SECOND-MAXIMAL SUBALGEBRAS OF LEIBNIZ ALGEBRAS

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Abstract. In this work we study Leibniz algebras whose second-maximal subalgebras are ideals. We provide a classification based on solvability, nilpotency, and the size of the derived algebra. We give specific descriptions of those Leibniz algebras whose derived algebra has codimension zero or one. This includes several algebras which are only possible over certain fields.

1. Introduction

Leibniz algebras were defined by Loday in 1993 [10, 11]. Leibniz algebras can be considered to be a generalization of Lie algebras, removing the restriction that the product must be anti-commutative (or equivalently that the square of an element must be zero). This means that some algebraic structures can appear in Leibniz algebras that are not not possible in Lie algebras. For instance, the subalgebra generated by a single element in a Lie algebra is necessarily one-dimensional, whereas in a Leibniz algebra it generates a so-called cyclic subalgebra whose dimension is the highest nonzero power of the generator.

Much of the recent work in Leibniz algebras has focused on classifying Leibniz algebras based on either their dimension [1, 6, 13], their nilradical [5, 7, 8], or their maximal and Frattini subalgebras [3, 12]. The present body of work falls into this last category. We study Leibniz algebras based on their second-maximal subalgebras, that is the maximal subalgebras of the maximal subalgebras of the algebra, by generalizing the work of Stitzinger on Lie algebras [14]. We consider several cases of Leibniz algebras all of whose second-maximal subalgebras are ideals based on solvablility and nilpotency. In many cases our results mirror the results for Lie algebras but with a single cyclic subalgebra structure that interacts with the rest of the algebra in interesting ways.

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Several authors have classified low dimensional Leibniz algebras over the field of complex numbers [1, 6, 13]. Our results agree with theirs for the few low dimensional cases where every second-maximal subalgebra is an ideal, but we find additional algebras over other fields.

2. PRELIMINARIES

A Leibniz algebra L is a vector space over a field F which satisfies the Jacobi identity

$$a(bc) = (ab)c + b(ac).$$

Equivalently we could say that left-multiplication by a, or ℓ_a, is a derivation. We define $Leib(L)$ to be the subalgebra of L generated by the squares of elements in L. Note that $Leib(L)$ is an ideal of L and $L/Leib(L)$ is a Lie algebra. The Frattini subalgebra of L is the intersection of all maximal subalgebras of L, denoted $Frat(L)$.

Of particular importance to the study of Leibniz algebras whose maximal or second-maximal subalgebras are ideals is the following proposition, proven in [9].

Proposition 2.1. *A Leibniz algebra L is nilpotent if and only if every maximal subalgebra of L is an ideal of L.*

The next two results follow similar work for Lie algebras in [14].

Lemma 2.2. *Let L be a Leibniz algebra. Then L has only one maximal subalgebra if and only if L is cyclic and nilpotent.*

Proof. Let M be the unique maximal subalgebra of L. Then let $x \in L\backslash M$. Then the subalgebra generated by x, $\langle x \rangle$, is either contained in a maximal subalgebra (hence $x \in M$, a contradiction) or it is all of L. Thus $L = \langle x \rangle$ is cyclic. In this case the only maximal subalgebra is $M = L^2$, which is an ideal of L, so L is nilpotent by Proposition 2.1. For the converse, if L is cyclic, then the only maximal subalgebra of L is L^2. □

If L is a Lie algebra, then L cyclic implies that $\dim(L) = 1$. So this result generalizes the result in [14]: for a Lie algebra L, L has only one maximal subalgebra if and only if $\dim(L) = 1$.

Lemma 2.3. *Suppose every second-maximal subalgebra of the Leibniz algebra L is an ideal of L. Then every maximal subalgebra is nilpotent, and is either an ideal or cyclic.*

Proof. Let M be a maximal subalgebra. Since every maximal subalgebra of M is an ideal (of L), M is nilpotent by Proposition 2.1. Suppose M is not cyclic. Then by Lemma 2.2, M has two distinct maximal subalgebras, N_1 and N_2. Since N_1 and N_2 are second-maximal subalgebras of L, each is an ideal of L. Therefore, $M = N_1 + N_2$ is an ideal of L. ☐

The converse of this statement is not true. Consider the following example.

Example 2.4. Let L be the nilpotent cyclic Leibniz algebra $L = \langle x \rangle = \text{span}\{x, x^2, \dots, x^n\}$. The only maximal subalgebra of L is $M = L^2 = \text{span}\{x^2, \dots, x^n\}$, which is nilpotent and an ideal of L. Since M is abelian, $N = \text{span}\{x^2, \dots, x^{n-1}\}$ is a second-maximal subalgebra of L, but N is not an ideal of L since $[x, x^{n-1}] = x^n \notin N$.

Since the only cyclic subalgebras of a Lie algebra is one-dimensional, if we restrict ourselves to Lie algebras, Lemma 2.3 recovers the result in [14]. Specifically, if every second-maximal subalgebra of a Lie algebra L is an ideal of L, then every maximal subalgebra of L is nilpotent, and is either and ideal or one-dimensional.

3. CLASSIFICATION

In this sections we will classify Leibniz algebras all of whose sexond-maximal subalgebras are ideals. We begin by considering such algebras which are non-solvable.

Theorem 3.1. *Let L be a non-solvable Leibniz algebra. Then every second-maximal subalgebra of L is an ideal of L if and only if $L/Leib(L)$ is simple and every maximal subalgebra M of L is cyclic with $M^2 = Leib(L) = Frat(L)$.*

Proof. Let L be a non-solvable Leibniz algebra, and suppose that every second-maximal subalgebra of L is an ideal of L. Then $\bar{L} = L/Leib(L)$ is a Lie algebra. We know that $Leib(L)$ is abelian, hence sovable. If $\bar{L}$ is solvable then L would be solvable, a contradiction. By [14], since $\bar{L}$ is not solvable, it must be simple with every proper subalgebra one-dimensional.

Let $\bar{A}$ be a proper (hence maximal) subalgebra of $\bar{L}$, then $\bar{A} = \text{span}\{\bar{a}\}$ for some $\bar{a} \neq \bar{0} \in \bar{L}$. Then $A = \bar{A} + Leib(L)$ is a maximal subalgebra of L which properly contains $Leib(L)$ so $Leib(L)$ is not maximal. Now let M be any maximal subalgebra of L. If $M \leq Leib(L)$ then M is properly contained in A, a contradiction. Thus $M \not\leq Leib(L)$ and there exists an $m \in M \backslash Leib(L)$. Then $\langle m \rangle + Leib(L) = \text{span}\{m\} \oplus Leib(L)$, so $\langle m \rangle + Leib(L)/Leib(L)$ is a one-dimensional subalgebra of $\bar{L}$. Hence it is maximal in $\bar{L}$ and its preimage $M_2 = \langle m \rangle + Leib(L)$ is maximal in L.

We claim that $M_2 = \langle m \rangle$. Suppose that $M_2 \neq \langle m \rangle$, so $\langle m \rangle$ is contained in a maximal subalgebra N of M_2. Since N is second-maximal in L, N is an ideal of L, so $\bar{N} = N + Leib(L)/Leib(L)$ is an ideal of $\bar{L}$. But since $\bar{L}$ is simple and $\bar{m} \neq \bar{0} \in \bar{N}$, $\bar{N} = \bar{L}$. This is a contradiction since $\bar{N}$ is contained in $\bar{M}_2$, which is one-dimensional, and $\dim(\bar{L}) = \dim(L/Leib(L)) > 1$. Therefore it must be that $M_2 = \langle m \rangle + Leib(L) = \langle m \rangle$, so $Leib(L) \subsetneq M_2$. But $M_2 = \langle m \rangle \leq M$, so by the maximality of M_2, we have $M = M_2$. Thus every maximal subalgebra is cyclic and contains $Leib(L)$, so $Leib(L) = Frat(L) = M^2$ for each maximal subalgebra M. □

It should be noted that since each maximal subalgebra is nilpotent and cyclic with $M^2 = Leib(L)$, all maximal subalgebras are isomorphic. Since every proper subalgebra of a cyclic algebra is abelian, we also have the following corollary.

Corollary 3.2. *Let L be a non-solvable Leibniz algebra with every second-maximal subalgebra of L is an ideal of L. Then every subalgebra of L is an ideal and is either cyclic or abelian.*

Note that in Theorem 3.1 the algebra L is an interesting cyclic extension of a simple Lie algebra where every proper subalgebra is one-dimensional. Specifically, if $x \notin Leib(L)$ then the subalgebra generated by x must be contained in some maximal subalgebra M with

$M^2 = Leib(L)$. In characteristic zero, we can also say that x generates M. Thus every maximal subalgebra is a nilpotent, cyclic subalgebra generated by some $x \notin Leib(L)$ and with $M^2 = Leib(L)$.

Example 3.3. Let $V = \operatorname{span}\{\vec{v}_1, \dots \vec{v}_n\}$. Consider the Leibniz algebra $L = \mathbb{R}^3 + V$ with bracket structure given by $\left[\vec{a}, \vec{b}\right] = \vec{a} \times \vec{b}$ for $\vec{a} \neq \vec{b} \in \mathbb{R}^3$; $[\vec{a}, \vec{a}] = \vec{v}_1$ and $[\vec{a}, \vec{v}_i] = \vec{v}_{i+1}$ for each $\vec{a} \neq \vec{0} \in \mathbb{R}^3$ and $i = 1, 2, \dots, n-1$. Then $Leib(L) = V$, and $L/Leib(L) \cong \mathbb{R}^3$ is the simple Lie algebra whose bracket is the cross product. Since $\mathbb{R}^3$ is simple, every maximal subalgebra of L is of the form $\langle \vec{a} \rangle = \operatorname{span}\{\vec{a}, \vec{v}_1, \dots, \vec{v}_n\}$ for some $\vec{a} \in \mathbb{R}$. The only second-maximal subalgebra of L is V itself, which is an abelian ideal of L. The only other proper subalgebras of L are of the form $\operatorname{span}\{\vec{v}_i, \vec{v}_{i+1}, \dots, \vec{v}_n\}$, which are also abelian ideals.

Now that we have investigated non-solvable Leibniz algebras, we turn our attention to algebras which are solvable, but not nilpotent, that have the property that every second-maximal subalgebra is an ideal.

Theorem 3.4. *Let L be a finite-dimensional solvable non-nilpotent Leibniz algebra. Then every second-maximal subalgebra of L is an ideal of L if and only if L is one of the following:*

1. $L = \operatorname{span}\{x, a\}$, $ax = -xa = x$.
2. $L = \operatorname{span}\{x, a, a^2\}$, $ax = x + ca^2$, $xa = -x + da^2$, $x^2 = ea^2$, *where* $c, d, e \in F$ *such that* $\alpha^2 + (c+d)\alpha + e = 0$ *has* <u>*no*</u> *solutions for* α.
3. $L = \operatorname{span}\{x, a, a^2, a^3\}$, $ax = x + ca^3$, $xa = x + da^3$, $x^2 = a^3$, *where* $c, d \in F$ *and* F *is a field of characteristic 2.*

Proof. Let L be a solvable non-nilpotent Leibniz algebra with every second-maximal subalgebra an ideal. Since L is solvable, $\dim(L/L^2) \neq 0$. If $\dim(L/L^2) > 1$, then L is nilpotent (since this would imply that L is the sum of two nilpotent ideals), a contradiction. Thus $\dim(L/L^2) = 1$ and L^2 is a maximal subalgebra of L. If L^2 is the only maximal subalgebra, then by Lemma 2.2 L is nilpotent, which is a contradiction. Hence we can assume that there is another maximal subalgebra M of L. If M is an ideal, then again L is nilpotent, so we can assume that M is not an ideal. By Lemma 2.3, this means that M is cyclic.

So, there exists $a \in M$ with $M = \langle a \rangle$, and $L = L^2 + \langle a \rangle$. Notice $\langle a \rangle^2$ is maximal in $\langle a \rangle$ and second maximal in L, hence $\langle a \rangle^2$ is an ideal of L. Since L is finite-dimensional and solvable, by [9] L has a chain of ideals $0 = I_0 \leq I_1 \leq \cdots \leq I_n = L$ such that the dimension of $I_j = j$. There exists a smallest $1 \leq j \leq n-1$ such that $I_{j-1} \subseteq \langle a \rangle^2$, but $I_j \nsubseteq \langle a \rangle^2$. Thus, there exists a $b \in I_j$, such that $b \notin \langle a \rangle^2$. Since I_j/I_{j-1} is one-dimensional and $I_{j-1} \subseteq \langle a \rangle^2$, then I_j has a basis consisting of b and elements of $\langle a \rangle^2$.

Suppose $b \in \langle a \rangle$, then $\langle a \rangle^2 \subsetneq \langle a \rangle^2 + I_j \subseteq \langle a \rangle$. However, $\langle a \rangle^2$ is maximal in $\langle a \rangle$, and so $\langle a \rangle^2 + I_j = \langle a \rangle$. But now we have $\langle a \rangle^2$ and I_j both ideals of L, but $\langle a \rangle = \langle a \rangle^2 + I_j$ is not an ideal of L, so $b \notin \langle a \rangle$. Thus, $L = \langle a \rangle + I_j$, since $\langle a \rangle$ is maximal in L. Therefore, $L = \text{span}\{b, a, a^2, \ldots, a^n\}$ has a codimension one cyclic subalgebra.

By Lemma 2.3, $M = \langle a \rangle$ is nilpotent, so $a^{n+1} = 0$. Since $\langle a \rangle^2$ is a codimension 2 subspace of L contained in $Leib(L)$, $\dim(L/Leib(L)) \leq 2$. If $\dim(L/Leib(L)) = 0$ then L is abelian, hence Lie, a contradiction. If $\dim(L/Leib(L)) = 1$ then L is cyclic, which is a contradiction by Lemma 2.2. Thus $Leib(L) = \langle a \rangle^2 = \text{span}\{a^2, \ldots, a^n\}$. Since $\langle a \rangle^2$ is contained in L^2, we can expand $\{a^2, \ldots, a^n\}$ to a basis $\{x, a^2, \ldots, a^n\}$ of L^2 and assume $L = \text{span}\{x, a, a^2, \ldots, a^n\}$ with $x \in L^2$. Hence $L/Leib(L)$ is the Lie algebra $\text{span}\{\bar{a}, \bar{x}\}$ with $\bar{a}\bar{a} = \bar{x}\bar{x} = \bar{0}$, and $\bar{a}\bar{x} = -\bar{x}\bar{a} = c\bar{a} + d\bar{x}$. But $\text{span}\{\bar{x}\} = L^2/Leib(L)$ is an ideal of $L/Leib(L)$, hence $c = 0$. Performing a change of basis from a to $a' = a/d$ we can assume $d = 1$ and $\bar{a}\bar{x} = -\bar{x}\bar{a} = \bar{x}$.

Since M and L^2 are maximal in L, $Frat(L)$ is contained in $M \cap L^2 = \text{span}\{a^2, \ldots, a^n\} = Leib(L)$. We claim that $Frat(L) = Leib(L)$. If $Leib(L)$ is not contained in M_0 for some maximal subalgebra M_0 of L, then $M_0 + Leib(L) = L$. Hence $a \in M_0 + Leib(L)$ and $a = m + n$ where $m = c_0 x + c_1 a + \cdots + c_n a^n \in M_0$ and $n = d_2 a^2 + \cdots + d_n a^n \in Leib(L)$. Then the linear independence of the basis implies that $c_0 = 0$ and $c_1 = 1$. But $m = a + \cdots + c_n a^n \in M_0$ implies that $m^n = a^n \in M_0$, so $a^{n-1} = (a^{n-1} + c_2 a^n) - c_2 a^n = m^{n-1} - c_2 m^n \in M_0$. Continuing, we find $a \in M_0$ which contradicts our assumption $Leib(L) \nsubseteq M_0$. Thus $Leib(L)$ is contained in every maximal subalgebra, so $Leib(L) = Frat(L)$. Since $Leib(L)$ is codimension 2 in L this implies that all maximal subalgebras of L are given by L^2, M, or $M_\alpha = \text{span}\{\alpha a + x\} \oplus Leib(L)$.

Since every maximal subalgebra of $M = \langle a\rangle$ is an ideal, M is nilpotent, so $a^{n+1} = 0$. We will now use induction on the power of a to show that $xa^i = 0$ for all $i \geq 2$. Suppose that $xa^{i+1} = 0$. Recall the bracket structure of $L/Leib(L)$ is $\bar{a}\bar{x} = -\bar{x}\bar{a} = \bar{x}$, so $xa = -x + \sum_{j\geq 2} c_j a^j$ and $xa^i = \sum_{k\geq 2} d_k a^k$ for $i \geq 2$. Then the Jacobi identity on $0 = x(aa^i) = (xa)a^i + a(xa^i)$ gives that $0 = -d_2a^2 + (d_2 - d_3)a^3 + \cdots + (d_{n-1} - d_n)a^n$. But the powers of a are linearly independent, so $0 = d_2 = d_3 = \cdots = d_n$. Therefore $xa^i = 0$, so by induction $xa^i = 0$ for all $i = 2, \ldots, n$.

We now consider several cases for the size of n. If $n = 1$, then $L = \text{span}\{x, a\}$ with $ax = -xa = x$. Note that the only second-maximal subalgebra of L is $N = 0$ which is an ideal, so the converse holds for this case.

Now suppose that $n = 2$, so $L = \text{span}\{x, a, a^2\}$ with nontrivial products $ax = x + ca^2$, $xa = -x + da^2$, and $x^2 = ea^2$. Suppose that $e = 0$. Then $N_1 = \text{span}\{x\}$ is second-maximal in L, hence it is an ideal, which implies that $c = d = 0$. This is a contradiction since this implies that $N_2 = \text{span}\{x + a^2\}$ is second-maximal in L, but N_2 is not an ideal: $a(x + a^2) = x \notin N_2$. Thus we have that $e \neq 0$. Then $L^2 = \langle x\rangle$ and $M = \langle a\rangle$ each have only a single maximal subalgebra $N_1 = \text{span}\{a^2\} = Leib(L)$ which is an ideal of L. Consider $M_\alpha = \text{span}\{\alpha a + x, a^2\}$. Then $(\alpha a + x)^2 = p(\alpha)a^2$, where $p(\alpha) = \alpha^2 + (c + d)\alpha + e$ is a quadratic equation in terms of α. If $p(\alpha)$ has a root $\alpha \in F$, then L has a maximal subalgebra M_α and a second-maximal subalgebra $N_2 = \text{span}\{\alpha a + x\}$ which is not an ideal. If $p(\alpha)$ does not have a root in F, then every maximal subalgebra of L has $N_1 = \text{span}\{a^2\}$ as its only maximal subalgebra. Therefore in the $n = 2$ case, every second-maximal subalgebra of L is an ideal of L if and only if $p(\alpha)$ is irreducible over F.

Now suppose that $n = 3$, so $L = \text{span}\{x, a, a^2, a^3\}$ with nontrivial products $ax = x + c_2a^2 + c_3a^3$, $xa = -x + d_2a^2 + d_3a^3$, and $x^2 = e_2a^2 + e_3a^3$. If $e_3 = 0$, then $N = \text{span}\{x, a^2\} \lneq L^2 \lneq L$ is a second-maximal subalgebra, but not an ideal. If $e_2 \neq 0$, then $N = \text{span}\{x, x^2\} \lneq L^2 \lneq L$ is a second-maximal subalgebra, but not an ideal. Therefore we must have $e_2 = 0$ and $e_3 \neq 0$.

Then $N = \text{span}\{x, x^2\} = \text{span}\{x, a^3\} \lneq L^2 \lneq L$ is second-maximal, hence it must be an ideal. Thus $ax, xa \in N$ which implies that $c_2 = d_2 = 0$. The Jacobi identity on $a(xx)$ gives that $0 = -2e_3a^3$. If the characteristic of F is not 2, this implies $e_3 = 0$, a contradiction. If the characteristic of F is 2, then we obtain the family of algebras in (3). For the converse, the only second-maximal subalgebras of such an algebra are of the form $N_1 = \text{span}\{a^2, a^3\}$ and $N_2 = \text{span}\{x, x^2\} = \text{span}\{x, a^3\}$, both ideals of L.

Finally, consider the case $n \geq 4$, so $L = \text{span}\{x, a, a^2, \ldots, a^n\}$ with $ax, xa \neq 0$ and $x^2 = e_2a^2 + \cdots + e_na^n$. If $e_n = 0$, let $N = \text{span}\{x, a^2, \ldots, a^{n-2}, a^{n-1}\}$. If $e_{n-1} = 0$, let $N = \text{span}\{x, a^2, \ldots, a^{n-2}, a^n\}$. If $e_n \neq 0$ and $e_{n-1} \neq 0$, let $N = \text{span}\{x, a^2, \ldots, a^{n-1}, x^2\}$. In each case N is a subalgebra of L^2 since $(L^2)^2 = \text{span}\{x^2\} \subseteq N$ (hence it is a second-maximal subalgebra of L), but none is an ideal of L since $[a, N] \nsubseteq N$. Therefore there are no Leibniz algebras with $n \geq 4$ having every second-maximal subalgebra an ideal of L. $\square$

Since two of the three cases in the theorem are impossible over an algebraically closed field of characteristic zero, we immediately obtain the following result. We have an identical result (which agrees with [14]) if we restrict our attention to solvable non-nilpotent Lie algebra, since the same two cases are not Lie.

Corollary 3.5. *Let L be a finite-dimensional solvable non-nilpotent Leibniz algebra over an algebraically closed field of characteristic zero. Then every second-maximal subalgebra of L is an ideal of L if and only if $L = \text{span}\{x, a\}$, $xa = ax = -x$.*

Example 3.6. Consider the Leibniz algebra $L = \text{span}\{x, a, a^2\}$ over $\mathbb{R}$ with $ax = -xa = x$ and $x^2 = a^2$. Then $p(\alpha) = \alpha^2 + 1$, which is irreducible over $\mathbb{R}$. The maximal subalgebras of L are $M = \langle a \rangle$ and $M_\alpha = \text{span}\{\alpha a + x, a^2\}$ where $\alpha \in \mathbb{R}$. Notice $(\alpha a + x)^2 = (\alpha^2 + 1)a^2 = p(\alpha)a^2$ is nonzero for each α since $p(\alpha)$ is irreducible. Thus each M_α (including $M_0 = L^2$) is cyclic, and the only second-maximal subalgebra of L is $N = \text{span}\{a^2\} = Leib(L)$ which is contained in every maximal subalgebra and is an ideal of L.

Example 3.7. Consider the Leibniz algebra $L = \text{span}\{x, a, a^2, a^3\}$ over $\mathbb{Z}_2$ with $ax = xa = x$ and $x^2 = a^3$. Then L has three maximal subalgebras: $M = \langle a \rangle$, $L^2 = \text{span}\{x, a^2, a^3\}$, and $M_1 = \text{span}\{x + a, a^2, a^3\} = \langle x + a \rangle$. Since M and M_1 are cyclic, each has only one maximal subalgebra: $N_1 = Leib(L) = \text{span}\{a^2, a^3\}$. L^2 has two maximal subalgebras, N_1 and $N_2 = \text{span}\{x, x^2\} = \text{span}\{x, a^3\}$. Notice that both N_1 and N_2 are ideals of L.

We are now ready to combine our results and discuss arbitrary Leibniz algebras whose second-maximal subalgebras are ideals.

Theorem 3.8. *Let L be a finite-dimensional Leibniz algebra with $\dim(L/L^2) \leq 1$. Then every second-maximal subalgebra of L an ideal of L if and only if L is one of the following:*

1. *$L/Leib(L)$ is simple and every maximal subalgebra M of L is cyclic with $M^2 = Leib(L) = Frat(L)$.*
2. *$L = \text{span}\{x, a\}$, $ax = -xa = x$.*
3. *$L = \text{span}\{x, a, a^2\}$, $ax = x + ca^2$, $xa = -x + da^2$, $x^2 = ea^2$, where $c, d, e \in F$ such that $\alpha^2 + (c + d)\alpha + e = 0$ has no solutions for α.*
4. *$L = \text{span}\{x, a, a^2, a^3\}$, $ax = x + ca^3$, $xa = x + da^3$, $x^2 = a^3$, where $c, d \in F$ and F is a field of characteristic 2.*
5. *$L = \text{span}\{a\}$, $a^2 = 0$.*
6. *$L = \text{span}\{a, a^2\}$, $a^3 = 0$.*

Proof. Suppose that every second-maximal subalgebra of L is an ideal of L. If L is non-nilpotent, then by Theorems 3.4 and 3.1 we obtain the algebras described in (1)-(4), and the converse holds.

If L is nilpotent with $\dim(L/L^2) = 1$, then L is cyclic. In this case, we claim that $\dim(L)$ is at most 2. Suppose that $\dim(L) = n \geq 3$. Then, there exists a generator $a \in L$ such that $L = \text{span}\{a, a^2, \ldots, a^n\}$ with $a_n \neq 0$. Then, $N = \text{span}\{a^2, \ldots, a^{n-1}\} \neq 0$ is maximal in L^2, and so second maximal in L. Hence, by assumption, N is an ideal. But, N is not an ideal since $aa^{n-1} = a^n \notin N$. So, we must have $\dim(L) \leq 2$. The algebra $L = \text{span}\{a\}$

with $a^2 = 0$ has no second-maximal subalgebras, and the only second-maximal subalgebra of $L = \text{span}\{a, a^2\}$ with $a^3 = 0$ is $N = 0$ which is an ideal of L. Hence the converse holds.

□

Notice that in each of the algebras L described in Theorem 3.8, $L/Leib(L)$ is a Lie algebra. In case (1), $L/Leib(L)$ is simple with every maximal subalgebra (of $\bar{L}$) one-dimensional. In case (2), (3) and (4), $L/Leib(L)$ is the two-dimensional nonabelian Lie algebra. In case (5) and (6), $L/Leib(L)$ is the one-dimensional Lie algebra. Thus this theorem generalizes the main result of [14] for $\dim(L/L^2) \leq 1$.

If L is itself a Lie algebra, then L must be one of (1), (2), or (5). Additionally, in (1) since every cyclic subalgebra of a Lie algebra is one-dimensional, we have that every maximal subalgebra of L in that case is one-dimensional. This recovers the result of [14] for $\dim(L/L^2) \leq 1$.

Note that the algebras described in (1) can not have $\dim(L/L^2) = 1$, since that would imply that either $L/Leib(L)$ is abelian or its derived algebra is a proper ideal. Thus the Leibniz algebras with $\dim(L/L^2) = 1$ are exactly those algebras described in cases (2)-(6) of Theorem 3.8.

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