_\beta$, $\alpha + \beta = \overline{1}$ and satisfy (4.43)-(4.53). Define bilinear forms $\varphi_i : Coeff(R) \times Coeff(R) \to \mathbb{C}$ as follows:*

$$\varphi_0(a \otimes t^m, b \otimes t^n) = \alpha_0(a,b)\delta_{m+n+1,0}, \tag{4.61}$$

$$\varphi_1(a \otimes t^m, b \otimes t^n) = m\alpha_1(a,b)\delta_{m+n,0}, \tag{4.62}$$

$$\varphi_2(a \otimes t^m, b \otimes t^n) = m(m-1)\alpha_2(a,b)\delta_{m+n-1,0}, \tag{4.63}$$

$$\varphi_3(a \otimes t^m, b \otimes t^n) = m(m-1)(m-2)\alpha_3(a,b)\delta_{m+n-2,0}, \tag{4.64}$$

for a, $b \in V$, m, $n \in \mathbb{Z}$. Then, φ_0, φ_1, φ_2, φ_3 are 2-cocycles of Leibniz superalgebra $Coeff(R)$.

Proof. The proof is similar to that in Corollary 4.6. □

Example 4.12. *Let $R = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]W$ be a Lie conformal algebra with the λ-bracket as follows:*

$$[L_\lambda L] = (\partial + 2\lambda)L, \qquad [L_\lambda W] = (\partial + a\lambda)W, \qquad [W_\lambda W] = 0, \tag{4.65}$$

where $a \in \mathbb{C}$.

Obviously, R is a quadratic Lie conformal algebra corresponding to the Novikov algebra $(V = \mathbb{C}L \oplus \mathbb{C}W, \circ)$ given as follows:

$$L \circ L = L, \quad W \circ L = W, \quad L \circ W = (a-1)W, \quad W \circ W = 0. \tag{4.66}$$

Then, by a simple computation, we can get that any $\alpha_i(\cdot,\cdot)$ for $i \in \{0,1,2,3\}$ satisfying (4.55)-(4.60) are those given in Example 3.10 in [21] with $\beta = 0$. Therefore, any Leibniz central extension of R by a one-dimensional center is a Lie conformal algebraic central extension of R studied in Example 3.10 in [21].

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