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DERIVED BRACKETS FOR FAT LEIBNIZ ALGEBRAS

XIONGWEI CAI AND ZHANGJU LIU

Abstract. Given a Leibniz algebra L with left center Z, we work on $C(L, Z, S^\bullet(Z))$, the Z-standard complex of L with coefficients in $S^\bullet(Z)$. We construct the derived bracket for a fat Leibniz algebra in terms of a certain 3-cocycle and a Poisson algebra structure on the space of so-called "representable cochains".

1. Introduction

Leibniz algebras, objects that first appeared in Bloh's work [3] and named by Loday [9], can be viewed as the noncommutative analogue of Lie algebras. Some theorems and properties of Lie algebras have been proved to be still valid for Leibniz algebras, while many other questions are still open.

Courant algebroids, first introduced by Liu, Weinstein and Xu in [8], can be viewed as the geometric realization of Leibniz algebras. The algebraization of Courant algebroids, Courant-Dorfman algebras, are special examples of Leibniz algebras.

The derived bracket for a Lie algebra with an ad-invariant inner product is constructed by Lecomte-Roger [7] and Kosmann-Schwarzbach [5], in order to study the homological algebra of Lie bialgebras and quasi-Lie bialgebras, respectively. While the construction of derived bracket for a Courant algebroid was given by Kosmann-Schwarzbach [6], Roytenberg [11] and Alekseev-Xu [1].

It is a natural question to ask whether there is a derived bracket construction for Leibniz algebras. In this paper, we succeed to give a positive answer for fat Leibniz algebras. By a fat Leibniz algebra, we mean a Leibniz algebra whose naturally defined symmetric product is non-degenerate. Note that this is a different notion from a quadratic Leibniz algebra, defined by Benayadi-Hidri [2].

Given a Leibniz algebra L with left center Z, we will work on the H-standard complex (see Cai [4]) of L in the particular case when $H = Z$, $V = S^\bullet(Z)$. We will define a canonical 3-cocycle Θ and prove that the subcomplex consisting of the so-called "representable cochains" is a graded Poisson algebra. Finally we show that the Leibniz bracket of a fat Leibniz algebra can be represented by a derived bracket.

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2. Standard complex

In this section, we recall the definition of H-standard complex of a Leibniz algebra L with coefficients in V ([4]), and consider a 3-cocycle in the particular case when $H = Z$, $V = S^\bullet(Z)$.

Given a Leibniz algebra L with left center Z, let $H \supseteq Z$ be an isotropic ideal in L, and (V, τ) be an H-trivial representation of L (i.e. a left representation of L on which H acts trivially).

Key words and phrases. Leibniz algebras, representable cochains, derived brackets.

Denote by $C^n(L,H,V)$ the space of all sequences $\omega=(\omega_0,\cdots,\omega_{[\frac{n}{2}]})$, where ω_k is a linear map from $(\otimes^{n-2k}L)\otimes(\odot^k H)$ to V, $\forall k$, and is weakly skew-symmetric in arguments of L up to ω_{k+1}:

$$\begin{aligned} & \omega_k(e_1,\cdots e_i,e_{i+1},\cdots e_{n-2k};h_1,\cdots h_k)+\omega_k(e_1,\cdots e_{i+1},e_i,\cdots e_{n-2k};h_1,\cdots h_k) \\ = & -\omega_{k+1}(\cdots\widehat{e_i},\widehat{e_{i+1}},\cdots;(e_i,e_{i+1}),\cdots) \qquad \forall e\in L,\ h\in H \end{aligned}$$

$C(L,H,V)\triangleq\bigoplus_n C^n(L,H,V)$ becomes a cochain complex under the coboundary map $d=d_0+\delta+d'$, called the H-standard complex of L with coefficients in V, where d_0,δ,d' are defined for any $\omega\in C^n(L,H,V)$ respectively by:

$$\begin{aligned} (d_0\omega)_k(e_1,\cdots,e_{n+1-2k};h_1,\cdots h_k) &\triangleq \sum_a(-1)^{a+1}\rho(e_a)\omega_k(\cdots\widehat{e_a},\cdots;\cdots) \\ &\quad +\sum_{a<b}(-1)^a\omega_k(\cdots\widehat{e_a},\cdots e_a\circ e_b,\cdots;\cdots) \\ (\delta\omega)_k(e_1,\cdots,e_{n+1-2k};h_1,\cdots h_k) &\triangleq \sum_j\omega_{k-1}(\alpha_j,e_1,\cdots e_{n+1-2k};\cdots\widehat{h_j},\cdots) \\ (d'\omega)_k(e_1,\cdots e_{n+1-2k};h_1,\cdots h_k) &\triangleq \sum_{a,j}(-1)^{a+1}\omega_k(\cdots\widehat{e_a},\cdots;\cdots\widehat{h_j},h_j\circ e_a,\cdots). \end{aligned}$$

The Leibniz bracket of L induces a left action ρ of L on Z: $\rho(e)f\triangleq e\circ f$, $\forall e\in L,\ f\in Z$. And it can be extended by Leibniz rule to a left action of L on the symmetric tensor $S^\bullet(Z)$, still denoted by ρ. $(S^\bullet(Z),\rho)$ is obviously a Z-trivial representation of L, so we have the Z-standard complex $(C(L,Z,S^\bullet(Z)),d)$. Note that d' is 0, so $d=d_0+\delta$ in this case.

Definition 2.1. $(C(L,Z,S^\bullet(Z)),d=d_0+\delta)$ is called the standard complex of L.

For simplicity, we will denote $C(L,Z,S^\bullet(Z))$ by $C(L)$ from now on.

Proposition 2.2. *$C(L)$ is a differential graded commutative algebra, with the multiplication map defined for $\omega\in C^n(L)$, $\eta\in C^m(L)$ by:*

$$(\omega\cdot\eta)_k(e_1,\cdots,e_{n+m-2k};f_1,\cdots,f_k) \triangleq \sum_{\substack{i+j=k\\ \sigma\in sh(n-2i,m-2j)\\ \mu\in sh(i,j)}}(-1)^\sigma\omega_i(e_{\sigma(1)}\cdots e_{\sigma(n-2i)};f_{\mu(1)}\cdots f_{\mu(i)})\eta_j(e_{\sigma(n-2i+1)}\cdots;f_{\mu(i+1)}\cdots), \tag{2.1}$$

$\forall e\in L, f\in Z$, where $sh(\ ,\ )$ means the shuffle permutation.

Proof. The multiplication map above is obviously graded commutative, i.e.

$$\omega\cdot\eta=(-1)^{nm}\eta\cdot\omega.$$

We give the proof in 3 steps.

Step 1:

$C(L)$ is closed under the multiplication, i.e. $\omega\cdot\eta\in C^{n+m}(L)$:

$$\begin{aligned}
&(\omega\cdot\eta)_k(\cdots e_a,e_{a+1},\cdots;f_1,\cdots f_k)+(\omega\cdot\eta)_k(\cdots e_{a+1},e_a,\cdots;f_1,\cdots f_k)\\
=&\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(n-2i,m-2j)\\ \sigma^{-1}(a),\sigma^{-1}(a+1)\le n-2i}}(-1)^\sigma\big(\omega_i(\cdots e_a,e_{a+1},\cdots;\cdots)+\omega_i(\cdots e_{a+1},e_a,\cdots;\cdots)\big)\eta_j(\cdots)\\
&+\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(n-2i,m-2j)\\ \sigma^{-1}(a),\sigma^{-1}(a+1)>n-2i}}(-1)^\sigma\omega_i(\cdots)\big(\eta_j(\cdots e_a,e_{a+1},\cdots;\cdots)+\eta_i(\cdots e_{a+1},e_a,\cdots;\cdots)\big)\\
&+\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(n-2i,m-2j)\\ \sigma^{-1}(a)\le n-2i<\sigma^{-1}(a+1)}}(-1)^\sigma\big(\omega_i(\cdots e_a\cdots)\eta_j(\cdots e_{a+1}\cdots)+\omega_i(\cdots e_{a+1}\cdots)\eta_j(\cdots e_a\cdots)\big)\\
&+\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(n-2i,m-2j)\\ \sigma^{-1}(a+1)\le n-2i<\sigma^{-1}(a)}}(-1)^\sigma\big(\omega_i(\cdots e_{a+1}\cdots)\eta_j(\cdots e_a\cdots)+\omega_i(\cdots e_a\cdots)\eta_j(\cdots e_{a+1}\cdots)\big)
\end{aligned}$$

(note that the same sequence $(\cdots e_a,\cdots e_{a+1},\cdots)$ viewed as permutations of $(\cdots e_a,e_{a+1},\cdots)$ and $(\cdots e_{a+1},e_a,\cdots)$ have opposite signs)

$$\begin{aligned}
=&\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(n-2i,m-2j)\\ \sigma^{-1}(a),\sigma^{-1}(a+1)\le n-2i}}(-1)^{\sigma+1}\omega_{i+1}(\cdots,\widehat{e_a},\widehat{e_{a+1}},\cdots;(e_a,e_{a+1}),\cdots)\eta_j(\cdots)\\
&+\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(n-2i,m-2j)\\ \sigma^{-1}(a),\sigma^{-1}(a+1)>n-2i}}(-1)^{\sigma+1}\omega_i(\cdots)\eta_{j+1}(\cdots,\widehat{e_a},\widehat{e_{a+1}},\cdots;(e_a,e_{a+1}),\cdots)\\
=&\sum_{\substack{l+j=k+1\\ \sigma\in sh(n-2l,m-2j)}}\sum_{\substack{\tau\in sh(l,j)\\ \tau^{-1}((e_a,e_{a+1}))\le l}}(-1)^{\sigma+1}\omega_l(\cdots;(e_a,e_{a+1}),\cdots)\eta_j(\cdots)\\
&+\sum_{\substack{i+l=k+1\\ \sigma\in sh(n-2i,m-2l)}}\sum_{\substack{\tau\in sh(i,l)\\ \tau^{-1}((e_a,e_{a+1}))>i}}(-1)^{\sigma+1}\omega_i(\cdots)\eta_l(\cdots;(e_a,e_{a+1}),\cdots)\\
=&-(\omega\cdot\eta)_{k+1}(e_1,\cdots,\widehat{e_a},\widehat{e_{a+1}},\cdots;(e_a,e_{a+1}),\cdots)
\end{aligned}$$

Step 2:
The multiplication is associative:
$\forall\omega\in C^n(L),\ \eta\in C^m(L),\ \lambda\in C^l(L)$, by definition it is an easy calculation that, $((\omega\cdot\eta)\cdot\lambda)_k(e_1,\cdots,e_{n+m+l-2k};f_1,\cdots,f_k)$ and $(\omega\cdot(\eta\cdot\lambda))_k(e_1,\cdots,e_{n+m+l-2k};f_1,\cdots,f_k)$ both equal to:

$$\sum_{\substack{a+b+c=k\\ \sigma\in sh(n-2a,m-2b,l-2c)\\ \tau\in sh(a,b,c)}}(-1)^\sigma\omega_a(\cdots)\eta_b(\cdots)\lambda_c(\cdots)$$

Step 3:
The differential d is a graded derivation:

$$d(\omega\cdot\eta)=(d\omega)\cdot\eta+(-1)^n\omega\cdot(d\eta),\ \forall\omega\in C^n(L),\ \eta\in C^m(L).$$

Since $d=d_0+\delta$, it suffices to prove the equation for d_0,δ respectively.

For d_0, we only give the proof for the case of degree 0 here, since the proof is almost the same for cases of higher degrees (the only difference is that the sum should be taken over permutations of the arguments in Z as well).

$$
\begin{aligned}
& (d_0(\omega\cdot\eta))_0(e_1,\cdots,e_{n+m+1}) \\
= \ & \sum_a (-1)^{a+1}\rho(e_a)(\omega\cdot\eta)_0(\cdots\widehat{e_a}\cdots) + \sum_{a<b}(-1)^a(\omega\cdot\eta)_0(\cdots\widehat{e_a}\cdots\widehat{e_b},e_a\circ e_b\cdots) \\
= \ & \sum_a (-1)^{a+1}\rho(e_a)\big(\sum_{\sigma\in sh(n,m)\{\cdots,\hat{a},\cdots\}}(-1)^\sigma\omega_0(e_{\sigma(1)}\cdots e_{\sigma(n)})\eta_0(e_{\sigma(n+1)}\cdots e_{\sigma(n+m)})\big) \\
& +\sum_{a<b}(-1)^a\sum_{\substack{\sigma\in sh(n,m)\{\cdots\hat{a},\cdots\}\\ \sigma^{-1}(b)<n+1}}(-1)^\sigma\omega_0(e_{\sigma(1)},\cdots\widehat{e_b},e_a\circ e_b,\cdots e_{\sigma(n)})\eta_0(e_{\sigma(n+1)},\cdots e_{\sigma(n+m+1)}) \\
& +\sum_{a<b}(-1)^a\sum_{\substack{\sigma\in sh(n,m)\{\cdots\hat{a},\cdots\}\\ \sigma^{-1}(b)>n}}(-1)^\sigma\omega_0(e_{\sigma(1)},\cdots e_{\sigma(n)})\eta_0(e_{\sigma(n+1)},\cdots\widehat{e_b},e_a\circ e_b,\cdots e_{\sigma(n+m+1)}) \\
& \text{(let } \sigma_1,\ \sigma_2 \text{ be the permutations adding } a \text{ to } \sigma \text{ in front and at back respectively)} \\
= \ & \sum_a\sum_{\sigma_1\in sh(n+1,m)}(-1)^{a+1}(-1)^{\sigma_1+\sigma_1^{-1}(a)-a} \\
& \quad \big(\rho(e_a)\omega_0(e_{\sigma_1(1)}\cdots\widehat{e_a},e_{\sigma_1(\sigma_1^{-1}(a)+1)}\cdots e_{\sigma_1(n+1)})\big)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
& +\sum_a\sum_{\sigma_2\in sh(n,m+1)}(-1)^{a+1}(-1)^{\sigma_2+\sigma_2^{-1}(a)-a} \\
& \quad \omega_0(e_{\sigma_2(1)}\cdots e_{\sigma_2(n)})\big(\rho(e_a)\eta_0(e_{\sigma_2(n+1)}\cdots\widehat{e_a},e_{\sigma_2(\sigma_2^{-1}(a)+1)}\cdots e_{\sigma_2(n+m+1)})\big) \\
& +\sum_{a<b}\sum_{\substack{\sigma_1\in sh(n+1,m)\\ \sigma_1^{-1}(b)<n+2}}(-1)^a(-1)^{\sigma_1+\sigma_1^{-1}(a)-a} \\
& \quad \omega_0(e_{\sigma_1(1)}\cdots\widehat{e_a},e_{\sigma_1(\sigma_1^{-1}(a)+1)}\cdots\widehat{e_b},e_a\circ e_b\cdots)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
& +\sum_{a<b}\sum_{\substack{\sigma_2\in sh(n,m+1)\\ \sigma_2^{-1}(b)>n+1}}(-1)^a(-1)^{\sigma_2+\sigma_2^{-1}(a)-a} \\
& \quad \omega_0(e_{\sigma_2(1)}\cdots)\eta_0(e_{\sigma_2(n+1)}\cdots\widehat{e_a},e_{\sigma_2(\sigma_2^{-1}(a)+1)}\cdots\widehat{e_b},e_a\circ e_b\cdots e_{\sigma_2(n+m+1)}) \\
= \ & \sum_{\sigma_1}(-1)^{\sigma_1}\sum_{(a_1\triangleq\sigma_1^{-1}(a))<n+2}(-1)^{a_1+1} \\
& \quad \big(\rho(e_{\sigma_1(a_1)})\omega_0(e_{\sigma_1(1)}\cdots\widehat{e_{\sigma_1(a_1)}}\cdots e_{\sigma_1(n+1)})\big)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
& +\sum_{\sigma_1}(-1)^{\sigma_1}\sum_{(a_1\triangleq\sigma_1^{-1}(a))<(b_1\triangleq\sigma_1^{-1}(b))<n+2}(-1)^{a_1} \\
& \quad \omega_0(e_{\sigma_1(1)}\cdots\widehat{e_{\sigma_1(a_1)}}\cdots\widehat{e_{\sigma_1(b_1)}},e_{\sigma_1(a_1)}\circ e_{\sigma_1(b_1)}\cdots)\eta_0(e_{\sigma_1(n+2)}\cdots e_{\sigma_1(n+m+1)}) \\
& +\sum_{\sigma_2}(-1)^{\sigma_2+n}\omega_0(e_{\sigma_2(1)},\cdots,e_{\sigma_2(n)})\cdot\sum_{a_2\triangleq\sigma_2^{-1}(a)} \\
& \quad (-1)^{a_2-n+1}\rho(e_{\sigma_2(a_2)})\eta_0(e_{\sigma_2(n+1)},\cdots,\widehat{e_{\sigma_2(a_2)}},\cdots,e_{\sigma_2(n+m+1)}) \\
& +\sum_{\sigma_2}(-1)^{\sigma_2+n}\omega_0(e_{\sigma_2(1)},\cdots,e_{\sigma_2(n)})\cdot\sum_{n<(a_2\triangleq\sigma_2^{-1}(a))<(b_2\triangleq\sigma_2^{-1}(b))} \\
& \quad (-1)^{a_2-n}\eta_0(e_{\sigma_2(n+1)}\cdots\widehat{e_{\sigma_2(a_2)}}\cdots\widehat{e_{\sigma_2(b_2)}},e_{\sigma_2(a_2)}\circ e_{\sigma_2(b_2)}\cdots e_{\sigma_2(n+m+1)})
\end{aligned}
$$

$$= \Big((d_0\omega)\cdot\eta + (-1)^n\omega\cdot(d_0\eta)\Big)_0(e_1,\cdots,e_{n+m+1})$$

For δ,

$$\begin{aligned}
& (\delta(\omega\cdot\eta))_k(e_1,\cdots,e_{n+m+1-2k};f_1,\cdots,f_k)\\
= & \sum_i(\omega\cdot\eta)_{k-1}(f_i,e_1,\cdots,e_{n+m+1-2k};\cdots,\hat{f}_i,\cdots)\\
= & \sum_i\sum_{\substack{a+b=k-1\\ \tau\in sh(a,b)}}\sum_{\substack{\sigma\in sh(n-2a,m-2b)\\ \sigma^{-1}(f_i)\le n-2a}}(-1)^\sigma\omega_a(f_i,\cdots;\cdots\hat{f}_i\cdots)\eta_b(\cdots)\\
& +\sum_i\sum_{\substack{a+b=k-1\\ \tau\in sh(a,b)}}\sum_{\substack{\sigma\in sh(n-2a,m-2b)\\ \sigma^{-1}(f_i)>n-2a}}(-1)^\sigma\omega_a(\cdots)\eta_b(f_i,\cdots;\cdots\hat{f}_i\cdots)\\
& \text{(removing } f_i \text{ from } \sigma\text{, adding } f_i \text{ to } \tau \text{ in front and at back respectively)}\\
= & \sum_{\substack{a+b=k\\ \sigma\in sh(n+1-2a,m-2b)}}\sum_{\substack{\tau\in sh(a,b)\\ \tau^{-1}(i)\le a}}(-1)^\sigma\omega_{a-1}(f_i,e_{\sigma(1)},\cdots;\cdots,\widehat{f_{\tau(\tau^{-1}(i))}},\cdots)\eta_b(\cdots)\\
& +\sum_{\substack{a+b=k\\ \sigma\in sh(n+1-2a,m-2b)}}\sum_{\substack{\tau\in sh(a,b)\\ \tau^{-1}(i)>a}}(-1)^{\sigma+n}\omega_a(\cdots)\eta_{b-1}(f_i,e_{\sigma(n-2a+1)},\cdots;\cdots,\widehat{f_{\tau(\tau^{-1}(i))}},\cdots)\\
= & ((\delta\omega)\cdot\eta)_k(\cdots)+(-1)^n(\omega\cdot(\delta\eta))_k(\cdots)
\end{aligned}$$

The proof is finished. ■

Remark 2.3. In [4], we construct a Courant-Dorfman algebra structure on $S^\bullet(Z)\otimes L$ for any Leibniz algebra L with left center Z, and prove an isomorphism between H-standard complexes of them. So it is a direct conclusion that $C(L)$ is isomorphic to the standard complex of the Courant-Dorfman algebra $S^\bullet(Z)\otimes L$.

Next we consider a 3-cochain in $C(L)$. Let $\Theta_0 : L\otimes L\otimes L \to S^\bullet(Z)$ and $\Theta_1 : L\otimes Z\to S^\bullet(Z)$ be defined as:

$$\Theta_0(e_1,e_2,e_3) = (e_1\circ e_2, e_3)$$

$$\Theta_1(e;f) = -(e,f). \tag{2.2}$$

We can prove that $\Theta = (\Theta_0,\Theta_1)$ is a 3-cocycle by a direct calculation, but actually we have the following:

Proposition 2.4. *$\Theta = d\zeta$ is a 3-coboundary, where $\zeta = (\zeta_0,\zeta_1)\in C^2(L)$ is defined by:*

$$\zeta_0(e_1,e_2)\triangleq(e_1,e_2),\quad \zeta_1(f)\triangleq -2f.$$

Proof. Since

$$\zeta_0(e_1,e_2)+\zeta_0(e_2,e_1) = 2(e_1,e_2) = -\zeta_1((e_1,e_2)),$$

$\zeta = (\zeta_0, \zeta_1)$ is a 2-cochain in $C^2(L)$. By definition,

$$\begin{aligned}
& (d\zeta)_0(e_1, e_2, e_3) \\
= \; & \rho(e_1)\zeta_0(e_2, e_3) - \rho(e_2)\zeta_0(e_1, e_3) + \rho(e_3)\zeta_0(e_1, e_2) \\
& -\zeta_0(e_1 \circ e_2, e_3) - \zeta_0(e_2, e_1 \circ e_3) + \zeta_0(e_1, e_2 \circ e_3) \\
= \; & \rho(e_1)(e_2, e_3) - (e_1 \circ e_2, e_3) - (e_2, e_1 \circ e_3) \\
& -\rho(e_2)(e_1, e_3) + (e_1, e_2 \circ e_3) + \rho(e_3)(e_1, e_2) \\
= \; & -(e_2 \circ e_1, e_3) + \rho(e_3)\zeta_0(e_1, e_2) \\
= \; & (e_1 \circ e_2, e_3)
\end{aligned}$$

$$(d\zeta)_1(e; f) = \rho(e)\zeta_1(f) + \zeta_0(f, e) = -2(e, f) + (e, f) = -(e, f)$$

so $\Theta = d\zeta$ is a 3-coboundary. ■

Actually Θ is exactly the restriction of the canonical 3-cocycle of the Courant-Dorfman algebra $S^\bullet(Z) \otimes L$ (see Remark 2.3). We will call Θ the canonical 3-cocycle of L.

3. Poisson structure on a subcomplex

In this section, we consider a subcomplex, denoted by $\tilde{C}(L)$, consisting of the so-called "representable cochains", and construct a Poisson algebra structure on $\tilde{C}(L)$.

Let $L^\vee \triangleq Hom(L,\ S^\bullet(Z))$. In this paper, Hom always means $\mathfrak{k}$-linear homomorphisms.

$\forall \omega \in C^n(L)$, ω_k gives rise to a map $\bar{\omega}_k :\ L^{\otimes n-2k-1} \otimes S^k(Z) \to L^\vee$:

$$\bar{\omega}_k(e_1, \cdots, e_{n-2k-1}; f_1, \cdots f_k)(e) \triangleq (\iota_{f_k} \cdots \iota_{f_1} \iota_{e_{n-2k-1}} \cdots \iota_{e_1} \omega_k)(e) = \omega_k(e_1, \cdots, e_{n-2k-1}, e; f_1, \cdots, f_k).$$

The symmetric product $(\cdot, \cdot)$ of L can be $S^\bullet(Z)$-linearly extended to a symmetric product on $S^\bullet(Z) \otimes L$, thus inducing a map

$$\phi \triangleq (\cdot,\ ) :\ S^\bullet(Z) \otimes L \to L^\vee.$$

Definition 3.1. Given any $\omega \in C^n(L)$, if $Im(\bar{\omega}_k) \subseteq Im(\phi)$, $\forall k$, we call ω a "representable cochain". The graded subspace of $C(L)$ consisting of all representable cochains is denoted by $\tilde{C}(L)$.

By definition, $e^\flat \triangleq (e,\ ) :\ L \to Z$ is obviously a representable cochain.

Given $\omega \in \tilde{C}^n(L)$, ω_k induces a $\mathfrak{k}$-linear map

$$\tilde{\omega}_k :\ L^{\otimes n-2k-1} \to Hom(S^k(Z),\ S^\bullet(Z) \otimes L),$$

which is defined by

$$\tilde{\omega}_k(e_1, \cdots, e_{n-2k-1})(f_1, \cdots, f_k) \triangleq \phi^{-1}(\bar{\omega}_k(e_1, \cdots, e_{n-2k-1}; f_1, \cdots, f_k)).$$

Note that, to determine $\tilde{\omega}_k$, we only need to choose the preimage of $\bar{\omega}_k$ for given basis of L and Z, and then take the $\mathfrak{k}$-linear extension. So $\tilde{\omega}_k$ depends on the choices, it is not uniquely determined unless ϕ is injective (i.e. the bilinear product of L is non-degenerate).

Proposition 3.2. *$\tilde{C}(L)$ is a subcomplex of $C(L)$.*

Proof. $\forall \omega \in \tilde{C}^n(L)$, we need to prove that $d\omega \in \tilde{C}^{n+1}(L)$:

$$\begin{aligned}
& (d\omega)_k(e_1, \cdots e_{n+1-2k}; f_1, \cdots f_k) \\
= & \sum_a (-1)^{a+1}\rho(e_a)\omega_k(\cdots \widehat{e_a}, \cdots ; \cdots) + \sum_{a<b}(-1)^a \omega_k(\cdots \widehat{e_a}, \cdots e_a \circ e_b, \cdots ; \cdots) \\
& + \sum_i \omega_{k-1}(f_i, e_1, \cdots ; \cdots \widehat{f_i}, \cdots) \\
= & \sum_{a \le n-2k} (-1)^{a+1}\rho(e_a)(\tilde{\omega}_k(e_1, \cdots \widehat{e_a}, \cdots e_{n-2k})(f_1, \cdots f_k), e_{n+1-2k}) \\
& +(-1)^n \rho(e_{n+1-2k})\omega_k(e_1, \cdots e_{n-2k}; \cdots) \\
& + \sum_{a<b\le n-2k} (-1)^a (\tilde{\omega}_k(\cdots \widehat{e_a}, \cdots e_a \circ e_b, \cdots e_{n-2k})(f_1, \cdots f_k), e_{n+1-2k}) \\
& + \sum_{a \le n-2k} (-1)^a \omega_k(\cdots \widehat{e_a}, \cdots e_{n-2k}, e_a \circ e_{n+1-2k}; \cdots) \\
& + \sum_i (\tilde{\omega}_{k-1}(f_i, e_1, \cdots e_{n-2k})(\cdots \widehat{f_i}, \cdots), e_{n+1-2k}) \\
= & (\bullet, e_{n+1-2k}).
\end{aligned}$$

The proof is finished. ∎

Next, we will define a graded bracket on $\tilde{C}(L)$.

$\forall \alpha \in Hom(S^k(Z),\ S^\bullet(Z) \otimes L)$, $\beta \in Hom(S^l(Z),\ S^\bullet(Z) \otimes L)$, define $\langle \alpha \cdot \beta \rangle \in Hom(S^{k+l}(Z),\ S^\bullet(Z))$ as

$$\langle \alpha \cdot \beta \rangle (f_1, \cdots, f_{k+l}) \triangleq \sum_{\sigma \in sh(k,l)} (\alpha(f_{\sigma(1)}, \cdots, f_{\sigma(k)}),\ \beta(f_{\sigma(k+1)}, \cdots, f_{\sigma(k+l)})).$$

$\forall \gamma \in Hom(S^k(Z),\ S^\bullet(Z))$, $\delta \in Hom(S^l(Z),\ S^\bullet(Z))$, define $\gamma \circ \delta \in Hom(S^{k+l-1}(Z),\ S^\bullet(Z))$ as

$$\gamma \circ \delta (f_1, \cdots, f_{k+l-1}) \triangleq \sum_{\sigma \in sh(l,k-1)} \check{\gamma}(\delta(f_{\sigma(1)}, \cdots, f_{\sigma(l)}), f_{\sigma(l+1)}, \cdots, f_{\sigma(l+k-1)}),$$

where $\check{\gamma}:\ S^\bullet(Z) \otimes S^{k-1}(Z) \to S^\bullet(Z)$ is extended from γ by Leibniz rule in the first argument.

Now given $\omega \in \tilde{C}^n(L)$, $\eta \in \tilde{C}^m(L)$, we define the bracket $\{\omega, \eta\}$ as follows:

$$\{\omega, \eta\} \triangleq \omega \bullet \eta + \omega \diamond \eta - (-1)^{nm} \eta \diamond \omega, \tag{3.1}$$

where $\omega \bullet \eta = ((\omega \bullet \eta)_0, (\omega \bullet \eta)_1, \cdots)$, with $(\omega \bullet \eta)_k :\ \otimes^{n+m-2-2k} L \to Hom(S^k(Z),\ S^\bullet(Z))$ defined by

$$\begin{aligned}
& (\omega \bullet \eta)_k(e_1, \cdots, e_{n+m-2-2k}) \\
\triangleq\ & (-1)^{m-1} \sum_{\substack{i+j=k \\ \sigma \in sh(n-2i-1, m-2j-1)}} (-1)^\sigma \langle \tilde{\omega}_i(e_{\sigma(1)}, \cdots e_{\sigma(n-2i-1)}) \cdot \tilde{\eta}_j(e_{\sigma(n-2i)}, \cdots e_{\sigma(n+m-2-2k)}) \rangle,
\end{aligned}$$

(obviously the value does not depend on the choices of $\tilde{\omega}_i$ and $\tilde{\eta}_j$, so it is well-defined)

and $\omega \diamond \eta = ((\omega \diamond \eta)_0, (\omega \diamond \eta)_1, \cdots)$, with $(\omega \diamond \eta)_k :\ \otimes^{n+m-2-2k} L \to Hom(S^k(Z),\ S^\bullet(Z))$ defined by

$$\begin{aligned}
& (\omega \diamond \eta)_k(e_1, \cdots, e_{n+m-2-2k}) \\
\triangleq\ & \sum_{\substack{i+j=k \\ \sigma \in sh(n-2i-2, m-2j)}} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)} \cdots e_{\sigma(n-2i-2)}) \circ \eta_j(e_{\sigma(n-2i-1)} \cdots e_{\sigma(n+m-2-2k)}).
\end{aligned}$$

The following is the main theorem of this section:

Theorem 3.3. $(\tilde{C}(L), \{\cdot,\cdot\})$ *is a graded Poisson algebra.*

Before the proof of this theorem, we prove the following two lemmas first.

Lemma 3.4. $\tilde{C}(L)$ *is a subalgebra of* $C(L)$ *with the multiplication map defined in 2.1.*

Proof. Given $\eta \in \tilde{C}^m(L), \lambda \in \tilde{C}^l(L)$, we need to prove that $\eta\lambda \in \tilde{C}^{m+l}(L)$:

$$\begin{aligned}
&(\eta\lambda)_k(e_1,\cdots,e_{m+l-2k};f_1,\cdots,f_k)\\
= &\sum_{\substack{i+j=k\\ \sigma\in sh(m-2i,l-2j)\\ \tau\in sh(i,j)}} (-1)^\sigma \eta_i(e_{\sigma(1)}\cdots;f_{\tau(1)}\cdots f_{\tau(i)})\lambda_j(e_{\sigma(m-2i+1)}\cdots e_{\sigma(m+l-2k)};f_{\tau(i+1)}\cdots f_{\tau(k)})\\
= &\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(m-2i,l-2j)\\ \sigma^{-1}(m+l-2k)=m-2i}} (-1)^\sigma(\tilde{\eta}_i(\cdots,\widehat{e_{m+l-2k}};\cdots),e_{m+l-2k})\lambda_j(\cdots;\cdots)\\
&+\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\substack{\sigma\in sh(m-2i,l-2j)\\ \sigma^{-1}(m+l-2k)=m+l-2k}} (-1)^\sigma\eta_i(\cdots;\cdots)(\tilde{\lambda}_j(\cdots,\widehat{e_{m+l-2k}};\cdots),e_{m+l-2k})\\
= &(\{\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\bar{\sigma}\in sh(m-2i-1,l-2j)} (-1)^{\bar{\sigma}+l}\tilde{\eta}_i(e_{\bar{\sigma}(1)},\cdots;f_{\tau(1)},\cdots)\lambda_j(e_{\bar{\sigma}(m-2i)}\cdots;f_{\tau(i+1)},\cdots)\\
&+\sum_{\substack{i+j=k\\ \tau\in sh(i,j)}}\sum_{\bar{\sigma}\in sh(m-2i,l-2j-1)} (-1)^{\bar{\sigma}}\eta_i(e_{\bar{\sigma}(1)}\cdots;f_{\tau(1)},\cdots)\tilde{\lambda}_j(e_{\bar{\sigma}(m-2i+1)}\cdots;f_{\tau(i+1)},\cdots)\},e_{m+l-2k})
\end{aligned}$$

The lemma is proved. ■

Lemma 3.5. $\omega\bullet\eta,\ \omega\diamond\eta,\ \{\omega,\eta\}$ *are all cochains in* $C^{n+m-2}(L)$.

Proof. $\omega\bullet\eta$ is a cochain in $C^{n+m-2}(L)$ because:

$$\begin{aligned}
&(\omega\bullet\eta)_k(e_1\cdots e_a,e_{a+1}\cdots e_{n+m-2-2k};f_1\cdots f_k)+(\omega\bullet\eta)_k(\cdots e_{a+1},e_a\cdots;\cdots)\\
= &(-1)^{m-1}\sum_{i+j=k}\{\sum_{\sigma,a\in\omega,a+1\in\eta}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots e_a\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_{a+1}\cdots e_{\sigma(n+m-2-2k)})\rangle\\
&\quad+\sum_{\sigma,a\in\eta,a+1\in\omega}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots e_a\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_{a+1}\cdots e_{\sigma(n+m-2-2k)})\rangle\}\\
&+(-1)^{m-1}\sum_{i+j=k}\{\sum_{\sigma,a\in\eta,a+1\in\omega}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots e_{a+1},\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots,e_a\cdots e_{\sigma(n+m-2-2k)})\rangle\\
&\quad+\sum_{\sigma,a\in\omega,a+1\in\eta}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots e_{a+1}\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_a\cdots e_{\sigma(n+m-2-2k)})\rangle\}\\
&+(-1)^{m-1}\sum_{i+j=k}\{\sum_{\sigma,a\in\omega,a+1\in\omega}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots e_a,e_{a+1}\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_{\sigma(n+m-2-2k)})\rangle\\
&\quad+\sum_{\sigma,a\in\omega,a+1\in\omega}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots e_{a+1},e_a\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_{\sigma(n+m-2-2k)})\rangle\}\\
&+(-1)^{m-1}\sum_{i+j=k}\{\sum_{\sigma,a\in\eta,a+1\in\eta}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_a,e_{a+1}\cdots e_{\sigma(n+m-2-2k)})\rangle\\
&\quad+\sum_{\sigma,a\in\eta,a+1\in\eta}(-1)^\sigma\langle\tilde{\omega}_i(e_{\sigma(1)}\cdots)\cdot\tilde{\eta}_j(e_{\sigma(n-2i)}\cdots e_{a+1},e_a\cdots)\rangle\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & (-1)^{m-1}\sum_{i+j=k}\sum_{\substack{\sigma\in sh(n-2i-1,m-2j-1)\\ a\in\omega,a+1\in\omega}}(-1)^{\sigma}(-1)\\
&\langle\tilde{\omega_{i+1}}(e_{\sigma(1)}\cdots\widehat{e_a},\widehat{e_{a+1}}\cdots e_{\sigma(n-2i-1)};(e_a,e_{a+1}))\cdot\tilde{\eta_j}(e_{\sigma(n-2i)}\cdots)\rangle\\
&+(-1)^{m-1}\sum_{i+j=k}\sum_{\substack{\sigma\in sh(n-2i-1,m-2j-1)\\ a\in\omega,a+1\in\omega}}(-1)^{\sigma}(-1)\\
&\langle\tilde{\omega_i}(e_{\sigma(1)}\cdots)\cdot\tilde{\eta_{j+1}}(e_{\sigma(n-2i)}\cdots\widehat{e_a},\widehat{e_{a+1}}\cdots e_{\sigma(n+m-2-2k)};(e_a,e_{a+1}))\rangle\\
= \quad & (-1)^{m}\sum_{i'+j=k+1}\sum_{\sigma'\in sh(n-2i'-1,m-2j-1)}(-1)^{\sigma'}\sum_{\substack{\tau\in sh(i',j)\\ (e_a,e_{a+1})\in\omega}}\\
&(\tilde{\omega_{i'}}(e_{\sigma'(1)}\cdots)((e_a,e_{a+1}),f_{\tau(1)}\cdots),\tilde{\eta_j}(e_{\sigma'(n-2i')}\cdots)(f_{\tau(i')}\cdots f_{\tau(k)}))\\
&+(-1)^{m}\sum_{i+j'=k+1}\sum_{\sigma'\in sh(n-2i-1,m-2j'-1)}(-1)^{\sigma'}\sum_{\substack{\tau\in sh(i',j)\\ (e_a,e_{a+1})\in\eta}}\\
&(\tilde{\omega_i}(e_{\sigma'(1)}\cdots)(f_{\tau(1)}\cdots),\tilde{\eta_{j'}}(e_{\sigma'(n-2i)}\cdots)((e_a,e_{a+1}),f_{\tau(i+1)}\cdots f_{\tau(k)}))\\
= \quad & -(\omega\bullet\eta)_{k+1}(e_1,\cdots,\widehat{e_a},\widehat{e_{a+1}},\cdots,e_{n+m-2-2k};(e_a,e_{a+1}),f_1,\cdots,f_k).
\end{aligned}
$$

$\omega\diamond\eta$ is a cochain in $C^{n+m-2}(L)$ because:

$$
\begin{aligned}
&(\omega\diamond\eta)_k(\cdots,e_a,e_{a+1}\cdots;f_1\cdots f_k)+(\omega\diamond\eta)_k(\cdots e_{a+1},e_a\cdots;\cdots)\\
= \quad & \sum_{i+j=k}\{\sum_{\sigma,a\in\omega,a+1\in\eta}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots e_a\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_{a+1}\cdots)\\
&\quad+\sum_{\sigma,a\in\eta,a+1\in\omega}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots e_a\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_{a+1}\cdots)\}\\
&+\sum_{i+j=k}\{\sum_{\sigma,a\in\eta,a+1\in\omega}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots e_{a+1}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_a\cdots)\\
&\quad+\sum_{\sigma,a\in\omega,a+1\in\eta}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots e_{a+1}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_a\cdots)\}\\
&+\sum_{i+j=k}\{\sum_{\sigma,a\in\omega,a+1\in\omega}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots e_a,e_{a+1}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots)\\
&\quad+\sum_{\sigma,a\in\omega,a+1\in\omega}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots e_{a+1},e_a\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots)\}\\
&+\sum_{i+j=k}\{\sum_{\sigma,a\in\eta,a+1\in\eta}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_a,e_{a+1}\cdots)\\
&\quad+\sum_{\sigma,a\in\eta,a+1\in\eta}(-1)^{\sigma}\omega_{i+1}(e_{\sigma(1)}\cdots)\circ\eta_j(e_{\sigma(n-2i-1)}\cdots e_{a+1},e_a\cdots)\}\\
= \quad & \sum_{i+j=k}\sum_{\substack{\sigma\in sh(n-2i-2,m-2j)\\ a\in\omega,a+1\in\omega}}(-1)^{\sigma}(-1)\\
&\omega_{i+2}(e_{\sigma(1)}\cdots\widehat{e_a},\widehat{e_{a+1}}\cdots e_{\sigma(n-2i-2)};(e_a,e_{a+1}))\circ\eta_j(e_{\sigma(n-2i-1)}\cdots)\\
&+\sum_{i+j=k}\sum_{\substack{\sigma\in sh(n-2i-2,m-2j)\\ a\in\eta,a+1\in\eta}}(-1)^{\sigma}(-1)
\end{aligned}
$$

$$
\begin{aligned}
& \omega_{i+1}(e_{\sigma(1)}\cdots e_{\sigma(n-2i-2)})\circ\eta_{j+1}(e_{\sigma(n-2i-1)}\cdots\widehat{e_{a+1}},\widehat{e_a}\cdots;(e_a,e_{a+1}))\\
= & \sum_{i'+j=k+1}\ \sum_{\sigma'\in sh(n-2i'-2,m-2j)}(-1)^{\sigma'}\sum_{\substack{\tau\in sh(j,i'-1)\\(e_a,e_{a+1})\in\omega}}\\
& \omega_{i'+1}(e_{\sigma'(1)}\cdots;\eta_j(e_{\sigma'(n-2i'-1)}\cdots;f_{\tau(1)}\cdots),(e_a,e_{a+1}),f_{\tau(j+1)}\cdots f_{\tau(k)})\\
& +\sum_{i+j'=k+1}\ \sum_{\sigma'\in sh(n-2i-2,m-2j')}(-1)^{\sigma'}\sum_{\substack{\tau\in sh(j,i'-1)\\(e_a,e_{a+1})\in\eta}}\\
& \omega_{i+1}(e_{\sigma'(1)}\cdots;\eta_{j'}(e_{\sigma'(n-2i-1)}\cdots;(e_a,e_{a+1}),f_{\tau(1)}\cdots),f_{\tau(j')}\cdots f_{\tau(k)})\\
= & -(\omega\diamond\eta)_{k+1}(e_1,\cdots,\widehat{e_a},\widehat{e_{a+1}},\cdots,e_{n+m-2-2k};(e_a,e_{a+1}),f_1,\cdots,f_k).
\end{aligned}
$$

So $\{\omega,\eta\}=\omega\bullet\eta+\omega\diamond\eta-(-1)^{nm}\eta\diamond\omega$ is also a cochain in $C^{n+m-2}(L)$. ■

Proof of theorem 3.3:

Proof. 1) By the lemmas above, in order for $\tilde{C}(L)$ to be a graded Poisson algebra, we need to prove the following:

(1). For any two representable cochains ω,η, $\{\omega,\eta\}=-(-1)^{nm}\{\eta,\omega\}$,

(2). For any representable cochains ω,η,λ,

$$\{\omega,\eta\lambda\}=\{\omega,\eta\}\lambda+(-1)^{nm}\eta\{\omega,\lambda\},$$

(3). The bracket of any two representable cochains is still a representable cochain, and

$$\{\omega,\{\eta,\lambda\}=\{\{\omega,\eta\},\lambda\}+(-1)^{nm}\{\eta,\{\omega,\lambda\}\}.$$

For (1), it suffices to prove $\omega\bullet\eta=-(-1)^{nm}\eta\bullet\omega$.

$\forall\sigma\in sh(n-2i-1,m-2j-1)$, switching the first $n-2i-1$ arguments with the last $m-2j-1$ arguments results in a sign difference $(-1)^{(n-1)(m-1)}$, so by definition there is merely a sign difference between $\omega\bullet\eta$ and $\eta\bullet\omega$ of $(-1)^{n-m+(n-1)(m-1)}=(-1)^{nm+1}$.

Thus (1) is proved.

For (2), we need to prove that $\{\omega,\bullet\}$ is a graded derivative.

$$
\begin{aligned}
& \{\omega,\eta\lambda\}_k(e_1,\cdots,e_{n+m+l-2-2k};f_1,\cdots,f_k)\\
= & (\omega\bullet\eta\lambda)_k(\cdots)+(\omega\diamond\eta\lambda)_k(\cdots)+(-1)^{n(m+l)+1}(\eta\lambda\diamond\omega)_k(\cdots)
\end{aligned}
$$

We calculate the three parts above respectively:

$$
\begin{aligned}
&(\omega \bullet \eta\lambda)_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\
= \ & (-1)^{m+l+1} \sum_{\substack{a+b=k \\ \sigma\in sh(n-2a-1,m+l-2b-1) \\ \tau\in sh(a,b)}} (\tilde{\omega_a}(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots), \widetilde{(\eta\lambda)}_b(e_{\sigma(n-2a)}, \cdots; f_{\tau(a+1)}, \cdots)) \\
= \ & (-1)^{m+l+1} \sum_{\substack{a+b+c=k \\ \sigma\in sh(n-2a-1,m-2b-1,l-2c) \\ \tau\in sh(a,b,c)}} (-1)^{\sigma+l}(\tilde{\omega_a}(e_{\sigma(1)} \cdots), \tilde{\eta_b}(e_{\sigma(n-2a)} \cdots)\lambda_c(e_{\sigma(n+m-2a-2b-1)} \cdots)) \\
& +(-1)^{m+l+1} \sum_{\substack{a+b+c=k \\ \sigma\in sh(n-2a-1,m-2b,l-2c-1) \\ \tau\in sh(a,b,c)}} (-1)^{\sigma}(\tilde{\omega_a}(e_{\sigma(1)} \cdots), \eta_b(e_{\sigma(n-2a)} \cdots)\tilde{\lambda_c}(e_{\sigma(n+m-2a-2b)} \cdots)) \\
= \ & \sum_{\substack{a+c=k \\ \sigma\in sh(n+m-2a-2,l-2c) \\ \tau\in sh(a,c)}} (-1)^{\sigma}(\omega \bullet \eta)_a(e_{\sigma(1)}, \cdots)\lambda_c(e_{\sigma(n+m-2a-1)}, \cdots) \\
& + \sum_{\substack{b+a=k \\ \sigma\in sh(m-2b,n+l-2a-2) \\ \tau\in sh(b,a)}} (-1)^{\sigma+(n-1)m}(-1)^m \eta_b(e_{\sigma(1)}, \cdots)(\omega \bullet \lambda)_a(e_{\sigma(m-2b+1)}, \cdots) \\
= \ & \big((\omega \bullet \eta) \cdot \lambda\big)_k(\cdots) + (-1)^{nm}\big(\eta \cdot (\omega \bullet \lambda)\big)_k(\cdots)
\end{aligned}
$$

$$
\begin{aligned}
&(\omega \diamond \eta\lambda)_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\
= \ & \sum_{\substack{a+b=k \\ \sigma\in sh(n-2a-2,m+l-2b) \\ \tau\in sh(b,a)}} (-1)^{\sigma}\omega_{a+1}(e_{\sigma(1)} \cdots e_{\sigma(n-2a-2)}; (\eta\lambda)_b(e_{\sigma(n-2a-1)} \cdots; f_{\tau(1)} \cdots), f_{\tau(b+1)} \cdots) \\
= \ & \sum_{\substack{a+b+c=k \\ \sigma\in sh(n-2a-2,m-2b,l-2c) \\ \tau\in sh(b,c,a)}} (-1)^{\sigma}\omega_{a+1}(e_{\sigma(1)} \cdots; \eta_b(e_{\sigma(n-2a-1)} \cdots)\lambda_c(e_{\sigma(n+m-2a-2b-1)} \cdots), \cdots) \\
= \ & \sum_{\substack{a+b+c=k \\ \sigma\in sh(n-2a-2,m-2b,l-2c) \\ \tau\in sh(b,a,c)}} (-1)^{\sigma}\omega_{a+1}(e_{\sigma(1)} \cdots; \eta_b(e_{\sigma(n-2a-1)} \cdots), \cdots)\lambda_c(e_{\sigma(n+m-2a-2b-1)} \cdots) \\
& + \sum_{\substack{a+b+c=k \\ \sigma\in sh(m-2b,n-2a-2,l-2c) \\ \tau\in sh(b,c,a)}} (-1)^{\sigma+nm}\eta_b(e_{\sigma(1)} \cdots)\omega_{a+1}(e_{\sigma(m-2b+1)} \cdots; \lambda_c(e_{\sigma(n+m-2a-2b-1)} \cdots), \cdots) \\
= \ & \sum_{\substack{a+c=k \\ \sigma\in sh(n+m-2a-2,l-2c) \\ \tau\in sh(a,c)}} (-1)^{\sigma}(\omega \diamond \eta)_a(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots)\lambda_c(e_{\sigma(n+m-2a-1)}, \cdots; f_{\tau(a+1)}, \cdots) \\
& +(-1)^{nm} \sum_{\substack{a+b=k \\ \sigma\in sh(m-2b,n+l-2a-2) \\ \tau\in sh(b,a)}} (-1)^{\sigma}\eta_b(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots)(\omega \diamond \lambda)_a(e_{\sigma(m-2b+1)}, \cdots; f_{\tau(b+1)}, \cdots) \\
= \ & \big((\omega \diamond \eta) \cdot \lambda\big)_k(\cdots) + (-1)^{nm}\big(\eta \cdot (\omega \diamond \lambda)\big)_k(\cdots)
\end{aligned}
$$

$$
\begin{aligned}
& (\eta\lambda \diamond \omega)_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\
= & \sum_{\substack{a+c=k \\ \sigma \in sh(m+l-2a-2, n-2c) \\ \tau \in sh(c,a)}} (-1)^{\sigma} (\eta\lambda)_{a+1}(e_{\sigma(1)}, \cdots; \omega_c(e_{\sigma(m+l-2a-1)}, \cdots), \cdots) \\
= & \sum_{\substack{a+b+c=k \\ \sigma \in sh(m-2a-2, n-2c, l-2b) \\ \tau \in sh(c,a,b)}} (-1)^{\sigma+nl} \eta_{a+1}(e_{\sigma(1)} \cdots; \omega_c(e_{\sigma(m-2a-1)} \cdots), \cdots) \lambda_b(e_{\sigma(n+m-2a-2c-1)} \cdots) \\
& + \sum_{\substack{a+b+c=k \\ \sigma \in sh(m-2a, l-2b-2, n-2c) \\ \tau \in sh(a,c,b)}} (-1)^{\sigma} \eta_a(e_{\sigma(1)} \cdots) \lambda_{b+1}(e_{\sigma(m-2a+1)} \cdots; \omega_c(e_{\sigma(m+l-2a-2b-1)} \cdots), \cdots) \\
= & \sum_{\substack{a+b=k \\ \sigma \in sh(m+n-2a-2, l-2b) \\ \tau \in sh(a,b)}} (-1)^{\sigma} (-1)^{nl} (\eta \diamond \omega)_a(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots) \lambda_b(e_{\sigma(n+m-2a-1)}, \cdots; f_{\tau(a+1)}, \cdots) \\
& + \sum_{\substack{a+b=k \\ \sigma \in sh(m-2a, n+l-2b-2) \\ \tau \in sh(a,b)}} (-1)^{\sigma} \eta_a(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots) \cdot (\lambda \diamond \omega)_b(e_{\sigma(m-2a+1)}, \cdots; f_{\tau(a+1)}, \cdots) \\
= & (-1)^{nl} \big((\eta \diamond \omega) \cdot \lambda\big)_k(\cdots) + \big(\eta \cdot (\lambda \diamond \omega)\big)_k(\cdots)
\end{aligned}
$$

So

$$
\begin{aligned}
& \{\omega, \eta\lambda\}_k(e_1, \cdots, e_{n+m+l-2-2k}; f_1, \cdots, f_k) \\
= & \big((\omega \bullet \eta) \cdot \lambda\big)_k(\cdots) + (-1)^{nm}\big(\eta \cdot (\omega \bullet \lambda)\big)_k(\cdots) \\
& + \big((\omega \diamond \eta) \cdot \lambda\big)_k(\cdots) + (-1)^{nm}\big(\eta \cdot (\omega \diamond \lambda)\big)_k(\cdots) \\
& + (-1)^{nm+1}\big((\eta \diamond \omega) \cdot \lambda\big)_k(\cdots) + (-1)^{nm}(-1)^{nl+1}\big(\eta \cdot (\lambda \diamond \omega)\big)_k(\cdots) \\
= & (\{\omega, \eta\} \cdot \lambda)_k(\cdots) + (-1)^{nm}(\eta \cdot \{\omega, \lambda\})_k(\cdots)
\end{aligned}
$$

$\{\omega, \bullet\}$ is a graded derivative, (2) is proved.

For (3), in order for $\{\omega, \eta\}$ to be a representable cochain, we need to prove

$$\{\omega, \eta\}_k(e_1, \cdots, e_{n+m-2k}; f_1, \cdots, f_k) = (e_{n+m-2k}, \bullet), \ \forall k.$$

$$
\begin{aligned}
& \{\omega, \eta\}_k(e_1, \cdots, e_{n+m-2-2k}; f_1, \cdots, f_k) \\
= & (-1)^{m+1} \sum_{\substack{a+b=k \\ \sigma \in sh(n-2a-1, m-2b-1) \\ \tau \in sh(a,b)}} (-1)^{\sigma} (\tilde{\omega_a}(e_{\sigma(1)}, \cdots; f_{\tau(1)}, \cdots), \tilde{\eta_b}(e_{\sigma(n-2a)}, \cdots; f_{\tau(a+1)}, \cdots)) \\
& + \sum_{\substack{a+b=k \\ \sigma \in sh(n-2a-2, m-2b) \\ \tau \in sh(b,a)}} (-1)^{\sigma} \omega_{a+1}(e_{\sigma(1)}, \cdots; \eta_b(e_{\sigma(n-2a-1)}, \cdots; f_{\tau(1)}, \cdots), f_{\tau(b+1)}, \cdots) \\
& + (-1)^{nm+1} \sum_{\substack{a+b=k \\ \sigma \in sh(m-2a-2, n-2b) \\ \tau \in sh(b,a)}} (-1)^{\sigma} \eta_{a+1}(e_{\sigma(1)}, \cdots; \omega_b(e_{\sigma(m-2a-1)}, \cdots; f_{\tau(1)}, \cdots), f_{\tau(b+1)}, \cdots) \\
= & (-1)^{m+1} \sum_{\substack{a+b=k \\ \sigma \in sh(n-2a-2, m-2b-1) \\ \tau \in sh(b,a)}} (-1)^{\sigma+m+1} \\
& \omega_a(e_{\sigma(1)} \cdots e_{\sigma(n-2a-2)}, e_{n+m-2-2k}, \tilde{\eta_b}(e_{\sigma(n-2a-1)} \cdots; f_{\tau(1)} \cdots); f_{\tau(b+1)} \cdots)
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{m+1} \sum_{\substack{a+b=k \\ \sigma\in sh(m-2b-2,n-2a-1) \\ \tau\in sh(a,b)}} (-1)^{\sigma+(n-1)m} \\
&\quad \eta_b(e_{\sigma(1)}\cdots e_{\sigma(m-2b-2)}, e_{n+m-2-2k}, \tilde{\omega_a}(e_{\sigma(m-2b-1)}\cdots; f_{\tau(1)}\cdots); f_{\tau(a+1)}\cdots) \\
&+\{(e_{n+m-2-2k}, \bullet) + \sum_{\substack{a+b=k \\ \sigma\in sh(n-2a-2,m-2b-1) \\ \tau\in sh(b,a)}} (-1)^{\sigma} \\
&\quad \omega_{a+1}(e_{\sigma(1)},\cdots; (\tilde{\eta_b}(e_{\sigma(n-2a-1)},\cdots; f_{\tau(1)},\cdots), e_{n+m-2-2k}), f_{\tau(b+1)},\cdots)\} \\
&+\{(e_{n+m-2-2k}, \bullet) + (-1)^{nm+1} \sum_{\substack{a+b=k \\ \sigma\in sh(m-2a-2,n-2b-1) \\ \tau\in sh(b,a)}} (-1)^{\sigma} \\
&\quad \eta_{a+1}(e_{\sigma(1)},\cdots; (\tilde{\omega_b}(e_{\sigma(m-2a-1)},\cdots; f_{\tau(1)},\cdots), e_{n+m-2-2k}), f_{\tau(b+1)},\cdots)\} \\
= \;& (e_{n+m-2-2k}, \bullet) \\
&+ \sum_{\substack{a+b=k \\ \sigma\in sh(n-2a-2,m-2b-1) \\ \tau\in sh(b,a)}} (-1)^{\sigma}\omega_a(\cdots, e_{n+m-2-2k}, \tilde{\eta_b}(\cdots);\cdots) \\
&+ \sum_{\substack{a+b=k \\ \sigma\in sh(m-2b-2,n-2a-1) \\ \tau\in sh(a,b)}} (-1)^{\sigma+nm+1}\eta_b(\cdots, e_{n+m-2-2k}, \tilde{\omega_a}(\cdots);\cdots) \\
&+ \sum_{\substack{a+b=k \\ \sigma\in sh(n-2a-2,m-2b-1) \\ \tau\in sh(b,a)}} (-1)^{\sigma+1} \\
&\quad \{\omega_a(\cdots, e_{n+m-2-2k}, \tilde{\eta_b}(\cdots);\cdots) + \omega_a(\cdots, \tilde{\eta_b}(\cdots), e_{n+m-2-2k};\cdots)\} \\
&+ \sum_{\substack{a+b=k \\ \sigma\in sh(m-2a-2,n-2b-1) \\ \tau\in sh(b,a)}} (-1)^{\sigma+nm} \\
&\quad \{\eta_a(\cdots, e_{n+m-2-2k}, \tilde{\omega_b}(\cdots);\cdots) + \eta_a(\cdots, \tilde{\omega_b}(\cdots), e_{n+m-2-2k};\cdots)\} \\
= \;& (e_{n+m-2-2k}, \bullet)
\end{aligned}
$$

Thus (3) is proved. ■

If ϕ is an isomorphism(i.e. the symmetric product of $S^\bullet(Z)\otimes L$ is strongly non-degenerate), any $\omega \in C(L)$ is a representable cochain, so $C(L) = \tilde{C}(L)$ is a graded commutative Poisson algebra.

4. Derived brackets

In this section we prove that the Leibniz bracket of a fat Leibniz algebra is a derived bracket. First we prove the following:

Proposition 4.1. $\{\Theta, \eta\} = -d\eta, \ \forall \eta \in \tilde{C}(L)$.

Proof. It is obvious that Θ is a representable cochain.

$$
\begin{aligned}
& (\Theta\bullet\eta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
= & (-1)^{m-1}\sum_{\sigma\in sh(2,m-2k-1)}(-1)^{\sigma}(\tilde{\Theta_0}(e_{\sigma(1)},e_{\sigma(2)}),\tilde{\eta_k}(e_{\sigma(3)}\cdots e_{\sigma(m+1-2k)};\cdots))\\
& +(-1)^{m-1}\sum_{\tau\in sh(1,k-1)}(\tilde{\Theta_1}(f_{\tau(1)}),\tilde{\eta_{k-1}}(e_1,\cdots,e_{m+1-2k};f_{\tau(2)},\cdots,f_{\tau(k)}))\\
= & (-1)^{m-1}\sum_{a<b}(-1)^{a+b+1}(e_a\circ e_b,\tilde{\eta_k}(e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots e_{m+1-2k};f_1,\cdots f_k))\\
& +(-1)^{m-1}\sum_i(-1)(f_i,\tilde{\eta_{k-1}}(e_1,\cdots,e_{m+1-2k};f_1,\cdots,\hat{f_i},\cdots,f_k))\\
= & (-1)^m\sum_{a<b}(-1)^{a+b}\eta_k(e_1,\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots,e_{m+1-2k},e_a\circ e_b;f_1,\cdots,f_k)\\
& +(-1)^m\sum_i\eta_{k-1}(e_1,\cdots,e_{m+1-2k},f_i;f_1,\cdots,\hat{f_i},\cdots,f_k)
\end{aligned}
$$

$$
\begin{aligned}
& (\Theta\diamond\eta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
= & \sum_a(-1)^{a+1}\Theta_1(e_a)\circ\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k})\\
= & \sum_a(-1)^{a+1}(-1)(e_a,\eta_k(e_1,\cdots,\hat{e_a},\cdots e_{m+1-2k};f_1,\cdots,f_k))\\
= & \sum_a(-1)^a\rho(e_a)\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k};f_1,\cdots,f_k)
\end{aligned}
$$

$$
\begin{aligned}
& (-1)^{m+1}(\eta\diamond\Theta)_k(e_1,\cdots,e_{m+1-2k};f_1,\cdots,f_k)\\
= & (-1)^{m+1}\sum_{\sigma\in sh(m-2k-2,3)}(-1)^{\sigma}\\
& \eta_{k+1}(e_{\sigma(1)},\cdots,e_{\sigma(m-2k-2)})\circ\Theta_0(e_{\sigma(m-2k-1)},e_{\sigma(m-2k)},e_{\sigma(m-2k+1)})\\
& +(-1)^{m+1}\sum_a(-1)^{a+m+1}\eta_k(e_1,\cdots\hat{e_a},\cdots,e_{m+1-2k})\circ\Theta_1(e_a)\\
= & \sum_{a<b<c}(-1)^{a+b+c+1}\eta_{k+1}(e_1\cdots\hat{e_a}\cdots\hat{e_b}\cdots\hat{e_c}\cdots e_{m+1-2k};(e_a\circ e_b,e_c),f_1\cdots f_k)\\
& +\sum_a(-1)^a\sum_i\eta_k(e_1,\cdots,\hat{e_a},\cdots,e_{m+1-2k};-(e_a,f_i),f_1,\cdots,\hat{f_i},\cdots,f_k)\\
= & \sum_{a<b}(-1)^{a+b}\sum_{b<c<m+2-2k}(-1)^c\{\eta_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots,e_{c-1},e_a\circ e_b,e_c,\cdots;\cdots)\\
& +\eta_k(\cdots,\hat{e_a},\cdots,\hat{e_b},\cdots,e_{c-1},e_c,e_a\circ e_b,\cdots;\cdots)\}\\
& +\sum_{i,a}(-1)^a\{\eta_{k-1}(\cdots e_{a-1},f_i,e_a,\cdots;\hat{f_i}\cdots)+\eta_{k-1}(\cdots e_{a-1},e_a,f_i,\cdots;\hat{f_i}\cdots)\}\\
= & \sum_{a<b}(-1)^{a+1}\{\eta_k(\cdots\hat{e_a}\cdots e_a\circ e_b,\cdots;\cdots)+(-1)^{b+m}\eta_k(\cdots\hat{e_a}\cdots\hat{e_b}\cdots e_a\circ e_b;\cdots)\}
\end{aligned}
$$

$$-\sum_i \eta_{k-1}(f_i, e_1, \cdots ; \cdots \hat{f_i} \cdots) + (-1)^{m+1} \sum_i \eta_{k-1}(e_1 \cdots e_{m+1-2k}, f_i; \cdots \hat{f_i} \cdots)$$

The sum of the equations above is

$$\begin{aligned} & \{\Theta, \eta\}_k(e_1, \cdots, e_{m+1-2k}; f_1, \cdots, f_k) \\ = & \sum_a (-1)^a \rho(e_a) \eta_k(e_1, \cdots, \hat{e_a}, \cdots, e_{m+1-2k}; f_1, \cdots, f_k) \\ & + \sum_{a<b} (-1)^{a+1} \eta_k(\cdots, \hat{e_a}, \cdots, e_a \circ e_b, \cdots ; \cdots) \\ & - \sum_i \eta_{k-1}(f_i, e_1, \cdots, e_{m+1-2k}; \cdots, \hat{f_i}, \cdots) \\ = & -(d\eta)_k(e_1, \cdots, e_{m+1-2k}; f_1, \cdots, f_k) \end{aligned}$$

The proof is finished. ∎

Next, we give the definition of fat Leibniz algebras:

Definition 4.2. Given a Leibniz algebra L, if the symmetric product $(\cdot, \cdot)$ is non-degenerate, we call L a fat Leibniz algebra.

The omni Lie algebra $ol(V) = gl(V) \oplus V$ is obviously a fat Leibniz algebra. And the space of sections of any Courant algebroid is also a fat Leibniz algebra.

Actually given any Leibniz algebra L with trivial center, there is associated a fat Leibniz algebra $\tilde{L}$:

Proposition 4.3. *Suppose L is a Leibniz algebra with trivial center, then $\tilde{L} \triangleq L/K$ is a fat Leibniz algebra, where K is the kernel of the bilinear product of L, i.e. $K = \{k \in L | (k, e) = 0, \ \forall e \in L\}$.*

Proof. Since the product of L is invariant:

$$\tau(e_1)(k, e_2) = (e_1 \circ k, e_2) + (k, e_1 \circ e_2), \quad \forall e_1, e_2 \in L, \ \forall k \in K,$$

it follows that

$$(e_1 \circ k, e_2) = 0, \quad \forall e_2 \in L,$$

so $e_1 \circ k \in K$. Furthermore since $e_1 \circ k + k \circ e_1 = (k, e_1) = 0$, so $k \circ e_1 = -e_1 \circ k$ is also in K. Thus K is an ideal of L.

The Leibniz bracket of L naturally induces a bracket on L/K:

$$\bar{e}_1 \circ \bar{e}_2 \triangleq \overline{e_1 \circ e_2},$$

where $\bar{e}$ is the equivalent class of $e \in L$ in L/K. Suppose there exists $\bar{k} \in L/K$ such that $(\bar{k}, \bar{e}) = 0, \ \forall \bar{e} \in L/K$, i.e. $(k, e) \in K, \ \forall e \in L$. Since (k, e) is in the left center of L, $(k, e) \in K$ implies that (k, e) is also in the right center of L. So $(k, e) = 0$ by the assumption that the center of L is trivial. It follows that k itself is in K, $\bar{k} = 0 \in L/K$. As a result, the bilinear product on L/K is non-degenerate. ∎

Finally we give the main theorem of this section:

Theorem 4.4. *With the above notations, we have*

$$(e_1 \circ e_2)^\flat = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}.$$

In particular, if L is a fat Leibniz algebra, then the Leibniz bracket can be represented as a derived bracket:

$$e_1 \circ e_2 = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}^\sharp,$$

where $(\bullet)^\sharp : Im((\bullet)^\flat) \to L$ *is the (partial) inverse map of* $(\bullet)^\flat$*, i.e.*

$$((\phi)^\sharp)^\flat \triangleq \phi, \quad \forall \phi \in Im((\bullet)^\flat).$$

Proof. $\{\Theta, e_1^\flat\}$ is a 2 cochain:

$$\begin{aligned} & \{\Theta, e_1^\flat\}_0(e_2, e_3) \\ = \; & \langle \tilde{\Theta}_0(e_2, e_3), \tilde{e_1^\flat} \rangle + \Theta_1(e_2) \circ e_1^\flat(e_3) - \Theta_1(e_3) \circ e_1^\flat(e_2) \\ = \; & (e_2 \circ e_3, e_1) - (e_2, (e_1, e_3)) + (e_3, (e_1, e_2)) \\ = \; & -(e_2 \circ e_1, e_3) + (e_3, e_1 \circ e_2 + e_2 \circ e_1) \\ = \; & (e_1 \circ e_2, e_3) \end{aligned}$$

$$\{\Theta, e_1^\flat\}_1(f) = \langle \tilde{\Theta}_1(f), \tilde{e_1^\flat} \rangle = -(e_1, f)$$

We see that $\{\Theta, e_1^\flat\}$ is a representable cochain.
$\{\{\Theta, e_1^\flat\}, e_2^\flat\}$ is a 1 cochain:

$$\begin{aligned} & \{\{\Theta, e_1^\flat\}, e_2^\flat\}_0(e_3) \\ = \; & \langle \{\Theta, \tilde{e_1^\flat}\}_0(e_3), \tilde{e_2^\flat} \rangle + \{\Theta, e_1^\flat\}_1 \circ e_2^\flat(e_3) \\ = \; & (e_1 \circ e_3, e_2) - (e_1, (e_2, e_3)) \\ = \; & -(e_1 \circ e_2, e_3) \end{aligned}$$

So $(e_1 \circ e_2)^\flat(e_3) = -\{\{\Theta, e_1^\flat\}, e_2^\flat\}(e_3) = (e_1 \circ e_2, e_3)$.
The proof is finished. ■

Remark 4.5. As mentioned in Remark 2.3, $C(L)$ is isomorphic to the standard complex of the Courant-Dorfman algebra $S^\bullet(Z) \otimes L$. Actually theorem 4.4 is true for $e_1, e_2 \in S^\bullet(Z) \otimes L$. In [10], Roytenberg proved that the Dorfman bracket of a non-degenerate Courant-Dorfman algebra (i.e. the symmetric product is strongly non-degenerate) is a derived bracket. Our theorem 4.4 can be viewed as a generalization of his result, since the symmetric product of $S^\bullet(Z) \otimes L$ is only non-degenerate, but not strongly non-degenerate.

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Mathematics Research Unit, FSTC, University of Luxembourg, Luxembourg
E-mail address: shernvey@gmail.com

Department of Mathematics, Peking University, Beijing
E-mail address: liuzj@pku.edu.cn