H-STANDARD COHOMOLOGY FOR COURANT-DORFMAN ALGEBRAS AND LEIBNIZ ALGEBRAS

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Abstract. We introduce the notion of H-standard cohomology for Courant-Dorfman algebras and Leibniz algebras, by generalizing Roytenberg's construction. Then we generalize a theorem of Ginot-Grutzmann on transitive Courant algebroids, which was conjectured by Stienon-Xu. The relation between H-standard complexes of a Leibniz algebra and the associated crossed product is also discussed.

1. Introduction

The notion of Leibniz algebras, objects that date back to the work of "D-algebras" by Bloh [2], is due to Loday [7]. In literature, Leibniz algebras are sometimes also called Loday algebras.

A (left) Leibniz algebra L is a vector space over a field $\mathfrak{k}$ ($\mathfrak{k} = \mathbb{R}$ or $\mathbb{C}$) equipped with a bracket $\circ : L \otimes L \to L$, called the Leibniz bracket, satisfying the (left) Leibniz identity:

$$x \circ (y \circ z) = (x \circ y) \circ z + y \circ (x \circ z), \qquad \forall x, y, z \in L.$$

A concrete example is the omni Lie algebra $ol(V) \triangleq gl(V) \oplus V$, where V is a vector space. It is first introduced by Weinstein [16] as the linearization of the standard Courant algebroid $TV^* \oplus T^*V^*$. The Leibniz bracket of $ol(V)$ is given by:

$$(A + v) \circ (B + w) = [A, B] + Aw, \quad \forall A, B \in gl(V),\ v, w \in V.$$

In [8], Loday and Pirashvili introduced the notions of representations (corepresentations) and Leibniz homology (cohomology) for Leibniz algebras. They also studied universal enveloping algebras and PBW theorem for Leibniz algebras.

Leibniz algebras can be viewed as a non-commutative analogue of Lie algebras. Some theorems and properties of Lie algebras are still valid for Leibniz algebras, while some are not. The properties of Leibniz algebras are under continuous investigation by many authors, we can only mention a few works here [1, 4, 9, 13, 14].

Leibniz algebras have attracted more interest since the discovery of Courant algebroids, which can be viewed as the geometric realization of Leibniz algebras in certain sense. Courant algebroids are important objects in recent studies of Poisson geometry, symplectic geometry and generalized complex geometry.

The notion of Courant algebroids was first introduced by Liu, Weistein and Xu in [6], as an answer to an earlier question "what kind of object is the double of a Lie bialgebroid". In their original definition, a Courant algebroid is defined in terms of a skew-symmetric bracket, now known as "Courant bracket". In [10], Roytenberg proved that a Courant algebroid can be equivalently defined in terms of a Leibniz bracket, now known as "Dorfman bracket". And he defined standard complex and standard cohomology of Courant algebroids in the language of supermanifolds in [12]. In [15], Stienon and Xu defined naive cohomology of Courant algebroids , and conjectured that

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there is an isomorphism between standard and naive cohomology for a transitive Courant algebroid. Later this conjecture was proved by Ginot and Grutzmann in [5].

In [11], Roytenberg introduced the notion of Courant-Dorfman algebras, as an algebraic analogue of Courant algebroids. And he defined standard complex and standard cohomology for Courant-Dorfman algebras. Furthermore, he proved that there is an isomorphism of graded Poisson algebras between the standard complex of a Courant algebroid E and the associated Courant-Dorfman algebra $\mathcal{E} = \Gamma(E)$.

The main objective of this article is to develop a similar cohomology theory, the so-called H-standard cohomology, for Courant-Dorfman algebras as well as Leibniz algebras.

Given a Courant-Dorfman algebra $(\mathcal{E}, R, \langle\cdot,\cdot\rangle, \partial, \circ)$, and an R-submodule $\mathcal{H} \supseteq \partial R$ which is an isotropic ideal of $\mathcal{E}$, let $(\mathcal{V}, \nabla)$ be an $\mathcal{H}$-representation of $\mathcal{E}$ (a left representation of Leibniz algebra $\mathcal{E}$ such that ∇ is a covariant differential and $\mathcal{H}$ acts trivially on $\mathcal{V}$). By generalizing Roytenberg's construction, we shall define the $\mathcal{H}$-standard complex $(C^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V}), d)$ and $\mathcal{H}$-standard cohomology $H^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V})$ (see Theorem 3.3 and Definition 3.5). And when $\mathcal{E}/\mathcal{H}$ is projective, we shall prove the following result:

$$H^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V}) \cong H^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}).$$

Note that we don't require the symmetric bilinear form $\langle\cdot,\cdot\rangle$ to be non-degenerate here. In particular when $\mathcal{E}$ is the space of sections of a transitive Courant algebroid E (over M), and $\mathcal{H} = \rho^*(\Omega^1(M))$, $\mathcal{V} = C^\infty(M)$, the result above recovers Stienon and Xu's conjecture.

Given a Leibniz algebra L with left center Z, suppose $H \supseteq Z$ is an isotropic ideal of L, and (V, τ) is an H-representation of L (a left representation of L such that H acts trivially on V), similarly we can define the H-standard complex $(C^\bullet(L, H, V), d)$ and H-standard cohomology $H^\bullet(L, H, V)$. And we have the following result:

$$H^\bullet(L, H, V) \cong H^\bullet_{CE}(L/H, V).$$

This result can be proved directly, but in this paper we choose a roundabout way. We construct a Courant-Dorfman algebra structure on $\mathcal{L} = S^\bullet(Z) \otimes L$, and then prove there is an isomorphism between the H-standard complex of L and the $\mathcal{H} = S^\bullet(Z) \otimes H$-standard complex of $\mathcal{L}$. Finally based on the result for Courant-Dorfman algebras, we may obtain the result above by inference.

The structure of this paper is organized as follows:

In Section 2, we provide some basic knowledge about Leibniz algebras and Courant-Dorfman algebras. In Section 3, we give the definition of H-standard complex and cohomology for Courant-Dorfman algebras and Leibniz algebras. In Section 4, we prove the isomorphism theorem for Courant-Dorfman algebras, as a generalization of Stienon and Xu's conjecture. In Section 5, we associate a Courant-Dorfman algebra structure on $\mathcal{L}$ to any Leibniz algebra L, and discuss the relation between H-standard complexes of them, finally we prove an isomorphism theorem for Leibniz algebras.

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2. Preliminaries

In this section we list some basic notions and properties about Leibniz algebras and Courant-Dorfman algebras. For more details we refer to [8, 11].

Definition 2.1. A (left) Leibniz algebra is a vector space L over a field $\mathfrak{k}$ ($\mathfrak{k}=\mathbb{R}$ for our main interest), endowed with a bilinear map (called Leibniz bracket) $\circ: \ L\otimes L\to L$, which satisfies (left) Leibniz rule:

$$e_1\circ(e_2\circ e_3)=(e_1\circ e_2)\circ e_3+e_2\circ(e_1\circ e_3)\quad \forall e_1,e_2,e_3\in L$$

Example 2.2. Given any Lie algebra g and its representation (V,ρ), the semi-direct product $g\ltimes V$ with a bilinear operation $\circ$ defined by

$$(A+v)\circ(B+w)\triangleq[A,B]+\rho(A)w,\quad \forall A,B\in g,\ v,w\in V$$

forms a Leibniz algebra.

In particular, for any vector space V, $gl(V)\oplus V$ is a Leibniz algebra with Leibniz bracket

$$(A+v)\circ(B+w)=[A,B]+Aw,\quad \forall A,B\in gl(V),\ v,w\in V.$$

It is called an omni Lie algebra, and denoted by $ol(V)$ (see Weinstein [16]).

Definition 2.3. A representation of a Leibniz algebra L is a triple (V,l,r), where V is a vector space equipped with two linear maps: left action $l: \ L\to gl(V)$ and right action $r: \ L\to gl(V)$ satisfying the following equations:

$$l_{e_1\circ e_2}=[l_{e_1},l_{e_2}],\ r_{e_1\circ e_2}=[l_{e_1},r_{e_2}],\ r_{e_1}\circ l_{e_2}=-r_{e_1}\circ r_{e_2},\quad \forall e_1,e_2\in L, \tag{2.1}$$

where the brackets on the right hand side are the commutators in $gl(V)$.

If V is only equipped with left action $l: \ L\to gl(V)$ which satisfies $l_{e_1\circ e_2}=[l_{e_1},l_{e_2}]$, we call (V,l) a left representation of L.

For (V,l,r)(or (V,l)) a representation (or left representation) of L, we call V an L-module (or left L-module).

Given a left representation (V,l), there are two standard ways to extend V to an L-module. One is called symmetric extension, with the right action defined as $r_e=-l_e$; the other is called antisymmetric extension, with the right action defined as $r_e=0$. In this paper, we always take the symmetric extension $(V,l,-l)$.

Example 2.4. Denote by Z the left center of L, i.e.

$$Z\triangleq\{e\in L|e\circ e'=0,\ \forall e'\in L\}.$$

It is easily checked that

$$e_1\circ e_2+e_2\circ e_1\in Z,\qquad \forall e_1,e_2\in L$$

and Z is an ideal of L. Moreover, the Leibniz bracket of L induces a left action ρ of L on Z:

$$\rho(e)z\triangleq e\circ z\qquad \forall e\in L,z\in Z.$$

Definition 2.5. Given a Leibniz algebra L and an L-module (V,l,r), the Leibniz cohomology of L with coefficients in V is the cohomology of the cochain complex $C^n(L,V)=Hom(\otimes^nL,V)$ $(n\geq 0)$ with the coboundary operator $d_0: \ C^n(L,V)\to C^{n+1}(L,V)$ given by:

$$\begin{aligned}
&(d_0\eta)(e_1,\cdots,e_{n+1})\\
=\ &\sum_{a=1}^{n}(-1)^{a+1}l_{e_a}\eta(e_1,\cdots,\hat{e_a},\cdots,e_{n+1})+(-1)^{n+1}r_{e_{n+1}}\eta(e_1,\cdots,e_n)\\
&+\sum_{1\leq a<b\leq n+1}(-1)^a\eta(e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},e_a\circ e_b,\cdots,e_{n+1})
\end{aligned}\tag{2.2}$$

The resulting cohomology is denoted by $H^\bullet(L;V,l,r)$, or simply $H^\bullet(L,V)$ if it causes no confusion.

As a special type of Leibniz algebras, Courant-Dorfman algebras can be viewed as the algebraization of Courant algebroids:

Definition 2.6. A Courant-Dorfman algebra $(\mathcal{E}, R, \langle\cdot,\cdot\rangle, \partial, \circ)$ consists of the following data:
a commutative algebra R over a field $\mathfrak{k}$ ($\mathfrak{k} = \mathbb{R}$ for our main interest);
an R-module $\mathcal{E}$;
a symmetric bilinear form $\langle\cdot,\cdot\rangle:\ \mathcal{E}\otimes_R\mathcal{E}\to R$;
a derivation $\partial:\ R\to\mathcal{E}$;
a Dorfman bracket $\circ:\ \mathcal{E}\otimes\mathcal{E}\to\mathcal{E}$.
These data are required to satisfy the following conditions for any $e, e_1, e_2, e_3 \in \mathcal{E}$ and $f, g\in R$:
(1). $e_1\circ(fe_2) = f(e_1\circ e_2) + \langle e_1, \partial f\rangle e_2$;
(2). $\langle e_1, \partial(e_2, e_3)\rangle = \langle e_1\circ e_2, e_3\rangle + \langle e_2, e_1\circ e_3\rangle$
(3). $e_1\circ e_2 + e_2\circ e_1 = \partial\langle e_1, e_2\rangle$;
(4). $e_1\circ(e_2\circ e_3) = (e_1\circ e_2)\circ e_3 + e_2\circ(e_1\circ e_3)$;
(5). $\partial f\circ e = 0$;
(6). $\langle\partial f, \partial g\rangle = 0$.

Given a Courant-Dorfman algebra $\mathcal{E}$, we can recover the anchor map

$$\rho:\ \mathcal{E}\to\mathfrak{X}^1 = Der(R,R)$$

from the derivation ∂ by setting:

$$\rho(e)\cdot f \triangleq \langle e, \partial f\rangle. \tag{2.3}$$

Let Ω^1 be the R-module of Kahler differentials with the universal derivation $d_R: R\to\Omega^1$. By the universal property of Ω^1, there is a unique homomorphism of R-modules $\rho^*:\Omega^1\to\mathcal{E}$ such that

$$\rho^*(d_R f) \triangleq \partial f, \qquad \forall f\in R \tag{2.4}$$

ρ^* is called the coanchor map of $\mathcal{E}$. When the bilinear form of $\mathcal{E}$ is non-degenerate, ρ^* can be equivalently defined by

$$\langle\rho^*\alpha, e\rangle = \langle\alpha, \rho(e)\rangle, \qquad \forall\alpha\in\Omega^1, e\in\mathcal{E},$$

where $\langle\cdot,\cdot\rangle$ on the right handside is the natural pairing of Ω^1 and $\mathfrak{X}^1$.

In the following of this section, we always assume

$$e\in\mathcal{E},\ \alpha\in\Omega^1,\ f\in R.$$

Given a Courant-Dorfman algebra $\mathcal{E}$, denote by $C^n(\mathcal{E}, R)$ the space of all sequences $\omega = (\omega_0, \cdots, \omega_{[\frac{n}{2}]})$, where ω_k is a linear map from $(\otimes^{n-2k}\mathcal{E})\otimes(\odot^k\Omega^1)$ to R, $\forall k$, satisfying the following conditions:

1). Weak skew-symmetricity in arguments of $\mathcal{E}$.
$\forall k$, ω_k is weakly skew-symmetric up to ω_{k+1}:

$$\begin{aligned}
&\omega_k(e_1,\cdots e_a, e_{a+1},\cdots e_{n-2k};\alpha_1,\cdots\alpha_k) + \omega_k(e_1,\cdots e_{a+1}, e_a,\cdots e_{n-2k};\alpha_1,\cdots\alpha_k)\\
=\ \ & -\omega_{k+1}(e_1,\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots e_{n-2k}; d_R\langle e_a, e_b\rangle, \alpha_1,\cdots\alpha_k),
\end{aligned}$$

2). Weak R-linearity in arguments of $\mathcal{E}$.
$\forall k$, ω_k is weakly R-linear up to ω_{k+1}:

$$\begin{aligned}
&\omega_k(e_1,\cdots fe_a,\cdots e_{n-2k};\alpha_1,\cdots\alpha_k)\\
=\ \ & f\omega_k(e_1,\cdots e_a,\cdots e_{n-2k};\cdots) + \sum_{b>a}(-1)^{b-a}\langle e_a, e_b\rangle\omega_{k+1}(e_1,\cdots\widehat{e_a},\cdots\widehat{e_b},\cdots e_{n-2k}; d_R f,\cdots),
\end{aligned}$$

3). R-linearity in arguments of Ω^1.
$\forall k$, ω_k is R-linear in arguments of Ω^1:

$$\omega_k(e_1,\cdots e_{n-2k};\alpha_1,\cdots f\alpha_l,\cdots\alpha_k)=f\omega_k(e_1,\cdots e_{n-2k};\alpha_1,\cdots\alpha_l,\cdots\alpha_k),$$

Then $C^\bullet(\mathcal{E},R)=\bigoplus_n C^n(\mathcal{E},R)$ becomes a cochain complex, with coboundary map d given for any $\omega\in C^n(\mathcal{E},R)$ by:

$$\begin{aligned}
&(d\omega)_k(e_1,\cdots,e_{n+1-2k};\alpha_1,\cdots,\alpha_k)\\
=\;&\sum_a(-1)^{a+1}\rho(e_a)\omega_k(\cdots\widehat{e_a},\cdots;\cdots)+\sum_{a<b}(-1)^a\omega_k(\cdots\widehat{e_a},\cdots\widehat{e_b},e_a\circ e_b,\cdots;\cdots)\\
&+\sum_i\omega_{k-1}(\rho^*(\alpha_i),e_1,\cdots,e_{n+1-2k};\alpha_1,\cdots\widehat{\alpha_i},\cdots\alpha_k)\\
&+\sum_{a,i}(-1)^a\omega_k(\cdots,\widehat{e_a},\cdots;\cdots,\widehat{\alpha_i},\iota_{\rho(e_a)}d_R\alpha_i,\cdots).
\end{aligned}$$

Definition 2.7. The cochain complex $(C^\bullet(\mathcal{E},R),d)$ is called the standard (cochain) complex of the Courant-Dorfman algebra $\mathcal{E}$, the resulting cohomology is called the standard cohomology of $\mathcal{E}$, and denoted by $H^\bullet_{st}(\mathcal{E})$.

3. H-Standard cohomology

In this section, we give the definition of H-standard cohomology for Courant-Dorfman algebras and Leibniz algebras respectively. The inspiration comes from Roytenberg's construction of standard cohomology for Courant-Dorfman algebras [11].

3.1. **For Courant-Dorfman algebras.** Given a Courant-Dorfman algebra $(\mathcal{E},R,\langle\cdot,\cdot\rangle,\partial,\circ)$, suppose $\mathcal{H}$ is an R-submodule as well as an isotropic ideal of $\mathcal{E}$ containing ∂R, and $(\mathcal{V},\nabla)$ is an $\mathcal{H}$-representation of $\mathcal{E}$, which is defined as follows:

Definition 3.1. An $\mathcal{H}$-trivial representation (or $\mathcal{H}$-representation for short) of a Courant-Dorfman algebra $(\mathcal{E},R,\langle\cdot,\cdot\rangle,\partial,\circ)$ is a pair $(\mathcal{V},\nabla)$, where $\mathcal{V}$ is an R-module, and $\nabla:\mathcal{E}\to Der(\mathcal{V})$ is a homomorphism of Leibniz algebras such that:

$$\begin{aligned}
\nabla_\alpha v&=0\\
\nabla_{fe}v&=f\nabla_e v\\
\nabla_e(fv)&=(\rho(e)f)v+f(\nabla_e v)\qquad\forall\alpha\in\mathcal{H},e\in\mathcal{E},f\in R,v\in\mathcal{V}.
\end{aligned}$$

Example 3.2. Since

$$\rho(\alpha)f=\langle\alpha,\partial f\rangle=0,\qquad\forall\alpha\in\mathcal{H},f\in R,$$

$\mathcal{H}$ is actually an R-submodule of $ker\rho$. It is easily checked that (R,ρ) is an $\mathcal{H}$-representation of $\mathcal{E}$.

In the following of this subsection, we always assume

$$e\in\mathcal{E},\ \alpha\in\mathcal{H},\ f\in R.$$

Denote by $C^n(\mathcal{E},\mathcal{H},\mathcal{V})$ the space of all sequences

$$\omega=(\omega_0,\cdots,\omega_{[\frac{n}{2}]}),$$

where the $\mathfrak{k}$-linear maps

$$\omega_k:\ (\otimes^{n-2k}\mathcal{E})\otimes(\odot^k\mathcal{H})\to\mathcal{V}$$

satisfy the following conditions:

1). Weak skew-symmetricity in arguments of $\mathcal{E}$.
$\forall k$, ω_k is weakly skew-symmetric up to ω_{k+1}:

$$\begin{aligned} & \omega_k(e_1,\cdots e_a,e_{a+1},\cdots e_{n-2k};\alpha_1,\cdots\alpha_k)+\omega_k(e_1,\cdots e_{a+1},e_a,\cdots e_{n-2k};\alpha_1,\cdots\alpha_k) \\ = \; & -\omega_{k+1}(e_1,\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots e_{n-2k};\partial\langle e_a,e_b\rangle,\alpha_1,\cdots\alpha_k), \end{aligned}$$

2). Weak R-linearity in arguments of $\mathcal{E}$.
$\forall k$, ω_k is weakly R-linear up to ω_{k+1}:

$$\begin{aligned} & \omega_k(e_1,\cdots fe_a,\cdots e_{n-2k};\alpha_1,\cdots\alpha_k) \\ = \; & f\omega_k(e_1,\cdots e_a,\cdots e_{n-2k};\cdots)+\sum_{b>a}(-1)^{b-a}\langle e_a,e_b\rangle\omega_{k+1}(e_1,\cdots\widehat{e_a},\cdots\widehat{e_b},\cdots e_{n-2k};\partial f,\cdots), \end{aligned}$$

3). R-linearity in arguments of $\mathcal{H}$.
$\forall k$, ω_k is R-linear in arguments of $\mathcal{H}$:

$$\omega_k(e_1,\cdots e_{n-2k};\alpha_1,\cdots f\alpha_l,\cdots\alpha_k)=f\omega_k(e_1,\cdots e_{n-2k};\alpha_1,\cdots\alpha_l,\cdots\alpha_k).$$

Theorem 3.3. *$C^\bullet(\mathcal{E},\mathcal{H},\mathcal{V})\triangleq\bigoplus_n C^n(\mathcal{E},\mathcal{H},\mathcal{V})$ is a cochain complex under the coboundary map $d=d_0+\delta+d'$, where d_0 is the coboundary map (Equation 2.2) corresponding to the Leibniz cohomology of $\mathcal{E}$ with coefficients in $(\mathcal{V},\nabla,-\nabla)$, and δ,d' are defined for any $\omega\in C^n(\mathcal{E},\mathcal{H},\mathcal{V})$ respectively by:*

$$\begin{aligned} (\delta\omega)_k(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k) & \triangleq \sum_i\omega_{k-1}(\alpha_i,e_1,\cdots,e_{n+1-2k};\cdots\widehat{\alpha_i},\cdots), \\ (d'\omega)_k(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k) & \triangleq \sum_{a,i}(-1)^{a+1}\omega_k(\cdots\widehat{e_a}\cdots;\cdots\widehat{\alpha_i},\alpha_i\circ e_a\cdots). \end{aligned}$$

Lemma 3.4. *$C^\bullet(\mathcal{E},\mathcal{H},\mathcal{V})$ is closed under $d=d_0+\delta+d'$.*

Proof. $\forall\omega\in C^n(\mathcal{E},\mathcal{H},\mathcal{V})$, we need to prove that $d\omega\in C^{n+1}(\mathcal{E},\mathcal{H},\mathcal{V})$.

First, we prove the weak skew-symmetricity in arguments of $\mathcal{E}$. We will calculate d_0,δ,d' parts separately. The calculations are straightforward from the definitions but rather tedious. To save space, we omit the details, and only list the results of calculations here.

$$\begin{aligned} & (d_0\omega)_k(e_1,\cdots e_i,e_{i+1}\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)+(d_0\omega)_k(e_1,\cdots e_{i+1},e_i\cdots;\cdots) \\ = \; & -(d_0\omega)_{k+1}(e_1,\cdots\widehat{e_i},\widehat{e_{i+1}},\cdots;\partial\langle e_i,e_{i+1}\rangle,\cdots)-\omega_k(\partial\langle e_i,e_{i+1}\rangle,e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\cdots), \end{aligned} \tag{3.1}$$

$$\begin{aligned} & (\delta\omega)_k(e_1,\cdots e_i,e_{i+1}\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)+(\delta\omega)_k(e_1,\cdots e_{i+1},e_i\cdots;\cdots) \\ = \; & -(\delta\omega)_{k+1}(e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\partial\langle e_i,e_{i+1}\rangle,\cdots)+\omega_k(\partial\langle e_i,e_{i+1}\rangle,e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\cdots), \end{aligned}$$

$$\begin{aligned} & (d'\omega)_k(e_1\cdots e_i,e_{i+1}\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)+(d'\omega)_k(e_1,\cdots e_{i+1},e_i\cdots;\cdots) \\ = \; & -(d'\omega)_{k+1}(e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\partial\langle e_i,e_{i+1}\rangle,\cdots). \end{aligned} \tag{3.2}$$

The sum of the three equations above tells:

$$\begin{aligned} & (d\omega)_k(e_1,\cdots e_i,e_{i+1}\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)+(d\omega)_k(e_1,\cdots e_{i+1},e_i\cdots;\cdots) \\ = \; & -(d\omega)_{k+1}(e_1,\cdots,\widehat{e_i},\widehat{e_{i+1}},\cdots;\partial\langle e_i,e_{i+1}\rangle,\cdots), \end{aligned}$$

i.e. $(d\omega)_k$ is weakly skew-symmetric up to $(d\omega)_{k+1}$.

Next, we prove the weak R-linearity in arguments of $\mathcal{E}$. By direct calculations, we have the following:

$$\begin{aligned}
&(d_0\omega)_k(e_1,\cdots fe_i,\cdots e_{n+1-2k};\alpha_1,\cdots \alpha_k)-f(d_0\omega)_k(e_1,\cdots e_i,\cdots;\cdots)\\
=\ &\sum_{a>i}(-1)^{a-i}\langle e_i,e_a\rangle(d_0\omega)_{k+1}(\cdots\widehat{e_i},\cdots\widehat{e_a},\cdots;\partial f,\cdots)\\
&+\sum_{i<a}(-1)^{a-i}\langle e_i,e_a\rangle\omega_k(\partial f,e_1,\cdots\widehat{e_i},\cdots\widehat{e_a},\cdots;\cdots)
\end{aligned}\tag{3.3}$$

$$\begin{aligned}
&(\delta\omega)_k(e_1,\cdots fe_i,\cdots e_{n+1-2k};\alpha_1,\cdots \alpha_k)-f(\delta\omega)_k(e_1,\cdots e_i,\cdots;\cdots)\\
=\ &\sum_{i<a}(-1)^{a-i}\langle e_i,e_a\rangle(\delta\omega)_{k+1}(e_1,\cdots\widehat{e_i},\cdots\widehat{e_a},\cdots;\partial f,\cdots)\\
&-\sum_{i<a}(-1)^{a-i}\langle e_i,e_a\rangle\omega_k(\partial f,e_1,\cdots\widehat{e_i},\cdots\widehat{e_a},\cdots;\cdots)
\end{aligned}$$

$$\begin{aligned}
&(d'\omega)_k(e_1,\cdots fe_i,\cdots e_{n+1-2k};\alpha_1,\cdots \alpha_k)-f(d'\omega)_k(e_1,\cdots e_i,\cdots;\cdots)\\
=\ &\sum_{a>i}(-1)^{a-i}\langle e_i,e_a\rangle(d'\omega)_{k+1}(e_1,\cdots\widehat{e_i},\cdots\widehat{e_a},\cdots;\partial f,\cdots)
\end{aligned}\tag{3.4}$$

The sum of the three equations above is:

$$\begin{aligned}
&(d\omega)_k(e_1,\cdots fe_i,\cdots e_{n+1-2k};\alpha_1,\cdots \alpha_k)-f(d\omega)_k(e_1,\cdots e_i,\cdots;\cdots)\\
=\ &\sum_{a>i}(-1)^{a-i}\langle e_i,e_a\rangle(d\omega)_{k+1}(e_1,\cdots\widehat{e_i},\cdots\widehat{e_a},\cdots;\partial f,\cdots)
\end{aligned}$$

i.e. $(d\omega)_k$ is weakly R-linear up to $(d\omega)_{k+1}$.

Finally, we need to prove the R-linearity in arguments of $\mathcal{H}$. Again by direct calculations, we have the following:

$$\begin{aligned}
&(d_0\omega+d'\omega)_k(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots f\alpha_l,\cdots\alpha_k)-f(d_0\omega+d'\omega)_k(\cdots;\cdots\alpha_l,\cdots)\\
=\ &\sum_a(-1)^{a+1}\langle\alpha_l,e_a\rangle\omega_k(\cdots\widehat{e_a},\cdots;\partial f,\cdots\widehat{\alpha_l},\cdots),
\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}
&(\delta\omega)_k(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots f\alpha_l,\cdots\alpha_k)-f(\delta\omega)_k(\cdots;\cdots\alpha_l,\cdots)\\
=\ &\sum_a(-1)^{a}\langle\alpha_l,e_a\rangle\omega_k(\cdots\widehat{e_a},\cdots;\partial f,\cdots\widehat{\alpha_l},\cdots),
\end{aligned}$$

so

$$(d\omega)_k(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots f\alpha_l,\cdots\alpha_k)-f(d\omega)_k(\cdots;\cdots\alpha_l,\cdots)=0.$$

The lemma is proved. ■

Now we turn to the proof of Theorem 3.3:

Proof. By the lemma above, we only need to prove $d^2=0$.

d^2 can be divided into six parts

$$d^2=d_0^2+\delta^2+(d_0\circ\delta+\delta\circ d_0)+(d_0\circ d'+d'\circ d_0)+(\delta\circ d'+d'\circ\delta)+d'^2.$$

The first part equals 0, so we only need to compute the other five parts. By direct calculations, we have the following:

$$(\delta^2\omega)_k(e_1,\cdots,e_{n+2-2k};\alpha_1,\cdots,\alpha_k)=0,$$

$$((d_0\circ\delta+\delta\circ d_0)\omega)_k(e_1,\cdots;\alpha_1,\cdots)=-\sum_{i,a}\omega_{k-1}(\cdots\widehat{e_a},\alpha_i\circ e_a,\cdots;\cdots\widehat{\alpha_i},\cdots),$$

$$((d_0\circ d'+d'\circ d_0)\omega)_k(e_1,\cdots;\alpha_1,\cdots)=\sum_{j,a<c}(-1)^{a+c}\omega_k(\cdots\widehat{e_a},\cdots\widehat{e_c},\cdots;\cdots\alpha_j\circ(e_a\circ e_c),\cdots),$$

$$((\delta\circ d'+d'\circ\delta)\omega)_k(e_1,\cdots;\alpha_1,\cdots)=\sum_{j,a}(-1)^{a+1}\omega_{k-1}(\alpha_j\circ e_a,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_j},\cdots),$$

$$(d'^2\omega)_k(e_1,\cdots;\alpha_1,\cdots)=\sum_{a<b,i}(-1)^{a+b}\omega_k(\cdots\widehat{e_a},\cdots\widehat{e_b},\cdots;\cdots\widehat{\alpha_i},(\alpha_i\circ e_b)\circ e_a-(\alpha_i\circ e_a)\circ e_b,\cdots),$$

The sum of the above equations is:

$$\begin{aligned}
&(d^2\omega)_k(e_1,\cdots e_{n+2-2k};\alpha_1,\cdots\alpha_k)\\
=\;&\sum_{j,a}(-1)^{a+1}\omega_{k-1}(\alpha_j\circ e_a,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_j},\cdots)-\sum_{i,a}\omega_{k-1}(\cdots\alpha_i\circ e_a,\cdots;\cdots\widehat{\alpha_i},\cdots)\\
&+\sum_{j,a<b}(-1)^{a+b}\omega_k(\cdots\widehat{e_a},\cdots\widehat{e_b},\cdots;\cdots\alpha_j\circ(e_a\circ e_b),\cdots)\\
&+\sum_{a<b,i}(-1)^{a+b}\omega_k(\cdots\widehat{e_a},\cdots\widehat{e_b},\cdots;\cdots\widehat{\alpha_i},(\alpha_i\circ e_b)\circ e_a-(\alpha_i\circ e_a)\circ e_b,\cdots)\\
=\;&\sum_{i,b<a}(-1)^{a+b}\big(\omega_{k-1}(\cdots\alpha_i\circ e_a,e_b,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_i},\cdots)\\
&\quad+\omega_{k-1}(\cdots e_b,\alpha_i\circ e_a,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_i},\cdots)\big)\\
&+\sum_{i,a<b}(-1)^{a+b}\omega_k(\cdots\widehat{e_a},\cdots\widehat{e_b},\cdots;\cdots\langle e_a,\alpha_i\circ e_b\rangle,\cdots)\\
=\;&0
\end{aligned}$$

The proof is finished. ■

Definition 3.5. With the notations above, $(C^\bullet(\mathcal{E},\mathcal{H},\mathcal{V}),d)$ is called $\mathcal{H}$-standard complex of $\mathcal{E}$ with coefficients in $\mathcal{V}$. The resulting cohomology, denoted by $H^\bullet(\mathcal{E},\mathcal{H},\mathcal{V})$, is called $\mathcal{H}$-standard cohomology of $\mathcal{E}$ with coefficients in $\mathcal{V}$.

Let's consider the standard cohomology in lower degrees:
Degree 0:
$H^0(\mathcal{E},\mathcal{H},\mathcal{V})$ is the submodule of $\mathcal{V}$ consisting of all invariants, i.e.

$$H^0(\mathcal{E},\mathcal{H},\mathcal{V})=\{v\in\mathcal{V}|\nabla_e v=0,\ \forall e\in\mathcal{E}\}.$$

Degree 1:
A cocycle ω in $C^1(\mathcal{E},\mathcal{H},\mathcal{V})$ is a map $\omega_0:\mathcal{E}\to\mathcal{V}$ satisfying:

$$\omega_0(e_1\circ e_2)=\nabla_{e_1}\omega_0(e_2)-\nabla_{e_2}\omega_0(e_1),\quad\forall e_1,e_2\in\mathcal{E}$$

and

$$\omega_0(\alpha)=0,\quad\forall\alpha\in\mathcal{H}.$$

The first equation above tells that ω_0 is a derivation from $\mathcal{E}$ to $\mathcal{V}$, while the second equation tells that ω_0 induces a map from $\mathcal{E}/\mathcal{H}$ to $\mathcal{V}$.

$\eta \in C^1(\mathcal{E}, \mathcal{H}, \mathcal{V})$ is a coboundary iff there exists $v \in \mathcal{V}$ such that:

$$\eta_0(e) = \nabla_e v, \quad \forall e \in \mathcal{E},$$

i.e. η_0 is an inner derivation from $\mathcal{E}$ to $\mathcal{V}$.

Thus $H^1(\mathcal{E}, \mathcal{H}, \mathcal{V})$ is the space of "outer derivations": {derivations}/{inner derivations} from $\mathcal{E}$ to $\mathcal{V}$ which act trivially on $\mathcal{H}$. Or equivalently, $H^1(\mathcal{E}, \mathcal{H}, \mathcal{V})$ is the space of outer derivations from $\mathcal{E}/\mathcal{H}$ to $\mathcal{V}$.

Degree 2:

$\omega = (\omega_0, \omega_1) \in C^2(\mathcal{E}, \mathcal{H}, \mathcal{V})$ is a 2-cocycle iff:

$$\begin{aligned} &\nabla_{e_1}\omega_0(e_2, e_3) - \nabla_{e_2}\omega_0(e_1, e_3) + \nabla_{e_3}\omega_0(e_1, e_2) \\ &-\omega_0(e_1 \circ e_2, e_3) - \omega_0(e_2, e_1 \circ e_3) + \omega_0(e_1, e_2 \circ e_3) = 0 \end{aligned} \tag{3.6}$$

and

$$\nabla_e \omega_1(\alpha) + \omega_0(\alpha, e) + \omega_1(\alpha \circ e) = 0 \tag{3.7}$$

$\forall e, e_1, e_2, e_3 \in \mathcal{E}, \alpha \in \mathcal{H}$.

Equation 3.6 holds iff the bracket on $\bar{\mathcal{E}} \triangleq \mathcal{E} \oplus \mathcal{V}$ defined for any $e_1, e_2 \in \mathcal{E}$, $v_1, v_2 \in \mathcal{V}$ by:

$$(e_1 + v_1)\bar{\circ}(e_2 + v_2) \triangleq e_1 \circ e_2 + \big(\nabla_{e_1} v_2 - \nabla_{e_2} v_1 + \omega_0(e_1, e_2)\big)$$

is a Leibniz bracket. Furthermore, if Equation 3.7 also holds, it is easily checked that $(\bar{\mathcal{E}}, R, \overline{\langle\cdot,\cdot\rangle}, \bar{\partial}, \bar{\circ})$ is a Courant-Dorfman algebra, where $\overline{\langle\cdot,\cdot\rangle}$ and $\bar{\partial}$ are defined as:

$$\begin{aligned} \overline{\langle e_1 + v_1, e_2 + v_2\rangle} &= \langle e_1, e_2\rangle \\ \bar{\partial} f &= \partial f - \omega_1(\partial f). \end{aligned}$$

Actually Equation 3.6 implies that

$$\bar{\mathcal{H}} \triangleq \{\alpha - \omega_1(\alpha) | \alpha \in \mathcal{H}\}$$

is an ideal of $\bar{\mathcal{E}}$.

In a summation, 2-cocycles are in 1-1 correspondence with central extensions of Courant-Dorfman algebras which are split as metric R-modules:

$$0 \to \mathcal{V} \to \bar{\mathcal{E}} \to \mathcal{E} \to 0$$

such that $\bar{\mathcal{H}}$ is an ideal of $\bar{\mathcal{E}}$.

The central extensions determined by 2-cocycles ω_1, ω_2 are isomorphic iff $\omega_1 - \omega_2 = d\lambda$, for some $\lambda \in C^1(\mathcal{E}, \mathcal{H}, \mathcal{V})$.

Thus $H^2(\mathcal{E}, \mathcal{H}, \mathcal{V})$ classifies isomorphism classes of central extensions of Courant-Dorfman algebras which are split as metric R-modules:

$$0 \to \mathcal{V} \to \bar{\mathcal{E}} \to \mathcal{E} \to 0$$

such that $\bar{\mathcal{H}}$ is an ideal of $\bar{\mathcal{E}}$.

3.2. **For Leibniz algebras.** Assume L is a Leibniz algebra with left center Z. There is a symmetric bilinear product $(\cdot,\cdot) : L \otimes L \to Z$ defined as:

$$(e_1, e_2) \triangleq e_1 \circ e_2 + e_2 \circ e_1, \qquad \forall e_1, e_2 \in L.$$

It is easily checked that such defined bilinear product is invariant, i.e.

$$\rho(e_1)(e_2, e_3) = (e_1 \circ e_2, e_3) + (e_2, e_1 \circ e_3), \qquad \forall e_1, e_2, e_3 \in L.$$

Let $H \supseteq Z$ be an isotropic ideal in L. Let (V, τ) be an H-representation of L, which is defined as follows:

Definition 3.6. An H-trivial left representation (or H-representation for short) of a Leibniz algebra L is a pair (V, τ), where V is vector space, and $\tau : L \to gl(V)$ is a homomorphism of Leibniz algebras such that:

$$\tau(h)v = 0, \quad \forall h \in H, v \in V.$$

Example 3.7. Since

$$\rho(h)z = h \circ z = (h, z) = 0, \qquad \forall h \in H, z \in Z,$$

(Z, ρ) is an H-representation.

Denote by $C^n(L, H, V)$ the space of all sequences $\omega = (\omega_0, \cdots, \omega_{[\frac{n}{2}]})$, where ω_k is a linear map from $(\otimes^{n-2k} L) \otimes (\odot^k H)$ to V, $\forall k$, and is weakly skew-symmetric in arguments of L up to ω_{k+1}:

$$\begin{aligned} & \omega_k(e_1, \cdots e_i, e_{i+1,} \cdots e_{n-2k}; h_1, \cdots h_k) + \omega_k(e_1, \cdots e_{i+1}, e_{i,} \cdots e_{n-2k}; h_1, \cdots h_k) \\ = \; & -\omega_{k+1}(\cdots \widehat{e_i}, \widehat{e_{i+1}}, \cdots ; (e_i, e_{i+1}), \cdots) \end{aligned}$$

$\forall e_j \in L, h_l \in H$.

Theorem 3.8. *$C^\bullet(L, H, V) \triangleq \bigoplus_n C^n(L, H, V)$ is a cochain complex, under the coboundary map $d = d_0 + \delta + d'$, where d_0 is the coboundary map (Equation 2.2) corresponding to the Leibniz cohomology of L with coefficients in $(V, \tau, -\tau)$, and δ, d' are defined for any $\omega \in C^n(L, H, V)$ respectively by:*

$$\begin{aligned} (\delta\omega)_k(e_1, \cdots, e_{n+1-2k}; h_1, \cdots h_k) &\triangleq \sum_j \omega_{k-1}(\alpha_j, e_1, \cdots, e_{n+1-2k}; h_1, \cdots \widehat{h_j}, \cdots h_k) \\ (d'\omega)_k(e_1, \cdots e_{n+1-2k}; h_1, \cdots h_k) &\triangleq \sum_{a,j} (-1)^{a+1} \omega_k(\cdots \widehat{e_a}, \cdots ; \cdots \widehat{h_j}, h_j \circ e_a, \cdots) \end{aligned}$$

$\forall e_a \in L, h_i \in H$. ($(\delta\omega)_0$ is defined to be 0.)

Proof. The proof of this theorem is quite similar to that of Theorem 3.3, so we omit it here. ∎

Definition 3.9. $(C^\bullet(L, H, V), d)$ is called the H-standard complex of L with coefficients in V. The resulting cohomology, denoted by $H^\bullet(L, H, V)$ is called the H-standard cohomology of L with coefficients in V.

The H-standard cohomology of L in degree $0, 1, 2$ have similar interpretations to the case of Courant-Dorfman algebras:
$H^0(L, H, V)$ is the submodule of V consisting of all invariants.
$H^1(L, H, V)$ is the space of outer derivations from L to V acting trivially on H.
$H^2(L, H, V)$ classfies the equivalence classes of abelian extensions of L by V:

$$0 \to V \to \bar{L} \to L \to 0$$

such that $\bar{H}$ is an ideal of $\bar{L}$.

4. Isomorphism theorem for Courant-Dorfman algebra

In this section, we present one of our main results in this paper, which is a generalization of a theorem of Ginot-Grutzmann (conjectured by Stienon-Xu) for transitive Courant algebroids.

Let $\mathcal{E}, \mathcal{H}, \mathcal{V}$ be as described in the last section. Since $\mathcal{H}$ is an ideal in $\mathcal{E}$ containing ∂R, it is easily checked that $\mathcal{E}/\mathcal{H}$ is a Lie-Rinehart algebra with the induced anchor map(still denoted by ρ):

$$\rho([e])f \triangleq \rho(e)f, \qquad \forall e \in \mathcal{E}, f \in R,$$

and induced bracket:

$$[e_1] \circ [e_2] \triangleq [e_1 \circ e_2], \qquad \forall e_1, e_2 \in \mathcal{E}.$$

Moreover, $\mathcal{V}$ becomes a representation of $\mathcal{E}/\mathcal{H}$ with the induced action (still denoted by τ):

$$\tau([e])v \triangleq \tau(e)v, \qquad \forall e \in \mathcal{E}, v \in \mathcal{V}.$$

As a result, we have the Chevalley-Eilenberg complex $(C^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}), d_{CE})$ of $\mathcal{E}/\mathcal{H}$ with coefficients in $\mathcal{V}$, and the corresponding cohomology $H^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V})$.

Theorem 4.1. *Given a Courant-Dorfman algebra $(\mathcal{E}, R, \langle\cdot,\cdot\rangle, \partial, \circ)$, an R-submodule $\mathcal{H} \supseteq \partial R$ which is an isotropic ideal of $\mathcal{E}$, and an $\mathcal{H}$-representation $(\mathcal{V}, \nabla)$. If the quotient module $\mathcal{E}/\mathcal{H}$ is projective, then we have:*

$$H^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V}) \cong H^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}).$$

Before proof of this theorem, we prove the following two lemmas first.

Lemma 4.2. *$(C^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}), d_{CE})$ is isomorphic to the following subcomplex of $(C^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V}), d)$:*

$$C^\bullet_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V}) \triangleq \{\omega \in C^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V}) | \omega_k = 0, \ \forall k \geq 1, \ \iota_\alpha \omega_0 = 0, \ \forall \alpha \in \mathcal{H}\}$$

Proof. Obviously $C^\bullet_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V})$ is a subcomplex of $C^\bullet(\mathcal{E}, \mathcal{H}, \mathcal{V})$.

And it is easily checked that the following two maps φ, ϕ are well-defined cochain maps and invertible to each other:

$$\begin{aligned} \varphi : C^\bullet_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V}) &\to C^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}) \\ \varphi(\eta)([e_1], \cdots [e_n]) &\triangleq \eta(e_1, \cdots e_n) \quad \forall \eta \in C^n_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V}), e_a \in \mathcal{E}, \end{aligned}$$

and

$$\begin{aligned} \phi : C^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}) &\to C^\bullet_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V}) \\ \phi(\zeta)(e_1, \cdots e_n) &\triangleq \zeta([e_1], \cdots [e_n]) \quad \forall \zeta \in C^\bullet_{CE}(\mathcal{E}/\mathcal{H}, \mathcal{V}), e_a \in \mathcal{E}. \end{aligned}$$

■

Lemma 4.3. *Given any $\omega \in C^n(\mathcal{E}, \mathcal{H}, \mathcal{V})$, if $(d\omega)_k = 0, \ \forall k \geq 1$, then there exists $\eta \in C^n_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V})$ and $\lambda \in C^{n-1}(\mathcal{E}, \mathcal{H}, \mathcal{V})$ such that $\omega = \eta + d\lambda$.*

Proof. Since the quotient $\mathcal{E}/\mathcal{H}$ is a projective module, there exists an R-module decomposition: $\mathcal{E} = \mathcal{H} \oplus \mathcal{X}$. We will give an inductive construction of λ and β. The construction depends on the decomposition, but the cohomology class of β doesn't depend on the decomposition.

Suppose $n = 2m$ *or* $2m - 1$, we will define $\lambda_{m-1}, \lambda_{m-2}, \cdots, \lambda_0$ one by one, so that each $\lambda_p : \otimes^{n-1-2p}\mathcal{E} \otimes \odot^p \mathcal{H} \to \mathcal{V}, \ 0 \leq p \leq m-1$ satisfies the following conditions, which we call "Lambda Conditions":

1). λ_p is weakly skew-symmetric in arguments of $\mathcal{E}$ up to λ_{p+1},

2). λ_p is weakly R-linear in arguments of $\mathcal{E}$ up to λ_{p+1},

3). λ_p is R-linear in arguments of R,

4). $\omega_{p+1} = (d\lambda)_{p+1}$,
5). $\sum_i(\omega_p - d_0\lambda_p - d'\lambda_p)(\alpha_i, e_1, \cdots e_{n-1-2p}; \alpha_1, \cdots \widehat{\alpha_i}, \cdots \alpha_{p+1}) = 0$, $\forall \alpha_j \in \mathcal{H}, e_a \in \mathcal{E}$.
The construction of $\lambda_{m-1}, \lambda_{m-2}, \cdots, \lambda_0$ is done in the following four steps.
Step 1:
Construction of λ_{m-1}:
When $n = 2m-1$ is odd, let

$$\lambda_{m-1}(\alpha_1, \cdots \alpha_{m-1}) = 0, \qquad \forall \alpha_j \in \mathcal{H}.$$

When $n = 2m$ is even, let

$$\lambda_{m-1}(\beta; \alpha_1, \cdots \alpha_{m-1}) = \frac{1}{m}\omega_m(\beta, \alpha_1, \cdots \alpha_{m-1}),\ \forall \beta, \alpha_j \in \mathcal{H}$$

and

$$\lambda_{m-1}(x; \alpha_1, \cdots \alpha_{m-1}) = 0,\ \forall x \in \mathcal{X}, \alpha_j \in \mathcal{H}.$$

It is obvious that λ_{m-1} defined above satisfies Lambda Conditions 1) - 4). So we only need to prove Lambda Condition 5):

When $n = 2m-1$,

$$\sum_i(\omega_{m-1} - d_0\lambda_{m-1} - d'\lambda_{m-1})(\alpha_i; \alpha_1, \cdots \widehat{\alpha_i}, \cdots \alpha_m) = (d\omega)_m(\alpha_1, \cdots \alpha_m) = 0.$$

When $n = 2m$, the left hand side in condition 3) equals

$$\begin{aligned}
& \sum_i(\omega_{m-1} - d_0\lambda_{m-1} - d'\lambda_{m-1})(\alpha_i, e; \alpha_1, \cdots \widehat{\alpha_i}, \cdots \alpha_m) \\
= \quad & (\delta\omega)_m(e; \alpha_1, \cdots, \alpha_m) + \sum_i \nabla_e\lambda_{m-1}(\alpha_i; \cdots \widehat{\alpha_i}, \cdots) + \sum_i \lambda_{m-1}(\alpha_i \circ e; \cdots \widehat{\alpha_i}, \cdots) \\
& + \sum_{j\neq i}(-1)\lambda_{m-1}(e; \cdots \widehat{\alpha_i}, \cdots \alpha_j \circ \alpha_i, \cdots) + \sum_{j\neq i}\lambda_{m-1}(\alpha_i; \cdots \widehat{\alpha_i}, \cdots, \alpha_j \circ e, \cdots) \\
= \quad & (\delta\omega)_m(e; \alpha_1, \cdots \alpha_m) + \nabla_e\omega_m(\alpha_1, \cdots \alpha_m) + \frac{1}{m}\sum_i \omega_m(\alpha_1, \cdots \alpha_i \circ e, \cdots \alpha_m) \\
& + \sum_{i<j}(-1)\lambda_{m-1}(e; \alpha_i \circ \alpha_j + \alpha_j \circ \alpha_i, \cdots \widehat{\alpha_i}, \cdots \widehat{\alpha_j}, \cdots) + \frac{1}{m}\sum_j\sum_{i\neq j}\omega_m(\cdots \alpha_j \circ e, \cdots) \\
= \quad & (\delta\omega)_m(e; \alpha_1, \cdots \alpha_m) + (d_0\omega)_m(e; \alpha_1, \cdots \alpha_m) + (\frac{1}{m} + \frac{m-1}{m})\sum_i \omega_m(\cdots \alpha_i \circ e, \cdots) \\
= \quad & (d\omega)_m(e; \alpha_1, \cdots \alpha_m) \\
= \quad & 0
\end{aligned}$$

Step 2:
Suppose $\lambda_{m-1}, \cdots, \lambda_k (k > 0)$ are already defined so that they satisfy Lambda Conditions, we will construct λ_{k-1}, so that it also satisfies Lambda Conditions.

By $\mathfrak{k}$-linearity and the decomposition $\mathcal{E} = \mathcal{H} \oplus \mathcal{X}$, in order to determine λ_{k-1}, we only need to define the value of $\lambda_{k-1}(e_1, \cdots e_{n+1-2k}; \alpha_1, \cdots \alpha_k)$ in which each e_a is either in $\mathcal{H}$ or in $\mathcal{X}$.

First we let

$$\begin{aligned}
& \lambda_{k-1}(\beta_1, \cdots \beta_l, x_1, \cdots x_{n+1-2k-l}; \alpha_1, \cdots \alpha_{k-1}) \\
\triangleq \quad & \frac{1}{k+l-1}\sum_{1\leq j\leq l}(-1)^{j+1}(\omega_k - d_0\lambda_k - d'\lambda_k)(\beta_1, \cdots \widehat{\beta_j}, \cdots \beta_l, \cdots; \beta_j, \cdots)
\end{aligned} \tag{4.1}$$

$\forall \beta_r, \alpha_s \in \mathcal{H}, x_a \in \mathcal{X}$.

(We call $(\beta, \cdots \beta, x, \cdots x)$ a regular permutation.)

Note that if $l = 0$, we simply let

$$\lambda_{k-1}(x_1, \cdots, x_{n+1-2k}; \alpha_1, \cdots \alpha_{k-1}) = 0.$$

For a general permutation σ, an $x \in \mathcal{X}$ in σ is called an irregular element iff there exists at least one element of $\mathcal{H}$ standing behind x in σ. The value of $\lambda_{k-1}(\sigma; \cdots)$ is determined inductively as follows:

If the number of irregular elements in σ is 0, σ is a regular permutation. So the value of $\lambda_{k-1}(\sigma; \cdots)$ is determined by Equation 4.1.

Suppose the value of $\lambda_{k-1}(\sigma; \cdots)$ is already determined for σ with irregular elements less than t $(t \geq 1)$. Now for a permutation σ with t irregular elements, assume the last of them is $y \in \mathcal{X}$, and $\sigma = (\bullet, y, \beta_1, \cdots \beta_r, x_1, \cdots x_s)$, $\beta_i \in \mathcal{H}, x_j \in \mathcal{X}$. Switching y with $\beta_1, \cdots \beta_r$ one by one, finally we will get a permutation $\tilde{\sigma} = (\bullet, \beta_1, \cdots \beta_r, y, x_1, \cdots, x_s)$, which has $t-1$ irregular elements. The value of $\lambda_{k-1}(\tilde{\sigma}; \cdots)$ is already determined. By weak skew-symmetricity we let

$$\lambda_{k-1}(\sigma; \cdots) \triangleq (-1)^r \lambda_{k-1}(\tilde{\sigma}; \cdots) + \sum_{1 \leq i \leq r} (-1)^i \lambda_k(\bullet, \beta_1, \cdots \widehat{\beta_i}, \cdots \beta_r, x_1, \cdots x_s; \partial\langle y, \beta_i \rangle, \cdots).$$

As a summary, we have extended λ_{k-1} from regular permutations to general permutations by weak skew-symmetricity. The extension could be written as a formula:

$$\lambda_{k-1}(\sigma; \cdots) = (\pm 1)\lambda_{k-1}(\bar{\sigma}; \cdots) + \sum (\pm 1)\lambda_k(\bullet; \bullet),$$

where $\bar{\sigma}$ is the regular permutation corresponding to σ.We observe that, for different k, if we do exactly the same switchings, then the extension formulas should be similar (each summand has the same sign, with the subscripts of λ modified correspondingly). For example, if we have an extension formula for k:

$$\lambda_k(\sigma; \cdots) = (\pm 1)\lambda_k(\bar{\sigma}; \cdots) + \sum (\pm 1)\lambda_{k+1}(\bullet; \bullet),$$

then for $k-1$, we have similar formula:

$$\lambda_{k-1}(\beta, \sigma, x; \cdots) = (\pm 1)\lambda_{k-1}(\beta, \bar{\sigma}, x; \cdots) + \sum (\pm 1)\lambda_k(\beta, \bullet, x; \bullet), \quad \forall \beta \in \mathcal{H}, x \in \mathcal{X}.$$

Step 3:

We need to prove that λ_{k-1} constructed above satisfies Lambda Conditions:

Proof of Lambda Condition 1):

First we prove that λ_{k-1} for regular permutations is weakly skew-symmetric up to λ_k for the arguments in $\mathcal{H}$ and $\mathcal{X}$ respectively.

When the number of arguments in $\mathcal{H}$ is 0, the result is obvious.

Otherwise, for the arguments in $\mathcal{H}$,

$$
\begin{aligned}
& \lambda_{k-1}(\beta_1,\cdots\beta_r,\beta_{r+1},\cdots x_1,\cdots;\alpha_1,\cdots\alpha_{k-1})+\lambda_{k-1}(\beta_1,\cdots\beta_{r+1},\beta_r,\cdots x_1,\cdots;\cdots)\\
= & \frac{1}{k+l-1}((-1)^{r+1}+(-1)^r)(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_r},\beta_{r+1}\cdots;\beta_r,\cdots)\\
& +\frac{1}{k+l-1}((-1)^r+(-1)^{r+1})(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\beta_r,\widehat{\beta_{r+1}},\cdots;\beta_{r+1},\cdots)\\
& +\frac{1}{k+l-1}\sum_{j\neq r,r+1}(-1)^{j+1}\{(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta_r,\beta_{r+1}\cdots;\beta_j,\cdots)\\
& \quad +(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta_{r+1},\beta_r\cdots;\beta_j,\cdots)\}\\
= & \frac{1}{k+l-1}\sum_{j\neq r,r+1}(-1)^j\\
& \{(d_0\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta_r,\beta_{r+1}\cdots;\beta_j,\cdots)+(d_0\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta_{r+1},\beta_r\cdots;\beta_j,\cdots)\\
& +(d'\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta_r,\beta_{r+1}\cdots;\beta_j,\cdots)+(d'\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta_{r+1},\beta_r\cdots;\beta_j,\cdots)\}\\
& \text{(by equation 3.1 and 3.2)}\\
= & \frac{1}{k+l-1}\sum_{j\neq r,r+1}(-1)^j\{-d_0\lambda_{k+1}(\cdots\widehat{\beta_j},\cdots\widehat{\beta_r},\widehat{\beta_{r+1}},\cdots;\partial\langle\beta_r,\beta_{r+1}\rangle,\beta_j,\cdots)\\
& \quad -\lambda_k(\partial\langle\beta_r,\beta_{r+1}\rangle,\cdots\widehat{\beta_j},\cdots\widehat{\beta_r},\widehat{\beta_{r+1}},\cdots;\beta_j,\cdots)\\
& \quad -d'\lambda_{k+1}(\cdots\widehat{\beta_j},\cdots\widehat{\beta_r},\widehat{\beta_{r+1}},\cdots;\partial\langle\beta_r,\beta_{r+1}\rangle,\beta_j,\cdots)\}\\
= & \ 0
\end{aligned}
$$

For the arguments in $\mathcal{X}$,

$$
\begin{aligned}
& \lambda_{k-1}(\beta_1\cdots\beta_l,x_1\cdots x_a,x_{a+1}\cdots;\cdots)+\lambda_{k-1}(\beta_1\cdots\beta_l,x_1\cdots x_{a+1},x_a\cdots;\cdots)\\
& +\lambda_k(\beta_1\cdots\beta_l,x_1,\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\partial\langle x_a,x_{a+1}\rangle,\cdots)\\
= & \frac{1}{k+l-1}\sum_j(-1)^{j+1}\{(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots x_a,x_{a+1}\cdots;\beta_j,\cdots)\\
& \quad +(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots x_{a+1},x_a,\cdots;\beta_j,\cdots)\}\\
& +\frac{1}{k+l}\sum_j(-1)^{j+1}(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\partial\langle x_a,x_{a+1}\rangle,\cdots)\\
& \text{(by equation 3.1 and 3.2)}\\
= & \frac{1}{k+l-1}\sum_j(-1)^{j+1}\{-\omega_{k+1}(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\partial\langle x_a,x_{a+1}\rangle,\cdots)\\
& \quad +d_0\lambda_{k+1}(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\partial\langle x_a,x_{a+1}\rangle,\cdots)\\
& \quad +\lambda_k(\langle x_a,x_{a+1}\rangle,\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\cdots)\\
& \quad +d'\lambda_{k+1}(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\partial\langle x_a,x_{a+1}\rangle,\cdots)\}\\
& +\frac{1}{k+l}\sum_j(-1)^{j+1}(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\partial\langle x_a,x_{a+1}\rangle,\cdots)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(k+l-1)(k+l)}\sum_j (-1)^j(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\partial\langle x_a,x_{a+1}\rangle,\cdots)\\
&\quad +\frac{1}{k+l-1}\sum_j(-1)^{j+1}\lambda_k(\partial\langle x_a,x_{a+1}\rangle,\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_j,\cdots)\\
&= \frac{1}{(k+l-1)(k+l)}\sum_j(-1)^{j+1}\sum_{i<j}(-1)^i\\
&\qquad (\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\partial\langle x_a,x_{a+1}\rangle,\cdots\widehat{\beta_i},\cdots\widehat{\beta_j},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_i,\beta_j,\cdots)\\
&\quad +\frac{1}{(k+l-1)(k+l)}\sum_j(-1)^{j+1}\sum_{i>j}(-1)^{i+1}\\
&\qquad (\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\partial\langle x_a,x_{a+1}\rangle,\cdots\widehat{\beta_j},\cdots\widehat{\beta_i},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\beta_i,\beta_j,\cdots)\\
&= 0
\end{aligned}
$$

Next, for general permutation σ, we give the proof in the following three cases:
(1). $\lambda_{k-1}(\sigma_1,\beta_1,\beta_2,\sigma_2;\cdots)+\lambda_{k-1}(\sigma_1,\beta_2,\beta_1,\sigma_2;\cdots)=0,\ \forall\beta_1,\beta_2\in\mathcal{H}$
If every element in σ_1 is in $\mathcal{H}$, then

$$
\begin{aligned}
&\lambda_{k-1}(\sigma_1,\beta_1,\beta_2,\sigma_2;\cdots)+\lambda_{k-1}(\sigma_1,\beta_2,\beta_1,\sigma_2;\cdots)\\
=\ &(\pm1)\lambda_{k-1}(\sigma_1,\beta_1,\beta_2,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,\beta_1,\beta_2,\bullet;\bullet)\\
&+(\pm1)\lambda_{k-1}(\sigma_1,\beta_2,\beta_1,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,\beta_2,\beta_1,\bullet;\bullet)\\
=\ &(\pm1)\big(\lambda_{k-1}(\sigma_1,g_1,g_2,\bar{\sigma_2};\cdots)+\lambda_{k-1}(\sigma_1,g_2,g_1,\bar{\sigma_2};\cdots)\big)\\
&+\sum(\pm1)\big(\lambda_k(\sigma_1,g_1,g_2,\bullet;\bullet)+\lambda_k(\sigma_1,g_2,g_1,\bullet;\bullet)\big)\\
=\ &0
\end{aligned}
$$

Now suppose (1) holds for σ_1 containing at most m elements in $\mathcal{X}$, consider the case when σ_1 contains $m+1$ elements in $\mathcal{X}$, suppose x is the last one of them, move x to the last position in σ_1 and denote the elements in front of x as $\tilde{\sigma_1}$, $\tilde{\sigma_1}$ contains m elements in X.

$$
\begin{aligned}
&\lambda_{k-1}(\sigma_1,\beta_1,\beta_2,\sigma_2;\cdots)+\lambda_{k-1}(\sigma_1,\beta_2,\beta_1,\sigma_2;\cdots)\\
=\ &(\pm1)\lambda_{k-1}(\sigma_1,\beta_1,\beta_2,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,\beta_1,\beta_2,\bullet;\bullet)\\
&+(\pm1)\lambda_{k-1}(\sigma_1,\beta_2,\beta_1,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,\beta_2,\beta_1,\bullet;\bullet)\\
=\ &(\pm1)\big(\lambda_{k-1}(\sigma_1,\beta_1,\beta_2,\bar{\sigma_2};\cdots)+\lambda_{k-1}(\sigma_1,\beta_2,\beta_1,\bar{\sigma_2};\cdots)\big)\\
=\ &(\pm1)\big((\pm1)\lambda_{k-1}(\tilde{\sigma_1},x,\beta_1,\beta_2,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\bullet,\beta_1,\beta_2,\bar{\sigma_2};\bullet)\\
&\quad+(\pm1)\lambda_{k-1}(\tilde{\sigma_1},x,\beta_2,\beta_1,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\bullet,\beta_2,\beta_1,\bar{\sigma_2};\bullet)\big)\\
=\ &(\pm1)\big(\lambda_{k-1}(\tilde{\sigma_1},x,\beta_1,\beta_2,\bar{\sigma_2};\cdots)+\lambda_{k-1}(\tilde{\sigma_1},x,\beta_2,\beta_1,\bar{\sigma_2};\cdots)\big)\\
=\ &(\pm1)\big(-\lambda_k(\tilde{\sigma_1},\beta_2,\bar{\sigma_2};\partial\langle x,\beta_1\rangle,\cdots)+\lambda_k(\tilde{\sigma_1},\beta_1,\bar{\sigma_2};\partial\langle x,\beta_2\rangle,\cdots)+\lambda_{k-1}(\tilde{\sigma_1},\beta_1,\beta_2,x,\bar{\sigma_2};\cdots)\\
&-\lambda_k(\tilde{\sigma_1},\beta_1,\bar{\sigma_2};\partial\langle x,\beta_2\rangle,\cdots)+\lambda_k(\tilde{\sigma_1},\beta_2,\bar{\sigma_2};\partial\langle x,\beta_1\rangle,\cdots)+\lambda_{k-1}(\tilde{\sigma_1},\beta_2,\beta_1,x,\bar{\sigma_2};\cdots)\big)\\
=\ &(\pm1)\big(\lambda_{k-1}(\tilde{\sigma_1},\beta_1,\beta_2,x,\bar{\sigma_2};\cdots)+\lambda_{k-1}(\tilde{\sigma_1},\beta_2,\beta_1,x,\bar{\sigma_2};\cdots)\big)\\
=\ &0
\end{aligned}
$$

By mathematical induction, (1) is proved.

(2). $\lambda_{k-1}(\sigma_1, \beta, y, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, y, \beta, \sigma_2; \cdots) = -\lambda_k(\sigma_1, \sigma_2; \partial\langle\beta, y\rangle, \cdots),\ \forall \beta \in \mathcal{H},\ y \in \mathcal{X}$

$$
\begin{aligned}
& \lambda_{k-1}(\sigma_1, \beta, y, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, y, \beta, \sigma_2; \cdots) \\
= \;& (\pm 1)\lambda_{k-1}(\sigma_1, \beta, y, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, \beta, y, \bullet; \bullet) \\
& +(\pm 1)\lambda_{k-1}(\sigma_1, y, \beta, \bar{\sigma_2}; \cdots) + \sum (\pm 1)\lambda_k(\sigma_1, \beta, g, \bullet; \bullet) \\
= \;& (\pm 1)\big(\lambda_{k-1}(\sigma_1, \beta, y, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\sigma_1, y, \beta, \bar{\sigma_2}; \cdots)\big) \\
& + \sum (\pm 1)\big(\lambda_k(\sigma_1, \beta, y, \bullet; \bullet) + \lambda_k(\sigma_1, y, \beta, \bullet; \bullet)\big) \\
= \;& (\pm 1)\big(\lambda_{k-1}(\sigma_1, \beta, y, \bar{\sigma_2}; \cdots) + (-\lambda_k(\sigma_1, \bar{\sigma_2}; \partial\langle\beta, y\rangle, \cdots) - \lambda_{k-1}(\sigma_1, \beta, y, \bar{\sigma_2}; \cdots))\big) \\
& - \sum (\pm 1)\lambda_{k+1}(\sigma_1, \bullet; \partial\langle\beta, y\rangle, \bullet) \\
= \;& -\big((\pm 1)\lambda_k(\sigma_1, \bar{\sigma_2}; \partial\langle\beta, y\rangle, \cdots) + \sum (\pm 1)\lambda_{k+1}(\sigma_1, \bullet; \partial\langle\beta, y\rangle, \bullet)\big) \\
= \;& -\lambda_k(\sigma_1, \sigma_2; \partial\langle\beta, y\rangle, \cdots)
\end{aligned}
$$

(3). $\lambda_{k-1}(\sigma_1, y_1, y_2, \sigma_2; \cdots) + \lambda_{k-1}(\sigma_1, y_2, y_1, \sigma_2; \cdots) = -\lambda_k(\sigma_1, \sigma_2; \partial\langle y_1, y_2\rangle, \cdots)$, $\forall y_1, y_2 \in X$.
Suppose $\bar{\sigma_2} = (\beta_1, \cdots \beta_a, x_1, \cdots x_b)$, then

$$
\begin{aligned}
& \lambda_{k-1}(\sigma_1, y_1, y_2, \bar{\sigma_2}; \cdots) + \lambda_{k-1}(\sigma_1, y_2, y_1, \bar{\sigma_2}; \cdots) \\
= \;& \Big(\sum_{1\le i\le a} (-1)^i \lambda_k(\sigma_1, y_1, \beta_1, \cdots \widehat{\beta_i}, \cdots \beta_a, x_1, \cdots x_b; \partial\langle y_2, \beta_i\rangle, \cdots) \\
& \quad +(-1)^a \lambda_{k-1}(\sigma_1, y_1, \beta_1, \cdots \beta_a, y_2, x_1, \cdots x_b; \cdots)\Big) \\
& +\Big(\sum_{1\le j\le a} (-1)^j \lambda_k(\sigma_1, y_2, \beta_1, \cdots \widehat{\beta_j}, \cdots \beta_a, x_1, \cdots x_b; \partial\langle y_1, \beta_j\rangle, \cdots) \\
& \quad +(-1)^a \lambda_{k-1}(\sigma_1, y_2, \beta_1, \cdots \beta_a, y_1, x_1, \cdots x_b; \cdots)\Big) \\
= \;& \sum_i (-1)^i \Big(\sum_{1\le j<i} (-1)^j \lambda_{k+1}(\sigma_1, \beta_1, \cdots \widehat{\beta_j}, \cdots \widehat{\beta_i}, \cdots \beta_a, x_1, \cdots x_b; \partial\langle y_1, \beta_j\rangle, \partial\langle y_2, \beta_i\rangle, \cdots) \\
& \quad + \sum_{j>i} (-1)^{j+1} \lambda_{k+1}(\sigma_1, \beta_1, \cdots \widehat{\beta_i}, \cdots \widehat{\beta_j}, \cdots \beta_a, x_1 \cdots x_b; \partial\langle y_1, \beta_j\rangle, \partial\langle y_2, \beta_i\rangle, \cdots) \\
& \quad +(-1)^{a+1} \lambda_k(\sigma_1, \beta_1, \cdots \widehat{\beta_i}, \cdots \beta_a, y_1, x_1, \cdots x_b; \partial\langle y_2, \beta_i\rangle, \cdots)\Big) \\
& +(-1)^a \Big(\sum_{1\le j\le a} (-1)^j \lambda_k(\sigma_1, \beta_1, \cdots \widehat{\beta_j}, \cdots \beta_a, y_2, x_1, \cdots x_b; \partial\langle y_1, \beta_j\rangle, \cdots) \\
& \quad +(-1)^a \lambda_{k-1}(\sigma_1, \beta_1, \cdots \beta_a, y_1, y_2, x_1, \cdots x_b; \cdots)\Big) \\
& + \sum_j (-1)^j \Big(\sum_{1\le i<j} (-1)^i \lambda_{k+1}(\sigma_1, \beta_1, \cdots \widehat{\beta_i}, \cdots \widehat{\beta_j}, \cdots \beta_a, x_1, \cdots x_b; \partial\langle y_1, \beta_j\rangle, \partial\langle y_2, \beta_i\rangle, \cdots) \\
& \quad + \sum_{i>j} (-1)^{i+1} \lambda_{k+1}(\sigma_1, \beta_1, \cdots \widehat{\beta_j}, \cdots \widehat{\beta_i}, \cdots \beta_a, x_1, \cdots x_b; \partial\langle y_1, \beta_j\rangle, \partial\langle y_2, \beta_i\rangle, \cdots) \\
& \quad +(-1)^{a+1} \lambda_k(\sigma_1, \beta_1, \cdots \widehat{\beta_j}, \cdots \beta_a, y_2, x_1, \cdots x_b; \partial\langle y_1, \beta_j\rangle, \cdots)\Big) \\
& +(-1)^a \Big(\sum_{1\le i\le a} (-1)^i \lambda_k(\sigma_1, \beta_1, \cdots \widehat{\beta_i}, \cdots \beta_a, y_1, x_1, \cdots x_b; \partial\langle y_2, \beta_i\rangle, \cdots) \\
& \quad +(-1)^a \lambda_{k-1}(\sigma_1, \beta_1, \cdots \beta_a, y_2, y_1, x_1, \cdots x_b; \cdots)\Big)
\end{aligned}
$$

$$= \lambda_{k-1}(\sigma_1,\beta_1,\cdots\beta_a,y_1,y_2,x_1,\cdots x_b;\cdots)+\lambda_{k-1}(\sigma_1,\beta_1,\cdots\beta_a,y_2,y_1,x_1,\cdots x_b;\cdots)$$

So

$$\begin{aligned}
& \lambda_{k-1}(\sigma_1,y_1,y_2,\sigma_2;\cdots)+\lambda_{k-1}(\sigma_1,y_2,y_1,\sigma_2;\cdots) \\
= & (\pm1)\lambda_{k-1}(\sigma_1,y_1,y_2,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,y_1,y_2,\bullet;\bullet) \\
& +(\pm1)\lambda_{k-1}(\sigma_1,y_2,y_1,\bar{\sigma_2};\cdots)+\sum(\pm1)\lambda_k(\sigma_1,y_2,y_1,\bullet;\bullet) \\
= & (\pm1)\big(\lambda_{k-1}(\sigma_1,y_1,y_2,\bar{\sigma_2};\cdots)+\lambda_{k-1}(\sigma_1,y_2,y_1,\bar{\sigma_2};\cdots)\big) \\
& +\sum(\pm1)\big(\lambda_k(\sigma_1,y_1,y_2,\bullet;\bullet)+\lambda_k(\sigma_1,y_2,y_1,\bullet;\bullet)\big) \\
= & (\pm1)\big(\lambda_{k-1}(\sigma_1,\beta_1,\cdots\beta_a,y_1,y_2,x_1,\cdots x_b;\cdots)+\lambda_{k-1}(\sigma_1,\beta_1,\cdots\beta_a,y_2,y_1,x_1,\cdots x_b;\cdots)\big) \\
& -\sum(\pm1)\lambda_{k+1}(\sigma_1,\widehat{y_1},\widehat{y_2},\bullet;\partial\langle y_1,y_2\rangle,\bullet) \\
& \text{(denote by } \mu \text{the permutation } (\sigma_1,\beta_1,\cdots,\beta_a)) \\
= & (\pm1)\big((\pm1)\lambda_{k-1}(\bar{\mu},y_1,y_2,x_1,\cdots x_b;\cdots)+\sum(\pm1)\lambda_k(\bullet,y_1,y_2,x_1,\cdots x_b;\bullet) \\
& \quad +(\pm1)\lambda_{k-1}(\bar{\mu},y_2,y_1,x_1,\cdots,x_b;\cdots)+\sum(\pm1)\lambda_k(\bullet,y_2,y_1,x_1,\cdots,x_b;\bullet)\big) \\
& -\sum(\pm1)\lambda_{k+1}(\sigma_1,\widehat{y_1},\widehat{y_2},\bullet;\partial\langle y_1,y_2\rangle,\bullet) \\
= & -(\pm1)\big((\pm1)\lambda_k(\bar{\mu},\widehat{y_1},\widehat{y_2},x_1,\cdots;\partial\langle y_1,y_2\rangle,\cdots)+\sum(\pm1)\lambda_{k+1}(\bullet,\widehat{y_1},\widehat{y_2},x_1,\cdots;\partial\langle y_1,y_2\rangle,\bullet)\big) \\
& -\sum(\pm1)\lambda_{k+1}(\sigma_1,\widehat{y_1},\widehat{y_2},\bullet;\partial\langle y_1,y_2\rangle,\bullet) \\
= & -\big((\pm1)\lambda_k(\mu,\widehat{y_1},\widehat{y_2},x_1,\cdots x_b;\partial\langle y_1,y_2\rangle,\cdots)+\sum(\pm1)\lambda_{k+1}(\sigma_1,\widehat{y_1},\widehat{y_2},\bullet;\partial\langle y_1,y_2\rangle,\bullet)\big) \\
= & -\big((\pm1)\lambda_k(\sigma_1,\widehat{y_1},\widehat{y_2},\bar{\sigma_2};\partial\langle y_1,y_2\rangle,\cdots)+\sum(\pm1)\lambda_{k+1}(\sigma_1,\widehat{y_1},\widehat{y_2},\bullet;\partial\langle y_1,y_2\rangle,\bullet)\big) \\
= & -\lambda_k(\sigma_1,\widehat{y_1},\widehat{y_2},\sigma_2;\partial\langle y_1,y_2\rangle,\cdots)
\end{aligned}$$

Combining (1)(2)(3) above, Lambda condition 1) for λ_{k-1} is proven.

Proof of Lambda Condition 2):

Since the weak skew-symmetricity of λ_{k-1} is already proven, we only need to prove that λ_{k-1} is R-linear in the last argument of $\mathcal{E}$.

When the last argument is in $\mathcal{X}$:

$$\begin{aligned}
& \lambda_{k-1}(\sigma,fx;\cdots)-f\lambda_{k-1}(\sigma,x;\cdots) \\
= & \big((\pm1)\lambda_{k-1}(\bar{\sigma},fx;\cdots)+\sum(\pm1)\lambda_k(\bullet,fx;\bullet)\big)-f\big((\pm1)\lambda_{k-1}(\bar{\sigma},x;\cdots)+\sum(\pm1)\lambda_k(\bullet,x;\bullet)\big) \\
& \text{(suppose } \bar{\sigma}=(\beta_1,\cdots\beta_l,x_1,\cdots x_{n-2k-l})) \\
= & \frac{\pm1}{k+l-1}\sum_j(-1)^{j+1}\{(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots fx;\beta_j,\cdots) \\
& \quad -f(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots x;\beta_j,\cdots)\} \\
= & 0 \text{ (By Equation 3.3 and 3.4)}.
\end{aligned}$$

When the last argument is in $\mathcal{H}$:

$$\begin{aligned}
& \lambda_{k-1}(\sigma,f\beta;\cdots)-f\lambda_{k-1}(\sigma,\beta;\cdots) \\
= & \big((\pm1)\lambda_{k-1}(\bar{\sigma},f\beta;\cdots)+\sum(\pm1)\lambda_k(\bullet,f\beta;\bullet)\big)-f\big((\pm1)\lambda_{k-1}(\bar{\sigma},\beta;\cdots)+\sum(\pm1)\lambda_k(\bullet,\beta;\bullet)\big) \\
& \text{(suppose } \bar{\sigma}=(\beta_1,\cdots\beta_{l-1},x_1,\cdots x_{n+1-2k-l}))
\end{aligned}$$

$$
\begin{aligned}
= \;& (\pm1)\big((-1)^{n+1+l}\lambda_{k-1}(\beta_1,\cdots\beta_{l-1},f\beta,x_1,\cdots;\cdots)\\
&\quad+\sum_a(-1)^{a+n+l}\lambda_k(\beta_1,\cdots\beta_{l-1},x_1,\cdots\widehat{x_a},\cdots x_{n+1-2k-l};\partial\langle x_a,f\beta\rangle,\cdots)\big)\\
& -(\pm1)f\big((-1)^{n+1+l}\lambda_{k-1}(\beta_1,\cdots\beta_{l-1},\beta,x_1,\cdots;\cdots)\\
&\quad+\sum_a(-1)^{a+n+l}\lambda_k(\beta_1,\cdots\beta_{l-1},x_1,\cdots\widehat{x_a},\cdots x_{n+1-2k-l};\partial\langle x_a,\beta\rangle,\cdots)\big)\\
= \;& \frac{(\pm1)(-1)^{n+1+l}}{k+l-1}\sum_j(-1)^{j+1}\big((\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots f\beta,\cdots;\beta_j,\cdots)\\
&\quad -f(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots\beta,\cdots;\beta_j,\cdots)\big)\\
& +\frac{(\pm1)(-1)^{n}}{k+l-1}\big((\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta},\cdots;f\beta,\cdots)-f(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta},\cdots;\beta,\cdots)\big)\\
& +(\pm1)\sum_a(-1)^{a+n+l}\langle x_a,\beta\rangle\lambda_k(\cdots\widehat{\beta},\cdots\widehat{x_a},\cdots;\partial f,\cdots)\\
& \text{(By Equation 3.3, 3.4 and 3.5)}\\
= \;& \frac{\pm1}{k+l-1}\sum_{j,a}(-1)^{n+l+j+a}\langle\beta,x_a\rangle\big((\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_j},\cdots\widehat{\beta},\cdots\widehat{x_a},\cdots;\partial f,\beta_j,\cdots)\\
&\quad -\lambda_k(\partial f,\cdots\widehat{\beta_j},\cdots\widehat{\beta},\cdots\widehat{x_a},\cdots;\beta_j,\cdots)\big)\\
& +\frac{\pm1}{k+l-1}\sum_a(-1)^{n+1+l+a}\langle\beta,x_a\rangle\lambda_k(\cdots\widehat{\beta},\cdots\widehat{x_a},\cdots;\partial f,\cdots)\\
& +\frac{\pm1}{k+l-1}\sum_{j,a}(-1)^{a+n+l+j+1}\langle\beta,x_a\rangle(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_j},\cdots\widehat{\beta},\cdots\widehat{x_a},\cdots;\partial f,\beta_j,\cdots)\\
= \;& \frac{\pm1}{k+l-1}\sum_a\sum_{0\le j\le l-1}(-1)^{n+l+j+a+1}\langle\beta,x_a\rangle\lambda_k(\beta_0(\triangleq\partial f),\cdots\widehat{\beta_j},\cdots\beta_{l-1},\cdots\widehat{x_a},\cdots;\beta_j,\cdots)\\
= \;& \frac{\pm1}{(k+l-1)^2}\sum_a(-1)^{n+l+a+1}\langle\beta,x_a\rangle\\
& \big(\sum_{0\le i<j\le l-1}(-1)^{i+j}(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\beta_0,\cdots\widehat{\beta_i},\cdots\widehat{\beta_j},\cdots\widehat{x_a},\cdots;\beta_i,\beta_j,\cdots)\\
& +\sum_{0\le j<i\le l-1}(-1)^{i+j+1}(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\beta_0,\cdots\widehat{\beta_j},\cdots\widehat{\beta_i},\cdots\widehat{x_a},\cdots;\beta_i,\beta_j,\cdots)\big)\\
= \;& 0
\end{aligned}
$$

Proof of Lambda Condition 3):

$$
\begin{aligned}
& \lambda_{k-1}(\beta_1,\cdots\beta_l,x_1,\cdots x_{n+1-2k-l};f\alpha_1,\cdots\alpha_{k-1})-f\lambda_{k-1}(\cdots;\alpha_1,\cdots\alpha_{k-1})\\
= \;& \frac{1}{k+l-1}\sum_j(-1)^{j+1}\big((\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots;\beta_j,f\alpha_1,\cdots)\\
&\quad -f(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{\beta_j},\cdots;\beta_j,\alpha_1,\cdots)\big)\\
& \text{(By Equation 3.5)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{k+l-1}\sum_j(-1)^j\sum_a(-1)^{l+a+1}\langle x_a,\alpha_1\rangle\lambda_k(\cdots\widehat{\beta_j},\cdots\widehat{x_a},\cdots;\beta_j,\partial f,\widehat{\alpha_1},\cdots)\\
&= \frac{1}{(k+l-1)^2}\sum_a(-1)^{l+a+1}\langle x_a,\alpha_1\rangle\\
&\quad \big(\sum_{i<j}(-1)^{j+i+1}(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_i},\cdots\widehat{\beta_j},\cdots\widehat{x_a},\cdots;\beta_i,\beta_j,\partial f,\widehat{\alpha_1},\cdots)\\
&\quad +\sum_{i>j}(-1)^{j+i}(\omega_{k+1}-d_0\lambda_{k+1}-d'\lambda_{k+1})(\cdots\widehat{\beta_j},\cdots\widehat{\beta_i},\cdots\widehat{x_a},\cdots;\beta_i,\beta_j,\partial f,\widehat{\alpha_1},\cdots)\big)\\
&= 0
\end{aligned}
$$

Proof of Lambda Condition 4):
For regular permutations:

$$
\begin{aligned}
&(\delta\lambda)_k(\beta_1,\cdots\beta_l,x_1,\cdots x_{n-2k-l};\alpha_1,\cdots\alpha_k)\\
= &\sum_i\lambda_{k-1}(\alpha_i,\beta_1,\cdots\beta_l,x_1,\cdots x_{n-2k-l};\cdots\widehat{\alpha_i},\cdots)\\
= &\frac{1}{k+l}\sum_i(\omega_k-d_0\lambda_k-d'\lambda_k)(\beta_1,\cdots\beta_l,\cdots;\alpha_1,\cdots\alpha_k)\\
&+\frac{1}{k+l}\sum_{i,j}(-1)^j(\omega_k-d_0\lambda_k-d'\lambda_k)(\alpha_i,\beta_1\cdots\widehat{\beta_j},\cdots\beta_l\cdots;\beta_j,\cdots\widehat{\alpha_i},\cdots)\\
&\text{(By Lambda Condition 5) for } \lambda_k)\\
= &\frac{k}{k+l}(\omega_k-d_0\lambda_k-d'\lambda_k)(\beta_1,\cdots\beta_l,\cdots;\alpha_1,\cdots\alpha_k)\\
&+\frac{1}{k+l}\sum_j(-1)^j(-1)(\omega_k-d_0\lambda_k-d'\lambda_k)(\beta_j,\beta_1\cdots\widehat{\beta_j},\cdots\beta_l\cdots;\alpha_1,\cdots\alpha_k)\\
= &\frac{k+l}{k+l}(\omega_k-d_0\lambda_k-d'\lambda_k)(\beta_1,\cdots\beta_l,\cdots;\alpha_1,\cdots\alpha_k)
\end{aligned}
$$

So $\omega_k=(d\lambda)_k$ for regular permutation $(\beta_1,\cdots\beta_l,x_1,\cdots x_{n-2k-l})$.
Since ω_k and $(d\lambda)_k$ are both weakly skew-symmetric up to $\omega_{k+1}=(d\lambda)_{k+1}$, so $\omega_k=(d\lambda)_k$ holds for general permutations.
Proof of Lambda Condition 5):
We only need to prove the following:

$$
\begin{aligned}
&\sum_i(\omega_{k-1}-d_0\lambda_{k-1}-d'\lambda_{k-1})(\alpha_i,e_1,\cdots,e_{n+1-2k};\alpha_1,\cdots\widehat{\alpha_i},\cdots\alpha_k)\\
= &(d\omega)_k(e_1,\cdots,e_{n+1-2k};\alpha_1,\cdots\alpha_k)
\end{aligned}
$$

or equivalently,

$$
\begin{aligned}
&(d_0\omega_k+d'\omega_k)(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)+\sum_i(d_0\lambda_{k-1}+d'\lambda_{k-1})(\alpha_i,e_1,\cdots;\cdots\widehat{\alpha_i},\cdots)\\
= &(d_0\omega_k+d'\omega_k)(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)\\
&+\sum_{i,a}(-1)^a\nabla_{e_a}\lambda_{k-1}(\alpha_i,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_i},\cdots)+\sum_{i,a}(-1)\lambda_{k-1}(\cdots\widehat{e_a},\alpha_i\circ e_a,\cdots;\cdots\widehat{\alpha_i},\cdots)
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{i,a<b}(-1)^{a+1}\lambda_{k-1}(\alpha_i,\cdots\widehat{e_a},\cdots e_a\circ e_b,\cdots;\cdots\widehat{\alpha_i},\cdots)+\sum_{j\neq i}\lambda_{k-1}(\cdots;\cdots\widehat{\alpha_i},\cdots\alpha_j\circ\alpha_i,\cdots)\\
& +\sum_{j\neq i}\lambda_{k-1}(\cdots;\cdots\widehat{\alpha_i},\cdots\alpha_j\circ\alpha_i,\cdots)+\sum_{a,j\neq i}(-1)^a\lambda_{k-1}(\alpha_i,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_i},\cdots\alpha_j\circ e_a,\cdots)\\
= \; & (d_0\omega_k+d'\omega_k)(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)+\sum_a(-1)^a\nabla_{e_a}(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{e_a},\cdots;\cdots)\\
& +\sum_{a<b}(-1)^{a+1}(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{e_a},\cdots e_a\circ e_b,\cdots)-\sum_{a,i}\lambda_{k-1}(\cdots\widehat{e_a},\alpha_i\circ e_a,\cdots;\cdots\widehat{\alpha_i},\cdots)\\
& +\sum_{i<j}\lambda_{k-1}(e_1,\cdots e_{n+1-2k};\alpha_i\circ\alpha_j+\alpha_j\circ\alpha_i,\cdots\widehat{\alpha_i},\cdots\widehat{\alpha_j},\cdots)\\
& +\Big(\sum_{a,j}(-1)^a(\omega_k-d_0\lambda_k-d'\lambda_k)(\cdots\widehat{e_a},\cdots;\cdots\alpha_j\circ e_a,\cdots)\\
& \quad -\sum_{a,j}(-1)^a\lambda_{k-1}(\alpha_j\circ e_a,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_j},\cdots)\Big)\\
= \; & ((d_0^2+d_0\circ d')\lambda)_k(e_1,\cdots;\alpha_1,\cdots)-\sum_{a,i}\lambda_{k-1}(\cdots\widehat{e_a},\alpha_i\circ e_a,\cdots;\cdots\widehat{\alpha_i},\cdots)\\
& +((d'\circ d_0+d'^2)\lambda)_k(e_1,\cdots;\alpha_1,\cdots)+\sum_{a,j}(-1)^{a+1}\lambda_{k-1}(\alpha_j\circ e_a,\cdots\widehat{e_a},\cdots;\cdots\widehat{\alpha_j},\cdots)\\
= \; & ((d_0+d'+\delta)^2\lambda)_k(e_1,\cdots e_{n+1-2k};\alpha_1,\cdots\alpha_k)\\
= \; & 0
\end{aligned}
$$

Thus λ_{k-1} satisfies all Lambda Conditions.

Step 4:

By mathematical induction, finally we obtain $(\lambda_0,\cdots,\lambda_{m-1})$ with each λ_p satisfying Lambda Conditions.

Let $\lambda\triangleq(\lambda_0,\cdots,\lambda_{m-1})$, Lambda Conditions 1), 2), 3) imply that λ is a cochain in $C^{n-1}(\mathcal{E},\mathcal{H},\mathcal{V})$.

Let $\eta\triangleq\omega-d\lambda\in C^n(\mathcal{E},\mathcal{H},\mathcal{V})$.

By Lambda Condition 4), $\eta=(\omega_0-(d\lambda)_0,0,\cdots,0)=(\omega_0-d_0\lambda_0-d'\lambda_0,0,\cdots,0)$.

By Lambda Condition 5), $(\omega_0-d_0\lambda_0-d'\lambda_0)(\alpha,e_1,\cdots,e_{n-1})=0,\ \forall\alpha\in\mathcal{H},\ e_i\in\mathcal{E}$.

So $\iota_\alpha\eta_0=\iota_\alpha(\omega_0-d_0\lambda_0-d'\lambda_0)=0,\ \forall\alpha\in\mathcal{H}$, which implies that $\eta\in C^n_{nv}(\mathcal{E},\mathcal{H},\mathcal{V})$.

Thus $\omega=\eta+d\lambda,\ \eta\in C^n_{nv}(\mathcal{E},\mathcal{H},\mathcal{V}),\ \lambda\in C^{n-1}(\mathcal{E},\mathcal{H},\mathcal{V})$, the proof is finished. ■

Now we turn to the proof of Theorem 4.1:

Proof. By Lemma 4.2, we only need to prove that the inclusion map

$$\psi: C_{nv}(\mathcal{E},\mathcal{H},\mathcal{V})\to C(\mathcal{E},\mathcal{H},\mathcal{V})$$

induces an isomorphism

$$\psi_*: H^\bullet_{nv}(\mathcal{E},\mathcal{H},\mathcal{V})\to H^\bullet(\mathcal{E},\mathcal{H},\mathcal{V}).$$

1). ψ_* is surjective.

Given any $[\omega]\in H^n(\mathcal{E},\mathcal{H},\mathcal{V})$, since ω is closed, by Lemma 4.3, there exists $\eta\in C^n_{nv}(\mathcal{E},\mathcal{H},\mathcal{V})$ and $\lambda\in C^{n-1}(\mathcal{E},\mathcal{H},\mathcal{V})$ such that $\omega=\eta+d\lambda$. We see that $\eta=\omega-d\lambda$ is closed, so $\psi_*([\eta])=[\eta]=[\omega]$.

2). ψ_* is injective.

Suppose $\zeta\in C^n_{nv}(\mathcal{E},\mathcal{H},\mathcal{V})$ is exact in $C^n(\mathcal{E},\mathcal{H},\mathcal{V})$, i.e. $\exists\omega\in C^{n-1}(\mathcal{E},\mathcal{H},\mathcal{V}),\ d\omega=\zeta$, we need to prove that ζ is exact in $C^n_{nv}(\mathcal{E},\mathcal{H},\mathcal{V})$.

Since $(d\omega)_k = \zeta_k = 0$, $\forall k > 0$, by Lemma 4.3, there exists $\eta \in C^{n-1}_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V})$ and $\lambda \in C^{n-2}(\mathcal{E}, \mathcal{H}, \mathcal{V})$ such that $\omega = \eta + d\lambda$. So

$$\zeta = d\omega = d\eta$$

is exact in $C^n_{nv}(\mathcal{E}, \mathcal{H}, \mathcal{V})$.

The proof is finished. ■

Remark 4.4. When ρ^* is injective (in this case we call $\mathcal{E}$ a transitive Courant-Dorfman algebra), and $\mathcal{H} = \rho^*(\Omega^1)$, $\mathcal{V} = R$, Definition 3.5 recovers the ordinary standard cohomology (Definition 2.7). Moreover, if $\mathcal{E} = \Gamma(E)$ is the space of sections of a transitive Courant algebroid E, Theorem 4.1 recovers the isomorphism between the standard cohomology and naive cohomology of E, as conjectured by Stienon-Xu [15] and first proved by Ginot-Grutzmann [5]. So Theorem 4.1 is a generalization of their result.

Example 4.5. Suppose $\mathcal{G}$ is a bundle of quadratic Lie algebras on M. Given a standard Courant algebroid structure (see Chen, Stienon and Xu [3]) on

$$E = TM \oplus \mathcal{G} \oplus T^*M,$$

let $\mathcal{E} = \Gamma(E)$. As mentioned in the remark above, if we take $\mathcal{H} = \Gamma(T^*M) = \Omega^1(M)$ and $\mathcal{V} = C^\infty(M)$, the $\Omega^1(M)$-standard cohomology $H^\bullet(\mathcal{E}, \Omega^1(M), C^\infty(M))$ coincides with the standard cohomology of E, and is isomorphic to the cohomology of Lie algebroid $TM \oplus \mathcal{G}$ with coefficients in $C^\infty(M)$.

Now suppose $\mathcal{K}$ is an isotropic ideal in $\mathcal{G}$, then $\mathcal{H} = \Gamma(\mathcal{K} \oplus T^*M) \supseteq \Gamma(T^*M)$ is an isotropic ideal in $\mathcal{E}$. Given a $\Gamma(\mathcal{K} \oplus T^*M)$-representation $\mathcal{V}$ (e.g. $C^\infty(M)$), we have the $\Gamma(\mathcal{K} \oplus T^*M)$-standard cohomology $H^\bullet(\mathcal{E}, \Gamma(\mathcal{K} \oplus T^*M), C^\infty(M))$. By Theorem 4.1, it is isomorphic to the cohomology of Lie algebroid $TM \oplus (\mathcal{G}/\mathcal{K})$ with coefficients in $\mathcal{V}$.

5. Crossed products of Leibniz algebras

In this section, we associate a Courant-Dorfman algebra to any Leibniz algebra and consider the relation between H-standard complexes of them. At last we prove an isomorphism theorem for Leibniz algebras.

Given a Leibniz algebra L with left center Z, let $S^\bullet(Z)$ be the algebra of symmetric tensors of Z. We construct a Courant-Dorfman algebra structure on the tensor product

$$\mathcal{L} \triangleq S^\bullet(Z) \otimes L$$

as follows:

let R be $S^\bullet(Z)$;

let the $S^\bullet(Z)$-module structure of $\mathcal{L}$ be given by multiplication of $S^\bullet(Z)$, i.e.

$$f_1 \cdot (f_2 \otimes e) \triangleq (f_1 f_2) \otimes e, \qquad \forall f_1, f_2 \in S^\bullet(Z), e \in L;$$

(For simplicity, we will write $f \otimes e$ as fe from now on.)

let the symmetric bilinear form $\langle \cdot, \cdot \rangle$ of $\mathcal{L}$ be the $S^\bullet(Z)$-bilinear extension of the symmetric product $(\cdot, \cdot)$ of L, i.e.

$$\langle f_1 e_1, f_2 e_2 \rangle \triangleq f_1 f_2 (e_1, e_2), \qquad \forall f_1, f_2 \in S^\bullet(Z), e_1, e_2 \in L$$

(since $\langle e_1, e_2 \rangle = (e_1, e_2)$, in the following we always use the notation $\langle \cdot, \cdot \rangle$);

let the derivation $\partial : S^\bullet(Z) \to \mathcal{L}$ be the extension of the inclusion map $Z \hookrightarrow L$ by Leibniz rule, i.e.

$$\partial(z_1 \cdots z_k) \triangleq \sum_{1 \le i \le k} (z_1 \cdots \widehat{z_i} \cdots z_k) \partial z_i, \qquad \forall z_i \in Z;$$

let the Dorfman bracket on $\mathcal{L}$, still denoted by $\circ$, be the extension of the Leibniz bracket of L:

$$f_1e_1 \circ f_2e_2 \triangleq f_1f_2(e_1 \circ e_2) + \langle e_1, e_2\rangle f_2\partial f_1 + \langle e_1, \partial f_2\rangle f_1e_2 - \langle e_2, \partial f_1\rangle f_2e_1 \tag{5.1}$$

$\forall f_1, f_2 \in S^\bullet(Z), e_1, e_2 \in L.$

Proposition 5.1. *With the above notations, $(\mathcal{L}, S^\bullet(Z), \langle\cdot,\cdot\rangle, \partial, \circ)$ becomes a Courant-Dorfman algebra (called the crossed product of L).*

Proof. We need to check all the six conditions in Definition 2.6.

1). $f_1e_1 \circ f(f_2e_2) = f(f_1e_1 \circ f_2e_2) + \langle f_1e_1, \partial f\rangle f_2e_2$

$$\begin{aligned}
& f_1e_1 \circ f(f_2e_2)\\
= \;& ff_1f_2(e_1 \circ e_2) + \langle e_1, e_2\rangle ff_2\partial f_1 + \langle e_1, \partial(ff_2)\rangle f_1e_2 - \langle e_2, \partial f_1\rangle ff_2e_1\\
= \;& ff_1f_2(e_1 \circ e_2) + \langle e_1, e_2\rangle ff_2\partial f_1 + f\langle e_1, \partial f_2\rangle f_1e_2 + f_2\langle e_1, \partial f\rangle f_1e_2 - \langle e_2, \partial f_1\rangle ff_2e_1\\
= \;& f(f_1e_1 \circ f_2e_2) + \langle f_1e_1, \partial f\rangle f_2e_2.
\end{aligned}$$

2). $\langle f_1e_1, \partial\langle f_2e_2, f_3e_3\rangle\rangle = \langle f_1e_1 \circ f_2e_2, f_3e_3\rangle + \langle f_2e_2, f_1e_1 \circ f_3e_3\rangle$

$$\begin{aligned}
& \langle f_1e_1, \partial\langle f_2e_2, f_3e_3\rangle\rangle\\
= \;& f_1\langle e_1, \partial(f_2f_3\langle e_2, e_3\rangle)\rangle\\
= \;& f_1f_2f_3\langle e_1, \partial\langle e_2, e_3\rangle\rangle + f_1f_2\langle e_2, e_3\rangle\langle e_1, \partial f_3\rangle + f_1f_3\langle e_2, e_3\rangle\langle e_1, \partial f_2\rangle\\
= \;& f_1f_2f_3\big(\langle e_1 \circ e_2, e_3\rangle + \langle e_2, e_1 \circ e_3\rangle\big) + f_1f_2\langle e_2, e_3\rangle\langle e_1, \partial f_3\rangle + f_1f_3\langle e_2, e_3\rangle\langle e_1, \partial f_2\rangle\\
= \;& f_1f_2f_3\langle e_1 \circ e_2, e_3\rangle + f_2f_3\langle e_1, e_2\rangle\langle e_3, \partial f_1\rangle + f_1f_3\langle e_2, e_3\rangle\langle e_1, \partial f_2\rangle - f_2f_3\langle e_1, e_3\rangle\langle e_2, \partial f_1\rangle\\
& +f_1f_2f_3\langle e_2, e_1 \circ e_3\rangle + f_2f_3\langle e_1, e_3\rangle\langle e_2, \partial f_1\rangle + f_1f_2\langle e_2, e_3\rangle\langle e_1, \partial f_3\rangle - f_2f_3\langle e_1, e_2\rangle\langle e_3, \partial f_1\rangle\\
= \;& \langle f_1f_2(e_1 \circ e_2) + \langle e_1, e_2\rangle f_2\partial f_1 + \langle e_1, \partial f_2\rangle f_1e_2 - \langle e_2, \partial f_1\rangle f_2e_1, f_3e_3\rangle\\
& +\langle f_2e_2, f_1f_3(e_1 \circ e_3) + \langle e_1, e_3\rangle f_3\partial f_1 + \langle e_1, \partial f_3\rangle f_1e_3 - \langle e_3, \partial f_1\rangle f_3e_1\rangle\\
= \;& \langle f_1e_1 \circ f_2e_2, f_3e_3\rangle + \langle f_2e_2, f_1e_1 \circ f_3e_3\rangle.
\end{aligned}$$

3). $f_1e_1 \circ f_2e_2 + f_2e_2 \circ f_1e_1 = \partial\langle f_1e_1, f_2e_2\rangle$

$$\begin{aligned}
& f_1e_1 \circ f_2e_2 + f_2e_2 \circ f_1e_1\\
= \;& f_1f_2(e_1 \circ e_2) + \langle e_1, e_2\rangle f_2\partial f_1 + \langle e_1, \partial f_2\rangle f_1e_2 - \langle e_2, \partial f_1\rangle f_2e_1\\
& +f_1f_2(e_2 \circ e_1) + \langle e_1, e_2\rangle f_1\partial f_2 + \langle e_2, \partial f_1\rangle f_2e_1 - \langle e_1, \partial f_2\rangle f_1e_2\\
= \;& f_1f_2\partial\langle e_1, e_2\rangle + \langle e_1, e_2\rangle f_2\partial f_1 + \langle e_1, e_2\rangle f_1\partial f_2\\
= \;& \partial\langle f_1e_1, f_2e_2\rangle.
\end{aligned}$$

Combining 1) and 3), we get the following:

$$\begin{aligned}
& f(f_1e_1) \circ f_2e_2\\
= \;& (f(f_1e_1) \circ f_2e_2 + f_2e_2 \circ f(f_1e_1)) - f_2e_2 \circ f(f_1e_1)\\
= \;& \partial\langle f(f_1e_1), f_2e_2\rangle - (f(f_2e_2 \circ f_1e_1) + \langle f_2e_2, \partial f\rangle f_1e_1\\
= \;& \langle f_1e_1, f_2e_2\rangle\partial f + f\partial\langle f_1e_1, f_2e_2\rangle - f(f_2e_2 \circ f_1e_1) - \langle f_2e_2, \partial f\rangle f_1e_1\\
= \;& f(f_1e_1 \circ f_2e_2) + \langle f_1e_1, f_2e_2\rangle\partial f - \langle f_2e_2, \partial f\rangle f_1e_1
\end{aligned}$$

4). $\langle \partial f, \partial f'\rangle = 0$.

We only need to consider the case of monomials, suppose

$$f = z_1z_2\cdots z_k,\;\; f' = z_1'z_2'\cdots z_l',\;\; z_i, z_j' \in Z,$$

$$
\begin{aligned}
& \langle \partial f, \partial f' \rangle \\
= \ & \langle \sum_i (z_1 \cdots \widehat{z_i} \cdots z_k) \partial z_i, \sum_j (z'_1 \cdots \widehat{z'_j} \cdots z'_l) \partial z'_j \rangle \\
= \ & \sum_{i,j} (z_1 \cdots \widehat{z_i} \cdots z_k z'_1 \cdots \widehat{z'_j} \cdots z'_l)(\partial z_i \circ \partial z'_j + \partial z'_j \circ \partial z_i) \\
= \ & 0
\end{aligned}
$$

5). $\partial f \circ (f'e) = 0$
First we prove that $\partial f \circ e = 0, \ \forall f \in S^{\bullet}(Z), \ e \in L$.
We only need to consider the case of monomials: suppose $f = z_1 z_2 \cdots z_k, \ z_i \in Z$.
When $k = 1$, i.e. $f = z_1 \in Z$, the equation is trivial.
Now suppose the equation holds for any $k \leq m$, let's consider the case of $k = m + 1$.

$$
\begin{aligned}
& \partial(z_1 z_2 \cdots z_{m+1}) \circ e \\
= \ & \big((z_1 \cdots z_m)\partial z_{m+1} + z_{m+1}\partial(z_1 \cdots z_m)\big) \circ e \\
= \ & (z_1 \cdots z_m)(\partial z_{m+1} \circ e) + \langle \partial z_{m+1}, e \rangle \partial(z_1 \cdots z_m) - \langle e, \partial(z_1 \cdots z_m) \rangle \partial z_{m+1} \\
& + z_{m+1}(\partial(z_1 \cdots z_m) \circ e) + \langle \partial(z_1 \cdots z_m), e \rangle \partial z_{m+1} - \langle e, \partial z_{m+1} \rangle \partial(z_1 \cdots z_m) \\
= \ & 0
\end{aligned}
$$

Thus by induction, $\partial f \circ e = 0$ holds for any $f \in S^{\bullet}(Z)$.
Then combining 1) and 4),

$$
\partial f \circ (f'e) \ = \ f'(\partial f \circ e) + \langle \partial f, \partial f' \rangle e = 0
$$

6). $f_1 e_1 \circ (f_2 e_2 \circ f_3 e_3) = (f_1 e_1 \circ f_2 e_2) \circ f_3 e_3 + f_2 e_2 \circ (f_1 e_1 \circ f_3 e_3)$
First we prove the equation for the case when $f_2 = f_3 = 1$:

$$
\begin{aligned}
& (f_1 e_1 \circ e_2) \circ e_3 + e_2 \circ (f_1 e_1 \circ e_3) \\
= \ & \big(f_1(e_1 \circ e_2) + \langle e_1, e_2 \rangle \partial f_1 - \langle e_2, \partial f_1 \rangle e_1\big) \circ e_3 + e_2 \circ \big(f_1(e_1 \circ e_3) + \langle e_1, e_3 \rangle \partial f_1 - \langle e_3, \partial f_1 \rangle e_1\big) \\
= \ & \big(f_1((e_1 \circ e_2) \circ e_3) + \langle e_1 \circ e_2, e_3 \rangle \partial f_1 - \langle e_3, \partial f_1 \rangle (e_1 \circ e_2)\big) \\
& + \big(\langle e_1, e_2 \rangle (\partial f_1 \circ e_3) + \langle \partial f_1, e_3 \rangle \partial \langle e_1, e_2 \rangle - \langle e_3, \partial \langle e_1, e_2 \rangle \rangle \partial f_1\big) \\
& - \big(\langle e_2, \partial f_1 \rangle (e_1 \circ e_3) + \langle e_1, e_3 \rangle \partial \langle e_2, \partial f_1 \rangle - \langle e_3, \partial \langle e_2, \partial f_1 \rangle \rangle e_1\big) \\
& + \big(f_1(e_2 \circ (e_1 \circ e_3) + \langle e_2, \partial f_1 \rangle (e_1 \circ e_3)\big) + \big(\langle e_1, e_3 \rangle (e_2 \circ \partial f_1) + \langle e_2, \partial \langle e_1, e_3 \rangle \rangle \partial f_1\big) \\
& - \big(\langle e_3, \partial f_1 \rangle (e_2 \circ e_1) + \langle e_2, \partial \langle e_3, \partial f_1 \rangle \rangle e_1\big) \\
= \ & f_1((e_1 \circ e_2) \circ e_3) + f_1(e_2 \circ (e_1 \circ e_3) + \big(\langle e_1 \circ e_2, e_3 \rangle - \langle e_3, \partial \langle e_1, e_2 \rangle \rangle + \langle e_2, \partial \langle e_1, e_3 \rangle \rangle\big) \partial f_1 \\
& + \big(\langle e_3, \partial \langle e_2, \partial f_1 \rangle \rangle - \langle e_2, \partial \langle e_3, \partial f_1 \rangle \rangle\big) e_1 + \langle e_1, e_3 \rangle \big(e_2 \circ \partial f_1 - \partial \langle e_2, \partial f_1 \rangle\big) \\
& + \langle e_3, \partial f_1 \rangle \big(\partial \langle e_1, e_2 \rangle - e_1 \circ e_2 - e_2 \circ e_1\big) + \langle e_1, e_2 \rangle (\partial f_1 \circ e_3) \\
& + \langle e_2, \partial f_1 \rangle (e_1 \circ e_3) - \langle e_2, \partial f_1 \rangle (e_1 \circ e_3) \\
= \ & f_1(e_1 \circ (e_2 \circ e_3)) + \big(\langle e_2, \partial \langle e_1, e_3 \rangle \rangle - \langle e_2 \circ e_1, e_3 \rangle\big) \partial f_1 - \langle e_2 \circ e_3, \partial f_1 \rangle e_1 \\
= \ & f_1 e_1 \circ (e_2 \circ e_3)
\end{aligned}
$$

Then we prove the equation for the case when $f_3 = 1$:

$$
\begin{aligned}
& (f_1e_1 \circ f_2e_2) \circ e_3 + f_2e_2 \circ (f_1e_1 \circ e_3) \quad (\text{let } x_1 \triangleq f_1e_1) \\
= \; & \big(f_2(x_1 \circ e_2) + \langle x_1, \partial f_2\rangle e_2\big) \circ e_3 + f_2(e_2 \circ (x_1 \circ e_3)) + \langle e_2, x_1 \circ e_3\rangle \partial f_2 - \langle x_1 \circ e_3, \partial f_2\rangle e_2 \\
= \; & \big(f_2((x_1 \circ e_2) \circ e_3) + \langle x_1 \circ e_2, e_3\rangle \partial f_2 - \langle e_3, \partial f_2\rangle (x_1 \circ e_2)\big) \\
& + \big(\langle x_1, \partial f_2\rangle (e_2 \circ e_3) + \langle e_2, e_3\rangle \partial \langle x_1, \partial f_2\rangle - \langle e_3, \partial \langle x_1, \partial f_2\rangle\rangle e_2\big) \\
& + f_2(e_2 \circ (x_1 \circ e_3)) + \langle e_2, x_1 \circ e_3\rangle \partial f_2 - \langle x_1 \circ e_3, \partial f_2\rangle e_2 \\
= \; & f_2((x_1 \circ e_2) \circ e_3) + f_2(e_2 \circ (x_1 \circ e_3)) + \langle x_1, \partial f_2\rangle (e_2 \circ e_3) + \langle e_2, e_3\rangle \partial \langle x_1, \partial f_2\rangle \\
& + \big(\langle x_1 \circ e_2, e_3\rangle + \langle e_2, x_1 \circ e_3\rangle\big) \partial f_2 - \big(\langle e_3, \partial f_2\rangle (x_1 \circ e_2) + (\langle e_3, \partial\langle x_1, \partial f_2\rangle\rangle + \langle x_1 \circ e_3, \partial f_2\rangle) e_2\big) \\
= \; & x_1 \circ \big(f_2(e_2 \circ e_3) + \langle e_2, e_3\rangle \partial f_2 - \langle e_3, \partial f_2\rangle e_2\big) \\
= \; & f_1e_1 \circ (f_2e_2 \circ e_3).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& (f_1e_1 \circ f_2e_2) \circ f_3e_3 + f_2e_2 \circ (f_1e_1 \circ f_3e_3) \quad (\text{let } x_1 \triangleq f_1e_1, \ x_2 \triangleq f_2e_2) \\
= \; & f((x_1 \circ x_2) \circ e_3) + \langle x_1 \circ x_2, \partial f_3\rangle e_3 \\
& + f(x_2 \circ (x_1 \circ e_3)) + \langle x_2, \partial f_3\rangle (x_1 \circ e_3) + \langle x_1, \partial f_3\rangle (x_2 \circ e_3) + \langle x_2, \partial\langle x_1, \partial f_3\rangle\rangle e_3 \\
= \; & x_1 \circ \big(f_3(x_2 \circ e_3) + \langle x_2, \partial f_3\rangle e_3\big) \\
= \; & f_1e_1 \circ (f_2e_2 \circ f_3e_3)
\end{aligned}
$$

Thus the proposition is proved. ■

By Equation 2.3, the anchor map

$$\rho : \mathcal{L} \to Der(S^\bullet(Z), S^\bullet(Z))$$

can be defined as follows:

$$\rho(fe)(z_1 \cdots z_k) \triangleq f \sum_i (z_1 \cdots \widehat{z_i} \cdots z_k)(\rho(e) z_i), \quad \forall f \in S^\bullet(Z), e \in L, z_i \in Z.$$

Proposition 5.2. *Suppose $H \supseteq Z$ is an isotropic ideal in L, and (V, τ) is an H-representation of L, let $\mathcal{V} \triangleq S^\bullet(Z) \otimes V$, then*
1). $\mathcal{H} \triangleq S^\bullet(Z) \otimes H$ is an isotropic ideal in $\mathcal{L}$
2). (V, τ) induces an $\mathcal{H}$-representation $(\mathcal{V}, \nabla)$ of $\mathcal{L}$, where $\nabla : \mathcal{L} \to Der(\mathcal{V})$ is defined as follows:

$$\nabla_{f_1 e}(f_2 v) \triangleq f_1\big(\langle e, \partial f_2\rangle v + f_2(\tau(e) v)\big), \ \forall f_1, f_2 \in S^\bullet(Z), \ e \in L, \ v \in V.$$

Proof. 1). Since

$$\langle f_1 h_1, f_2 h_2\rangle = f_1 f_2 \langle h_1, h_2\rangle = 0, \quad \forall f_1, f_2 \in S^\bullet(Z), h_1, h_2 \in H,$$

$\mathcal{H}$ is isotropic in $\mathcal{L}$. And it is easily observed from Equation 5.1 that $\mathcal{H}$ is an ideal.

2). From the definition of ∇, it is obvious that

$$
\begin{aligned}
\nabla_{f_1 h}(f_2 v) &= 0 \\
\nabla_{f_1 x}(f_2 v) &= f_1 \nabla_x (f_2 v) \\
\nabla_x (f_1 (f_2 v)) &= (\rho(x) f_1)(f_2 v) + f_1 \nabla_x (f_2 v)
\end{aligned}
$$

$\forall f_1, f_2 \in S^\bullet(Z), h \in H, x \in \mathcal{L}, v \in V$. So we only need to prove that ∇ is a homomorphism of Leibniz algebras:

$$\begin{aligned}
& [\nabla_{f_1e_1}, \nabla_{f_2e_2}](fv) \\
= \quad & \nabla_{f_1e_1}\big(f_2\langle e_2, \partial f\rangle v + f_2 f\tau(e_2)v\big) - \nabla_{f_2e_2}\big(f_1\langle e_1, \partial f\rangle v + f_1 f\tau(e_1)v\big) \\
= \quad & \big(f_1\langle e_1, \partial(f_2\langle e_2, \partial f\rangle)\rangle v + f_1f_2\langle e_2, \partial f\rangle\tau(e_1)v + f_1\langle e_1, \partial(f_2 f)\rangle\tau(e_2)v + f_1f_2f\tau(e_1)\tau(e_2)v\big) \\
& -\big(f_2\langle e_2, \partial(f_1\langle e_1, \partial f\rangle)\rangle v + f_2f_1\langle e_1, \partial f\rangle\tau(e_2)v + f_2\langle e_2, \partial(f_1 f)\rangle\tau(e_1)v + f_2f_1f\tau(e_2)\tau(e_1)v\big) \\
= \quad & f_1f_2\big(\langle e_1, \partial\langle e_2, \partial f\rangle\rangle - \langle e_2, \partial\langle e_1, \partial f\rangle\rangle\big)v + f_1f_2f(\tau(e_1)\tau(e_2)v - \tau(e_2)\tau(e_1)v) \\
& +f_1\langle e_2, \partial f\rangle\langle e_1, \partial f_2\rangle v - f_2\langle e_1, \partial f\rangle\langle e_2, \partial f_1\rangle v + f_1f\langle e_1, \partial f_2\rangle\tau(e_2)v - f_2f\langle e_2, \partial f_1\rangle\tau(e_1)v \\
= \quad & f_1f_2\nabla_{e_1\circ e_2}(fv) + \langle e_1, \partial f_2\rangle f_1\nabla_{e_2}(fv) - \langle e_2, \partial f_1\rangle f_2\nabla_{e_1}(fv) \\
= \quad & \nabla_{f_1f_2(e_1\circ e_2) + \langle e_1, \partial f_2\rangle f_1e_2 - \langle e_2, \partial f_1\rangle f_2e_1}(fv) \\
= \quad & \nabla_{f_1e_1\circ f_2e_2}(fv)
\end{aligned}$$

The proof is finished. ■

Obviously $\mathcal{V}$ with the restriction of ∇ to $L \subseteq \mathcal{L}$ is still an H-representation of L, we still denote it by (V, ∇).

In the following, we always assume that

$$f \in S^\bullet(Z),\ e \in L,\ x \in \mathcal{L},\ h \in H,\ \alpha \in \mathcal{H}.$$

Theorem 5.3. *The H-standard complex of L with coefficients in $\mathcal{V}$ is isomorphic to the $\mathcal{H}$-standard complex of $\mathcal{L}$ with coefficients in $\mathcal{V}$, i.e.*

$$C^\bullet(L, H, \mathcal{V}) \cong C^\bullet(\mathcal{L}, \mathcal{H}, \mathcal{V}).$$

Proof. For simplicity, let $(C_1^\bullet, d_1)$ be $C^\bullet(L, H, \mathcal{V})$, and $(C_2^\bullet, d_2)$ be $C^\bullet(\mathcal{L}, \mathcal{H}, \mathcal{V})$. Given any $\eta \in C_2^n$, we can obtain an associated cochain in C_1^n by restriction, denote this restriction map by ψ. ψ is obviously a cochain map.

Next, given any $\omega \in C_1^n$, we can extend it to a cochain $\varphi\omega \in C_2^n$ as follows:

for the degree 2 arguments, extend ω from H to $\mathcal{H}$ by $S^\bullet(Z)$-linearity;

for the degree 1 arguments, extend ω from L to $\mathcal{L}$, from the last argument to the first argument one by one, by the equation of weak $S^\bullet(Z)$-linearity:

$$\begin{aligned}
& (\varphi\omega)_k(e_1, \cdots e_{a-1}, fe_a, x_{a+1}, \cdots x_{n-2k}; \alpha_1, \cdots \alpha_k) \\
= \quad & f(\varphi\omega)_k(e_1, \cdots e_{a-1}, e_a, x_{a+1}, \cdots x_{n-2k}; \alpha_1, \cdots \alpha_k) \\
& +\sum_{b>a}(-1)^{b-a}\langle e_a, x_b\rangle(\varphi\omega)_{k+1}(e_1, \cdots e_{a-1}, \widehat{e_a}, x_{a+1}, \cdots \widehat{x_b}, \cdots x_{n-2k}; \partial f, \alpha_1, \cdots \alpha_k).
\end{aligned}$$

The proof that $\varphi\omega$ is a cochain in C_2^n is left to the lemma below 5.4.

Obviously, $\psi \circ \varphi = id_{C_1^\bullet},\ \varphi \circ \psi = id_{C_2^\bullet}$. And φ is also a cochain map:

$$\varphi(d_1\omega) = \varphi(d_1(\psi(\varphi\omega))) = \varphi(\psi(d_2(\varphi\omega))) = d_2(\varphi\omega),\ \forall \omega \in C_1^\bullet$$

The proof is finished. ■

Lemma 5.4. *$\eta \triangleq \varphi\omega$ as defined above is a cochain in C_2^n.*

Proof. η is $S^\bullet(Z)$-linear in the arguments of $\mathcal{H}$ by definition. So we only need to prove weak skew-symmetricity and weak $S^\bullet(Z)$-linearity in the arguments of $\mathcal{L}$.

Proof of weak skew-symmetricity:

$$\begin{aligned} &\eta_k(x_1,\cdots x_a,x_{a+1}\cdots x_{n-2k};\alpha_1,\cdots\alpha_k)+\eta_k(x_1,\cdots x_{a+1},x_a\cdots x_{n-2k};\cdots) \\ =\ &-\eta_{k+1}(x_1,\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots x_{n-2k};\partial\langle x_a,x_{a+1}\rangle,\alpha_1,\cdots\alpha_k). \end{aligned} \tag{5.2}$$

Suppose $x_b=f_be_b$, $\forall b$. First we prove Equation 5.2 for the case when $x_1,\cdots x_{a-1}\in L$:

$$\begin{aligned} &\eta_k(e_1,\cdots e_{a-1},x_a,x_{a+1},\cdots x_{n-2k};\cdots)+\eta_k(e_1,\cdots e_{a-1},x_{a+1},x_a,\cdots x_{n-2k};\cdots) \\
= \ & f_a\eta_k(e_1,\cdots e_a,f_{a+1}e_{a+1},\cdots x_{n-2k};\cdots)-\langle e_a,f_{a+1}e_{a+1}\rangle\eta_{k+1}(\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots;\partial f_a,\cdots) \\
&+\sum_{b>a+1}(-1)^{b+a}\langle e_a,x_b\rangle\eta_{k+1}(\cdots\widehat{e_a},f_{a+1}e_{a+1},\cdots\widehat{x_b},\cdots;\partial f_a,\cdots) \\
&+f_{a+1}\eta_k(\cdots e_{a+1},f_ae_a,\cdots;\cdots)-\langle e_{a+1},f_ae_a\rangle\eta_{k+1}(\cdots\widehat{e_{a+1}},\widehat{e_a},\cdots;\partial f_{a+1},\cdots) \\
&+\sum_{b>a+1}(-1)^{b+a}\langle e_{a+1},x_b\rangle\eta_{k+1}(\cdots\widehat{e_{a+1}},f_ae_a,\cdots\widehat{x_b},\cdots;\partial f_{a+1},\cdots) \\
= \ & f_af_{a+1}\eta_k(\ldots e_a,e_{a+1},\cdots)+f_a\sum_{b>a+1}(-1)^{b+a+1}\langle e_{a+1},x_b\rangle\eta_{k+1}(\cdots e_a,\widehat{e_{a+1}},\cdots\widehat{x_b},\cdots;\partial f_{a+1},\cdots) \\
&+f_{a+1}\sum_{b>a+1}(-1)^{b+a}\langle e_a,x_b\rangle\eta_{k+1}(\cdots\widehat{e_a},e_{a+1},\cdots\widehat{x_b},\cdots;\partial f_a,\cdots) \\
&+\sum_{a+1<b<c}(-1)^{c+a}(-1)^{b+a+1}\langle e_a,x_c\rangle\langle e_{a+1},x_b\rangle\eta_{k+2}(\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots\widehat{x_b},\cdots\widehat{x_c},\cdots;\partial f_a,\partial f_{a+1},\cdots) \\
&+\sum_{a+1<c<b}(-1)^{c+a}(-1)^{b+a}\langle e_a,x_c\rangle\langle e_{a+1},x_b\rangle\eta_{k+2}(\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots\widehat{x_c},\cdots\widehat{x_b},\cdots;\partial f_a,\partial f_{a+1},\cdots) \\
&+f_af_{a+1}\eta_k(\cdots e_{a+1},e_a,\cdots)+f_{a+1}\sum_{b>a+1}(-1)^{b+a+1}\langle e_a,x_b\rangle\eta_{k+1}(\cdots e_{a+1},\widehat{e_a},\cdots\widehat{x_b},\cdots;\partial f_a,\cdots) \\
&+f_a\sum_{b>a+1}(-1)^{b+a}\langle e_{a+1},x_b\rangle\eta_{k+1}(\cdots\widehat{e_{a+1}},e_a,\cdots\widehat{x_b},\cdots;\partial f_{a+1},\cdots) \\
&+\sum_{a+1<b<c}(-1)^{c+a}(-1)^{b+a+1}\langle e_{a+1},x_c\rangle\langle e_a,x_b\rangle\eta_{k+2}(\cdots\widehat{e_{a+1}},\widehat{e_a},\cdots\widehat{x_b},\cdots\widehat{x_c},\cdots;\partial f_a,\partial f_{a+1},\cdots) \\
&+\sum_{a+1<c<b}(-1)^{c+a}(-1)^{b+a}\langle e_{a+1},x_c\rangle\langle e_a,x_b\rangle\eta_{k+2}(\cdots\widehat{e_{a+1}},\widehat{e_a},\cdots\widehat{x_c},\cdots\widehat{x_b},\cdots;\partial f_a,\partial f_{a+1},\cdots) \\
&-\big(f_{a+1}\langle e_a,e_{a+1}\rangle\eta_{k+1}(\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots;\partial f_a,\cdots)+f_a\langle e_{a+1},e_a\rangle\eta_{k+1}(\cdots\widehat{e_{a+1}},\widehat{e_a},\cdots;\partial f_{a+1},\cdots)\big) \\
= \ & f_af_{a+1}\big(\eta_k(\ldots,e_a,e_{a+1},\cdots;\cdots)+\eta_k(\ldots,e_{a+1},e_a,\cdots;\cdots)\big) \\
&-\big(f_{a+1}\langle e_a,e_{a+1}\rangle\eta_{k+1}(\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots;\partial f_a,\cdots)+f_a\langle e_{a+1},e_a\rangle\eta_{k+1}(\cdots\widehat{e_{a+1}},\widehat{e_a},\cdots;\partial f_{a+1},\cdots)\big) \\
= \ & -\eta_{k+1}(\cdots\widehat{e_a},\widehat{e_{a+1}},\cdots,x_{n-2k};\partial\langle f_ae_a,f_{a+1}e_{a+1}\rangle,\cdots) \end{aligned} \tag{5.3}$$

We will prove Equation 5.2 by mathematical induction.
If $n=2l$ is even, when $k=l-1$, Equation 5.2 is equivalent to 5.3.
If $n=2l+1$ is odd, when $k=l-1$, 5.2 is the combination of 5.3 and the following:

$$\begin{aligned}
&\eta_{l-1}(f_1e_1,f_2e_2,f_3e_3;\alpha_1,\cdots\alpha_{l-1})+\eta_{l-1}(f_1e_1,f_3e_3,f_2e_2;\alpha_1,\cdots\alpha_{l-1})\\
=\;& f_1\eta_{l-1}(e_1,f_2e_2,f_3e_3;\cdots)-\langle e_1,f_2e_2\rangle\eta_l(f_3e_3;\partial f_1,\cdots)+\langle e_1,f_3e_3\rangle\eta_l(f_2e_2;\partial f_1,\cdots)\\
&+f_1\eta_{l-1}(e_1,f_3e_3,f_2e_2;\cdots)-\langle e_1,f_3e_3\rangle\eta_l(f_2e_2;\partial f_1,\cdots)+\langle e_1,f_2e_2\rangle\eta_l(f_3e_3;\partial f_1,\cdots)\\
=\;& -f_1\eta_l(e_1;\partial\langle f_2e_2,f_3e_3\rangle,\cdots)\\
=\;& -\eta_l(f_1e_1;\partial\langle f_2e_2,f_3e_3\rangle,\cdots).
\end{aligned}$$

Now suppose Equation 5.2 holds for $k>m$, consider the case when $k=m$. By Equation 5.3, we can further suppose that 5.2 holds for $x_1,\cdots x_i\in L,\ i<a$. We will prove 5.2 for the case when $k=m$ and $x_1,\cdots,x_{i-1}\in L$:

$$\begin{aligned}
&\eta_m(e_1,\cdots e_{i-1},f_ie_i,\cdots x_a,x_{a+1},\cdots;\cdots)+\eta_m(e_1,\cdots e_{i-1},f_ie_i,\cdots x_{a+1},x_a,\cdots;\cdots)\\
=\;& f_i\eta_m(\cdots e_i,\cdots x_a,x_{a+1},\cdots)+\sum_{b>i\neq a,a+1}(-1)^{b-i}\langle e_i,x_b\rangle\eta_{m+1}(\cdots\widehat{e_i},\cdots\widehat{x_b},\cdots x_a,x_{a+1},\cdots;\partial f_i\cdots)\\
& f_i\eta_m(\cdots e_i,\cdots x_{a+1},x_a,\cdots)+\sum_{b>i\neq a,a+1}(-1)^{b-i}\langle e_i,x_b\rangle\eta_{m+1}(\cdots\widehat{e_i},\cdots\widehat{x_b},\cdots x_{a+1},x_a,\cdots;\partial f_i\cdots)\\
&+((-1)^{a-i}+(-1)^{a+1-i})\langle e_i,x_a\rangle\eta_{m+1}(\cdots\widehat{e_i},\cdots\widehat{x_a},x_{a+1},\cdots;\partial f_i\cdots)\\
&+((-1)^{a+1-i}+(-1)^{a-i})\langle e_i,x_{a+1}\rangle\eta_{m+1}(\cdots\widehat{e_i},\cdots x_a,\widehat{x_{a+1}},\cdots;\partial f_i\cdots)\\
=\;& f_i\big(\eta_m(\cdots e_i,\cdots x_a,x_{a+1},\cdots;\cdots)+\eta_m(\cdots e_i,\cdots x_{a+1},x_a,\cdots;\cdots)\big)\\
&+\Big(\sum_{b>i\neq a,a+1}(-1)^{b-i}\langle e_i,x_b\rangle\eta_{m+1}(\cdots\widehat{e_i},\cdots\widehat{x_b},\cdots x_a,x_{a+1},\cdots;\partial f_i\cdots)\\
&\quad+\sum_{b>i\neq a,a+1}(-1)^{b-i}\langle e_i,x_b\rangle\eta_{m+1}(\cdots\widehat{e_i},\cdots\widehat{x_b},\cdots x_{a+1},x_a,\cdots;\partial f_i\cdots)\Big)\\
=\;& -f_i\eta_{m+1}(\cdots e_i,\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\partial\langle x_a,x_{a+1}\rangle,\cdots)\\
&-\sum_{b>i\neq a,a+1}(-1)^{b-i}\langle e_i,x_b\rangle\eta_{m+2}(\cdots\widehat{e_i},\cdots\widehat{x_b},\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\partial\langle x_a,x_{a+1}\rangle,\partial f_i\cdots)\\
=\;& -\eta_{m+1}(e_1,\cdots e_{i-1},f_ie_i,\cdots\widehat{x_a},\widehat{x_{a+1}},\cdots;\partial\langle x_a,x_{a+1}\rangle,\cdots)
\end{aligned}$$

By induction, 5.2 is proved.

Proof of weak $S^\bullet(Z)$-linearity:

$$\begin{aligned}
&\eta_k(x_1,\cdots x_{i-1},fx_i,\cdots x_{n-2k};\alpha_1,\cdots\alpha_k)\\
=\;& f\eta_k(\cdots x_i,\cdots;\alpha_1,\cdots\alpha_k)+\sum_{a>i}(-1)^{a-i}\langle x_i,x_a\rangle\eta_{k+1}(\cdots\widehat{x_i},\cdots\widehat{x_a},\cdots;\partial f,\alpha_1,\cdots\alpha_k)
\end{aligned}\tag{5.4}$$

When $x_1,\cdots,x_{i-1}\in L$,5.4 holds:

$$\begin{aligned}
&\eta_k(e_1,\cdots,e_{i-1},ff_ie_i,\cdots;\cdots)\\
=\;& ff_i\eta_k(\cdots e_i,\cdots;\cdots)+\sum_{a>i}(-1)^{a-i}\langle e_i,x_a\rangle\eta_{k+1}(\cdots\widehat{x_i},\cdots\widehat{x_a},\cdots;\partial(ff_i),\cdots)\\
=\;& f(\eta_k(\cdots f_ie_i,\cdots;\cdots)-\sum_{a>i}(-1)^{a-i}\langle e_i,x_a\rangle\eta_{k+1}(\cdots\widehat{x_i},\cdots\widehat{x_a},\cdots;\partial f_i,\cdots))\\
&+\sum_{a>i}(-1)^{a-i}\langle e_i,x_a\rangle\eta_{k+1}(\cdots\widehat{x_i},\cdots\widehat{x_a},\cdots;\partial(ff_i),\cdots)
\end{aligned}$$

$$= \quad f\eta_k(\cdots f_i e_i, \cdots; \cdots) + \sum_{a>i}(-1)^{a-i}\langle f_i e_i, x_a\rangle \eta_{k+1}(\cdots \widehat{x_i}, \cdots \widehat{x_a}, \cdots; \partial f, \cdots)$$

Now suppose 5.4 holds for any $k > m$, and for the case when $x_1, \cdots, x_j \in L(j < i),\ k = m$ as well. Consider the case when $x_1, \cdots, x_{j-1} \in L,\ k = m$:

$$\begin{aligned}
& \eta_k(e_1, \cdots e_{j-1}, f_j e_j, \cdots, f x_i, \cdots; \cdots) \\
= \ & f_j \eta_k(\cdots e_j, \cdots f x_i, \cdots; \cdots) + (-1)^{i-j}\langle e_j, f x_i\rangle \eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots; \partial f_j, \cdots) \\
& + \sum_{b>j, b\neq i}(-1)^{b+j}\langle e_j, x_b\rangle \eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_b}, \cdots, f x_i, \cdots; \partial f_j, \cdots) \\
= \ & f_j\big(f\eta_k(\cdots e_j, \cdots x_i, \cdots; \cdots) + \sum_{a>i}(-1)^{a+i}\langle x_i, x_a\rangle \eta_{k+1}(\cdots e_j, \cdots \widehat{x_i}, \cdots \widehat{x_a}, \cdots; \partial f, \cdots)\big) \\
& + \sum_{j<b<i}(-1)^{b+j}\langle e_j, x_b\rangle \big(f\eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_b}, \cdots, x_i, \cdots; \partial f_j, \cdots) \\
& \quad + \sum_{a>i}(-1)^{a+i}\langle x_i, x_a\rangle \eta_{k+2}(\cdots \widehat{e_j}, \cdots \widehat{x_b}, \cdots \widehat{x_i}, \cdots, \widehat{x_a}, \cdots; \partial f, \partial f_j, \cdots)\big) \\
& + \sum_{b>i}(-1)^{b+j}\langle e_j, x_b\rangle \big(f\eta_{k+1}(\cdots \widehat{e_j}, \cdots, x_i, \cdots \widehat{x_b}, \cdots; \partial f_j, \cdots) \\
& \quad + \sum_{i<a<b}(-1)^{a+i}\langle x_i, x_a\rangle \eta_{k+2}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots \widehat{x_a}, \cdots \widehat{x_b}, \cdots; \partial f, \partial f_j, \cdots) \\
& \quad + \sum_{b<a}(-1)^{a+i+1}\langle x_i, x_a\rangle \eta_{k+2}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots \widehat{x_b}, \cdots \widehat{x_a}, \cdots; \partial f, \partial f_j, \cdots)\big) \\
& + (-1)^{i+j}\langle e_j, f x_i\rangle \eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots; \partial f_j, \cdots) \\
= \ & f\big(f_j\eta_k(\cdots e_j, \cdots, x_i, \cdots; \cdots) + (-1)^{i+j}\langle e_j, x_i\rangle \eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots; \partial f_j, \cdots) \\
& \quad + \sum_{j<b<i}(-1)^{b+j}\langle e_j, x_b\rangle \eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_b}, \cdots, x_i, \cdots; \partial f_j, \cdots) \\
& \quad + \sum_{b>i}(-1)^{b+j}\langle e_j, x_b\rangle \eta_{k+1}(\cdots \widehat{e_j}, \cdots x_i, \cdots \widehat{x_b}, \cdots; \partial f_j, \cdots)\big) \\
& + \sum_{a>i}(-1)^{a+i}\langle x_i, x_a\rangle \big(f_j\eta_{k+1}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots \widehat{x_a}, \cdots; \partial f, \cdots) \\
& \quad + \sum_{j<b<i}(-1)^{b+j}\langle e_j, x_b\rangle \eta_{k+2}(\cdots e_j, \cdots \widehat{x_b}, \cdots \widehat{x_i}, \cdots x_a, \cdots; \partial f, \partial f_j, \cdots) \\
& \quad + \sum_{i<b<a}(-1)^{b+j+1}\langle e_j, x_b\rangle \eta_{k+2}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots \widehat{x_b}, \cdots \widehat{x_a}, \cdots; \partial f, \partial f_j, \cdots) \\
& \quad + \sum_{b>a}(-1)^{b+j}\langle e_j, x_b\rangle \eta_{k+2}(\cdots \widehat{e_j}, \cdots \widehat{x_i}, \cdots \widehat{x_a}, \cdots \widehat{x_b}, \cdots; \partial f, \partial f_j, \cdots)\big) \\
= \ & f\eta_k(\cdots f_j e_j, \cdots x_i, \cdots; \cdots) + \sum_{a>i}(-1)^{a-i}\langle x_i, x_a\rangle \eta_{k+1}(\cdots f_j e_j, \cdots \widehat{x_i}, \cdots, \widehat{x_a}, \cdots; \partial f, \cdots)
\end{aligned}$$

By mathematical induction, Equation 5.4 holds for any k.

The proof is finished. ■

It is easily proved that the Chevalley-Eilenberg complex of the Lie algebra L/H with coefficients in $\mathcal{V}$ is isomorphic to the Chevalley-Eilenberg complex of the Lie-Rinehart algebra $\mathcal{L}/\mathcal{H}$ with coefficients in $\mathcal{V}$. Whence, combining Theorem 5.3 and 4.1 (the quotient $\mathcal{L}/\mathcal{H} \cong S^\bullet(Z) \otimes (L/H)$ is a free module), we have the following:

Corollary 5.5. *With the notations above,*

$$H^\bullet(L, H, \mathcal{V}) \cong H^\bullet_{CE}(L/H, \mathcal{V}).$$

Actually this result is true for the H-representation (V, τ):

Theorem 5.6. *With the notations above,*

$$H^\bullet(L, H, V) \cong H^\bullet_{CE}(L/H, V).$$

Proof. Consider a subspace of $C^\bullet(\mathcal{L}, \mathcal{H}, \mathcal{V})$:

$$C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V}) \triangleq \bigoplus_n \{\omega \in C^n(\mathcal{L}, \mathcal{H}, \mathcal{V}) | \omega_k(e_1, \cdots e_{n-2k}; h_1, \cdots h_k) \in V, \ \forall k, \ \forall e \in L, h \in H\}.$$

It is easily checked from the definition of $d = d_0 + \delta + d'$ that $C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V})$ is a subcomplex.

Given any $\omega \in C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V})$ satisfying the condition of Lemma 4.3, we see that λ and η as constructed in the proof of this lemma are both in $C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V})$. Then by similar arguments to the proof of Theorem 4.1,

$$H^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V}) \triangleq H^\bullet(C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V}), d)$$

is isomorphic to the cohomology of the following subcomplex of $C^\bullet(\mathcal{L}, \mathcal{H}, \mathcal{V})$

$$\begin{aligned} C^\bullet_{nv\ 0}(\mathcal{L}, \mathcal{H}, \mathcal{V}) \quad \triangleq \quad & \bigoplus_n \{\omega \in C^n(\mathcal{L}, \mathcal{H}, \mathcal{V}) | \omega_k = 0, \ \forall k \geq 1, \ \iota_\alpha \omega_0 = 0, \ \forall \alpha \in \mathcal{H}, \\ & \omega_0(e_1, \cdots e_n) \in V, \ \forall e \in L\}, \end{aligned}$$

which is again isomorphic to $H^\bullet_{CE}(L/H, V)$.

On the other hand, in the proof of Theorem 5.3, if we restrict ψ from $C^\bullet(\mathcal{L}, \mathcal{H}, \mathcal{V})$ to $C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V})$ and φ from $C^\bullet(L, H, \mathcal{V})$ to $C^\bullet(L, H, V)$, we can get mutually invertible cochain maps between $C^\bullet(L, H, V)$ and $C^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V})$.

Thus

$$H^\bullet(L, H, \mathcal{V}) \cong H^\bullet_0(\mathcal{L}, \mathcal{H}, \mathcal{V}) \cong H^\bullet_{CE}(L/H, \mathcal{V}).$$

■

Example 5.7. If L is the omni-Lie algebra $gl(V) \oplus V$, V is the only isotropic ideal of L containing the left center V, and (V, τ) is a V-representation with τ being the standard action of $gl(V)$ on V. As introduced by Weinstein [16], the omni Lie algebra $gl(V) \oplus V$ can be viewed as the linearization of the standard Courant algebroid $TV^* \oplus T^*V^*$, where $gl(V)$ is identified with the space of linear vector fields and V is identified with the space of constant 1-forms. If we ignore the difference between $S^\bullet(V)$ and $C^\infty(V^*)$, the crossed product $\mathcal{L}$ as constructed in Proposition 5.1 can be viewed as a Courant-Dorfman subalgebra of $\Gamma(TV^* \oplus T^*V^*)$, in the sense that $\mathcal{L}$ consists of all polynomial vector fields (excluding constant ones) and polynomial 1-forms. By Theorem 5.3 and Corollary 5.5, the standard cohomology of $\mathcal{L}$ is isomorphic to the cohomology of Lie algebra $gl(V)$ with coefficients in $S^\bullet(V)$, which is trivial. Whence, although $\mathcal{L}$ is different from $\Gamma(TV^* \oplus T^*V^*)$, the standard cohomology of them are both trivial.

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