LOCAL AND 2-LOCAL DERIVATIONS AND AUTOMORPHISMS ON SIMPLE LEIBNIZ ALGEBRAS

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Abstract. The present paper is devoted to local and 2-local derivations and automorphism of complex finite-dimensional simple Leibniz algebras. We prove that all local derivations and 2-local derivations on a finite-dimensional complex simple Leibniz algebra are automatically derivations. We show that nilpotent Leibniz algebras as a rule admit local derivations and 2-local derivations which are not derivations. Further we consider automorphisms of simple Leibniz algebras. We prove that every 2-local automorphism on a complex finite-dimensional simple Leibniz algebra is an automorphism and show that nilpotent Leibniz algebras admit 2-local automorphisms which are not automorphisms. A similar problem concerning local automorphism on simple Leibniz algebras is reduced to the case of simple Lie algebras.

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1. Introduction

Let $\mathcal{A}$ be an associative algebra. Recall that a linear mapping Φ of $\mathcal{A}$ into itself is called a local automorphism (respectively, a local derivation) if for every $x \in \mathcal{A}$ there exists an automorphism (respectively, a derivation) Φ_x of $\mathcal{A}$, depending on x, such that $\Phi_x(x) = \Phi(x)$. These notions were introduced and investigated independently by Kadison [16] and Larson and Sourour [17]. Later, in 1997, P. Šemrl [23] introduced the concepts of 2-local automorphisms and 2-local derivations. A map $\Phi : \mathcal{A} \to \mathcal{A}$ (not linear in general) is called a 2-local automorphism (respectively, a 2-local derivation) if for every $x, y \in \mathcal{A}$, there exists an automorphism (respectively, a derivation) $\Phi_{x,y} : \mathcal{A} \to \mathcal{A}$ (depending on x, y) such that $\Phi_{x.y}(x) = \Phi(x)$, $\Phi_{x,y}(y) = \Phi(y)$.

The above papers gave rise to series of works devoted to description of mappings which are close to automorphisms and derivations of C*-algebras and operator algebras. For details we refer to the paper [4] and the survey [7].

Later, several papers have been devoted to similar notions and corresponding problems for derivations and automorphisms of Lie algebras.

Let $\mathcal{L}$ be a Lie algebra. A derivation (respectively, an automorphism) Φ of $\mathcal{L}$ is a linear (respectively, an invertible linear) map $\Phi : \mathcal{L} \to \mathcal{L}$ which satisfies the condition $\Phi([x, y]) = [\Phi(x), y] + [x, \Phi(y)]$ (respectively, $\Phi([x, y]) = [\Phi(x), \Phi(y)]$) for all $x, y \in \mathcal{L}$.

The notions of a local derivation (respectively, a local automorphism) and a 2-local derivation (respectively, a 2-local automorphism) for Lie algebras are defined as above, similar to the associative case. Every derivation (respectively, automorphism) of a Lie algebra $\mathcal{L}$ is a local derivation (respectively, local automorphism) and a 2-local derivation (respectively, 2-local automorphism). For a given Lie algebra $\mathcal{L}$, the main problem concerning these notions is to prove that they automatically become a derivation (respectively, an automorphism) or to give examples of local and 2-local derivations or automorphisms of $\mathcal{L}$, which are not derivations or automorphisms, respectively.

Solution of such problems for finite-dimensional Lie algebras over algebraically closed field of zero characteristic were obtained in [5, 6, 8] and [12]. Namely, in [8] it is proved that every 2-local derivation on a semi-simple Lie algebra $\mathcal{L}$ is a derivation and that each finite-dimensional nilpotent Lie algebra with dimension larger than two admits 2-local derivation which is not a derivation. In [5] we have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. Concerning 2-local automorphism, Chen and Wang in [12] prove that if $\mathcal{L}$ is a simple Lie algebra of type $\mathcal{A}_l$, $\mathcal{D}_l$ or $\mathcal{E}_k$ $(k = 6, 7, 8)$ over an algebraically closed field of characteristic zero, then every 2-local automorphism of $\mathcal{L}$ is an automorphism. Finally, in [6] Ayupov and Kudaybergenov generalized this result of [12] and proved that every 2-local automorphism of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism. Moreover, they show also that every nilpotent Lie algebra with finite dimension larger than two admits 2-local automorphisms which is not an automorphism. It should be noted that similar problems for local automorphism of finite-dimensional Lie algebras still remain open.

Leibniz algebras present a "non antisymmetric" extension of Lie algebras. In last decades a series of papers have been devoted to the structure theory and classification of finite-dimensional Leibniz algebras. Several classical theorems from Lie algebras theory have been extended to the Leibniz algebras case. For some details from the theory of Leibniz algebras we refer to the papers [1–3, 13, 18, 19]. In particular, for a finite-dimensional simple Leibniz algebras over an algebraically closed field of characteristic zero, derivations have been completely described in [21].

In the present paper we study local and 2-local derivations and automorphisms of finite-dimensional simple complex Leibniz algebras.

In Section 2 we give some preliminaries from the Leibniz algebras theory. In Section 3 we prove that every 2-local derivation on a simple Leibniz algebra $\mathcal{L}$ is a derivation. We also prove that all nilpotent Leibniz algebras (except so-called null-filiform Leibniz algebra) admit 2-local derivations which are not derivations. Similar results for local derivations on simple Leibniz algebras are obtained in Section 4. Namely, we show that every local derivation on a simple complex Leibniz algebra is a derivation and that each finite-dimensional filiform, Leibniz algebra $\mathcal{L}$ with $\dim \mathcal{L} \geq 3$ admits a local derivation which is not a derivation.

In Section 5 we study automorphisms of simple Leibniz algebras.

Finally, in Section 6 we consider 2-local and local automorphisms of finite-dimensional Leibniz algebras. First we show that every 2-local automorphism of a complex simple Leibniz algebra is an automorphism and prove that each n-dimensional nilpotent Leibniz algebra such that $n \geq 2$ and $\dim[\mathcal{L}, \mathcal{L}] \leq n - 2$, admits a 2-local automorphism which is not an automorphism. At the end of this Section 6 we show that the problem concerning local automorphisms of simple complex Leibniz algebras is reduced to the similar problem for simple Lie algebras, which is, unfortunately, still open.

2. Preliminaries

In this section we give some necessary definitions and preliminary results.

Definition 2.1. *An algebra $(\mathcal{L}, [\cdot,\cdot])$ over a field $\mathbb{F}$ is called a Leibniz algebra if it is defined by the identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \text{ for all } x, y \in \mathcal{L},$$

which is called Leibniz identity.

It is a generalization of the Jacobi identity since under the condition of anti-symmetricity of the product "$[\cdot,\cdot]$" this identity changes to the Jacobi identity. In fact, the definition above is the notion of the right Leibniz algebra, where "right" indicates that any right multiplication operator is a derivation of the algebra. In the present paper the term "Leibniz algebra" will always mean the "right Leibniz algebra". The left Leibniz algebra is characterized by the property that any left multiplication operator is a derivation.

Let $\mathcal{L}$ be a Leibniz algebra and $\mathcal{I}$ be the ideal generated by squares in $\mathcal{L} : \mathcal{I} = \mathrm{id}\langle [x,x] \mid x \in \mathcal{L}\rangle$. The quotient $\mathcal{L}/\mathcal{I}$ is called "the associated Lie algebra" of the Leibniz algebra $\mathcal{L}$. The natural epimorphism $\varphi : \mathcal{L} \to \mathcal{L}/\mathcal{I}$ is a homomorphism of Leibniz algebras. The ideal $\mathcal{I}$ is the minimal ideal with the property that the quotient algebra is a Lie algebra. It is easy to see that the ideal $\mathcal{I}$ coincides with the subspace of $\mathcal{L}$ spanned by the squares, and that $\mathcal{L}$ is the left annihilator of $\mathcal{I}$, i.e., $[\mathcal{L}, \mathcal{I}] = 0$.

Definition 2.2. *A Leibniz algebra $\mathcal{L}$ is called simple if its ideals are only $\{0\}, \mathcal{I}, \mathcal{L}$ and $[\mathcal{L}, \mathcal{L}] \neq \mathcal{I}$.*

This definition agrees with that of simple Lie algebra, where $\mathcal{I} = \{0\}$.

For a given Leibniz algebra $\mathcal{L}$ we define derived sequence as follows:

$$\mathcal{L}^{[1]} = \mathcal{L}, \quad \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}], \quad s \geq 1.$$

Definition 2.3. *A Leibniz algebra $\mathcal{L}$ is called solvable, if there exists $s \in \mathbb{N}$ such that $\mathcal{L}^{[s]} = \{0\}$.*

It is known that sum of solvable ideals of a Leibniz algebra is solvable ideal too. Therefore each Leibniz algebra contains a maximal solvable ideal which is called *solvable radical.*

The following theorem recently proved by D. Barnes [10] presents an analogue of Levi–Malcev's theorem for Leibniz algebras.

Theorem 2.4. *Let $\mathcal{L}$ be a finite dimensional Leibniz algebra over a field of characteristic zero and let $\mathcal{R}$ be its solvable radical. Then there exists a semisimple Lie subalgebra $\mathcal{S}$ of $\mathcal{L}$ such that $\mathcal{L} = \mathcal{S} \dot{+} \mathcal{R}$.*

This theorem applied to a simple Leibniz algebra $\mathcal{L}$ gives

Corollary 2.5. *Let $\mathcal{L}$ be a simple Leibniz algebra over a field of characteristic zero and let $\mathcal{I}$ be the ideal generated by squares in $\mathcal{L}$, then there exists a simple Lie algebra $\mathcal{G}$ such that $\mathcal{I}$ is an irreducible module over $\mathcal{G}$ and $\mathcal{L} = \mathcal{G} \dot{+} \mathcal{I}$.*

Further we shall use the following important result [14].

Theorem 2.6. *(Schur's Lemma) Let $\mathcal{G}$ be a complex Lie algebra and let $\mathcal{U}$ and $\mathcal{V}$ be irreducible $\mathcal{G}$-modules. Then*

(i) *Any $\mathcal{G}$-module homomorphism $\Theta : \mathcal{U} \to \mathcal{V}$ is either trivial or an isomorphism;*
(ii) *A linear map $\Theta : \mathcal{V} \to \mathcal{V}$ is a $\mathcal{G}$-module homomorphism if and only if $\Theta = \lambda id_{|\mathcal{V}}$ for some $\lambda \in \mathbb{C}$.*

The notion of derivation for a Leibniz algebras is defined similar to the the Lie algebras case as follows.

Definition 2.7. *A linear transformation d of a Leibniz algebra $\mathcal{L}$ is said to be a derivation if for any $x, y \in \mathcal{L}$ one has*

$$D([x,y]) = [D(x), y] + [x, D(y)].$$

Let a be an element of a Leibniz algebra $\mathcal{L}$. Consider the operator of right multiplication $R_a : \mathcal{L} \to \mathcal{L}$, defined by $R_a(x) = [x, a]$. The Leibniz identity which characterizes Leibniz algebras exactly means that every right multiplication operator R_a is a derivation. Such derivations are called inner derivation on $\mathcal{L}$. Denote by $Der(\mathcal{L})$ - the space of all derivations of $\mathcal{L}$.

Now we shall present the main subjects considered in this paper, so-called local and 2-local derivation.

Definition 2.8. *A linear operator $\Delta : \mathcal{L} \to \mathcal{L}$ is called a local derivation if for any $x \in \mathcal{L}$ there exists a derivation $D_x \in Der(\mathcal{L})$ such that*

$$\Delta(x) = D_x(x).$$

Definition 2.9. *A map $\Delta : \mathcal{L} \to \mathcal{L}$ (not necessary linear) is called 2-local derivation if for any $x, y \in \mathcal{L}$ there exists a derivation $D_{x,y} \in Der(\mathcal{L})$ such that*

$$\Delta(x) = D_{x,y}(x), \quad \Delta(y) = D_{x,y}(y).$$

From now on we assume that all algebras are considered over the field of complex numbers $\mathbb{C}$ and suppose that $\mathcal{L}$ is a non-Lie Leibniz algebra, i.e. $\mathcal{I} \neq \{0\}$.

Now we give a description of derivations on simple Leibniz algebras obtained in [21].

Let $\mathcal{L}$ be a simple Leibniz algebra with $\mathcal{L} = \mathcal{G} \dot{+} \mathcal{I}$. Consider a projection operator $\mathrm{pr}_{\mathcal{I}}$ from $\mathcal{L}$ onto $\mathcal{I}$, that is

$$\mathrm{pr}_{\mathcal{I}}(x + i) = i, \; x + i \in \mathcal{L} \dot{+} \mathcal{I}. \tag{2.1}$$

Suppose that $\mathcal{G}$ and $\mathcal{I}$ are not isomorphic as $\mathcal{G}$-modules. Then any derivation D on $\mathcal{L}$ can be represented as

$$D = R_a + \lambda \mathrm{pr}_{\mathcal{I}}, \tag{2.2}$$

where R_a is an inner derivation generated by an element $a \in \mathcal{G}$, $\mathrm{pr}_{\mathcal{I}}$ is a derivation of the form (2.1), $\lambda \in \mathbb{C}$.

Now let us assume that $\mathcal{G}$ and $\mathcal{I}$ are isomorphic as $\mathcal{G}$-modules. There exists a unique (up to multiplication by constant) isomorphism θ of linear spaces $\mathcal{G}$ and $\mathcal{I}$ such that $\theta([x, y]) = [\theta(x), y]$ for all $x, y \in \mathcal{G}$, i.e., θ is a module isomorphism of $\mathcal{G}$-modules $\mathcal{G}$ and $\mathcal{I}$. Let us extend θ onto $\mathcal{L}$ as

$$\theta(x + i) = \theta(x), \; x + i \in \mathcal{L} \dot{+} \mathcal{I}. \tag{2.3}$$

For a simple Leibniz algebra $\mathcal{L}$ with $\dim \mathcal{G} = \dim \mathcal{I}$ any derivation D on $\mathcal{L}$ can be represented as

$$D = R_a + \omega\theta + \lambda \mathrm{pr}_{\mathcal{I}}, \tag{2.4}$$

where $a \in \mathcal{G}$, $\mathrm{pr}_{\mathcal{I}}$ is a derivation of the form (2.1) and θ is a derivation of the form (2.3), $\lambda, \, \omega \in \mathbb{C}$.

3. 2-Local derivations on Leibniz algebras

3.1. **2-Local derivations on simple Leibniz algebras.**

The first main result of this section is the following.

Theorem 3.1. *Let $\mathcal{L}$ be a simple complex Leibniz algebra. Then any 2-local derivation on $\mathcal{L}$ is a derivation.*

For the proof of this Theorem we need several Lemmata.

From theory of representation of semisimple Lie algebras [14] we have that a Cartan subalgebra $\mathcal{H}$ of Lie algebra $\mathcal{G}$ acts diagonalizable on $\mathcal{G}$-module $\mathcal{I}$:

$$\mathcal{I} = \bigoplus_{\alpha\in\Gamma} \mathcal{I}_\alpha,$$

where

$$\begin{aligned} \mathcal{I}_\alpha &= \{i \in \mathcal{I} : [i,h] = \alpha(h)i, \ \forall\, h \in \mathcal{H}\}, \\ \Gamma &= \{\alpha \in \mathcal{H}^* : \mathcal{I}_\alpha \neq \{0\}\} \end{aligned}$$

and $\mathcal{H}^*$ is the space of all linear functionals on $\mathcal{H}$. Elements of Γ are called weights of $\mathcal{I}$.

For every $\beta \in \Gamma$ take a non zero element $i_\beta^{(0)} \in \mathcal{I}_\beta$. Set

$$i_0 = \sum_{\beta\in\Gamma} i_\beta^{(0)}.$$

Fix a regular element h_0 in $\mathcal{H}$, in particular

$$\{x \in \mathcal{G}; [x, h_0] = 0\} = \mathcal{H}.$$

In the following two Lemmata 3.2–3.3 we assume that $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ is a Leibniz algebra such that $\mathcal{G}$ and $\mathcal{I}$ are non isomorphic as $\mathcal{G}$-modules.

Lemma 3.2. *Let D be a derivation on $\mathcal{L}$ such that $D(h_0 + i_0) = 0$. Then $D|_{\mathcal{I}} \equiv 0$.*

Proof. By (2.2) there exist an element $a \in \mathcal{G}$ and a number $\lambda \in \mathbb{C}$ such that $D = R_a + \lambda \mathrm{pr}_{\mathcal{I}}$. We have

$$0 = D(h_0 + i_0) = [h_0 + i_0, a] + \lambda i_0 = [h_0, a] + [i_0, a] + \lambda i_0.$$

Since $[h_0, a] \in \mathcal{G}$ and $[i_0, a] + \lambda i_0 \in \mathcal{I}$, it follows that $[h_0, a] = 0$ and $[i_0, a] + \lambda i_0 = 0$. Since h_0 is a regular element, we have that $a \in \mathcal{H}$. Further

$$0 = [i_0, a] + \lambda i_0 = \left[\sum_{\beta\in\Gamma} i_\beta^{(0)}, a\right] + \lambda i_0 = \sum_{\beta\in\Gamma} \beta(a) i_\beta^{(0)} + \lambda \sum_{\beta\in\Gamma} i_\beta^{(0)}.$$

Thus $\beta(a) = -\lambda$ for all $\beta \in \Gamma$.

Let i be an arbitrary element of I, then it has a decomposition $i = \sum\limits_{\beta\in\Gamma} i_\beta$, where $i_\beta \in I_\beta$, $\beta \in \Gamma$. From $\beta(a) = -\lambda$ for all $\beta \in \Gamma$ we get

$$\begin{aligned} D(i) &= [i, a] + \lambda i = \left[\sum_{\beta\in\Gamma} i_\beta, a\right] + \lambda \sum_{\beta\in\Gamma} i_\beta = \\ &= \sum_{\beta\in\Gamma} \beta(a) i_\beta + \lambda \sum_{\beta\in\Gamma} i_\beta = 0. \end{aligned}$$

The proof is complete. $\square$

Lemma 3.3. *Let D be a derivation on $\mathcal{L}$ such that $D(h_0 + i_0) = 0$. Then $D = 0$.*

Proof. Let $y \in \mathcal{L}$ be an arbitrary element. Since $[y, y] \in \mathcal{I}$, Lemma 3.2 implies that $D([y, y]) = 0$. The derivation identity

$$D([y, y]) = [D(y), y] + [y, D(y)]$$

implies that

$$[D(y), y] + [y, D(y)] = 0. \tag{3.1}$$

Putting $y = x + i \in \mathcal{G} + \mathcal{I}$ in (3.1) we obtain that

$$[D(x + i), x + i] + [x + i, D(x + i)] = 0.$$

Taking into account $D(x + i) = D(x)$ we have that

$$[D(x), x] + [x, D(x)] + [i, D(x)] = 0.$$

Using (3.1) from the last equality we have $[\mathcal{I}, D(x)] = 0$. Further for arbitrary elements $z \in \mathcal{G}$ and $i \in \mathcal{I}$ we have

$$[i, [D(x), z]] \ = \ [[i, D(x)], z] - [[i, z], D(x)] = 0$$

and

$$[i, [z, D(x)]] \ = \ [[i, z], D(x)] - [[i, D(x)], z] = 0.$$

This means that $[\mathcal{I}, \mathcal{G}_{D(x)}] = 0$, where $\mathcal{G}_{D(x)}$ is an ideal in $\mathcal{G}$ generated by the element $D(x)$. Since $\mathcal{I}$ is an irreducible module over the simple Lie algebra $\mathcal{G}$, we obtain that $\mathcal{G}_{D(x)} = \{0\}$, i.e., $D(x) = 0$. Finally, $D(y) = D(x + i) = D(x) = 0$. The proof is complete. $\square$

Consider a decomposition for $\mathcal{G}$, called the *root decomposition*

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{G}_\alpha,$$

where

$$\begin{aligned} \mathcal{G}_\alpha &= \{x \in \mathcal{G} : [h, x] = \alpha(h)x, \forall\, h \in \mathcal{H}\}, \\ \Phi &= \{\alpha \in \mathcal{H}^* \setminus \{0\} : \mathcal{G}_\alpha \neq \{0\}\}. \end{aligned}$$

The set Φ is called the *root system* of $\mathcal{G}$, and subspaces $\mathcal{G}_\alpha$ are called the *root subspaces*.

Further if $\mathcal{G}$ and $\mathcal{I}$ are isomorphic $\mathcal{G}$-modules, then the decomposition for $\mathcal{I}$ can be written as

$$\mathcal{I} = \mathcal{I}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathcal{I}_\alpha,$$

where $\mathcal{I}_0 = \theta(\mathcal{H})$ and $\mathcal{I}_\beta = \theta(\mathcal{G}_\beta)$ for all $\beta \in \Phi$. This follows from

$$[\theta(x_\beta), h] = \theta([x_\beta, h]) = \beta(h)\theta(x_\beta)$$

and

$$[\theta(h'), h] = \theta([h', h]) = 0$$

for all $h, h' \in \mathcal{H}$ and $x_\beta \in \mathcal{G}_\beta$, $\beta \in \Phi$.

For every $\beta \in \Phi$ take a non zero element $x_\beta^{(0)} \in \mathcal{G}_\beta$ and put

$$x_0 = \sum_{\beta \in \Gamma} x_\beta^{(0)} \text{ and } i_0 = \theta(x_0).$$

It is clear that $[x_\beta^{(0)}, h] = \beta(h)x_\beta^{(0)}$ for all $h \in \mathcal{H}$ and $\beta \in \Phi$.

By [11, Lemma 2.2], there exists an element $h_0 \in \mathcal{H}$ such that $\alpha(h_0) \neq \beta(h_0)$ for every $\alpha, \beta \in \Phi, \alpha \neq \beta$. In particular, $\alpha(h_0) \neq 0$ for every $\alpha \in \Phi$. Such elements h_0 are called *strongly regular* elements of $\mathcal{G}$. Again by [11, Lemma 2.2], every strongly regular element h_0 is a *regular element*, i.e.

$$\{x \in \mathcal{G} : [h_0, x] = 0\} = \mathcal{H}.$$

Choose a fixed strongly regular element $h_0 \in \mathcal{H}$.

Lemma 3.4. *Suppose that $a, h \in \mathcal{H}$, $h \neq 0$ and $\lambda, \omega \in \mathbb{C}$ are such that*

$$\omega\theta(h) + [i_0, a] + \lambda i_0 = 0.$$

Then $a = 0$ and $\lambda = \omega = 0$.

Proof. We have

$$\begin{aligned} 0 &= [i_0, a] + \lambda i_0 + \omega\theta(h) = [\theta(x_0), a] + \lambda\theta(x_0) + \omega\theta(h) = \\ &= \theta([x_0, a]) + \lambda\theta(x_0) + \omega\theta(h) = \theta([x_0, a] + \lambda x_0 + \omega h). \end{aligned}$$

Since θ is an isomorphism of linear spaces $\mathcal{G}$ and $\mathcal{I}$, it follows that $[x_0, a] + \lambda x_0 + \omega h = 0$. Further

$$\begin{aligned} 0 &= [x_0, a] + \lambda x_0 + \omega h = \left[\sum_{\beta\in\Phi} x_\beta^{(0)}, a\right] + \lambda x_0 + \omega h = \\ &= \sum_{\beta\in\Phi} \beta(a)x_\beta^{(0)} + \lambda \sum_{\beta\in\Phi} x_\beta^{(0)} + \omega h. \end{aligned}$$

Now we multiply both sides of this equality by the regular element $h_0 \in \mathcal{H}$, then we get

$$0 = \sum_{\beta\in\Phi} \beta(a)\beta(h_0)x_\beta^{(0)} + \lambda \sum_{\beta\in\Phi} \beta(h_0)x_\beta^{(0)}.$$

From this we obtain $\beta(a)\beta(h_0) = -\lambda\beta(h_0)$ for all $\beta \in \Phi$. Since h_0 is a strongly regular element it follows that $\beta(h_0) \neq 0$ for all $\beta \in \Phi$, then we derive that $\beta(a) = -\lambda$. Putting in the last equality the root $-\beta$ instead of β we obtain $\beta(a) = \lambda$. Thus, $\lambda = 0$ and $\beta(a) = 0$ for all $\beta \in \Phi$. Since the set Φ contains $k = \dim \mathcal{H}$ linearly independent elements, it follows that Φ separates points of $\mathcal{H}$, and therefore we get $a = 0$. Further from

$$\omega\theta(h) + [i_0, a] + \lambda i_0 = 0$$

we obtain that $\omega\theta(h) = 0$. Since $\theta(h)$ is non zero, it follows that $\omega = 0$. The proof is complete. □

Lemma 3.5. *Let D be a derivation on $\mathcal{L}$ such that $D(h_0 + i_0) = 0$. Then $D = 0$.*

Proof. By (2.4) there exist an element $a \in \mathcal{G}$ and numbers $\lambda, \theta \in \mathbb{C}$ such that

$$D = R_a + \omega\theta + \lambda \mathrm{pr}_{\mathcal{I}}.$$

We have

$$0 = D(h_0 + i_0) = [h_0 + i_0, a] + \omega\theta(h_0) + \lambda i_0 = [h_0, a] + \omega\theta(h_0) + [i_0, a] + \lambda i_0.$$

Since $[h_0, a] \in \mathcal{G}$ and $\omega\theta(h_0) + [i_0, a] + \lambda i_0 \in \mathcal{I}$, it follows that $[h_0, a] = 0$ and $\omega\theta(h_0) + [i_0, a] + \lambda i_0 = 0$. Since h_0 is a strongly regular element, we have that $a \in \mathcal{H}$. Now Lemma 3.4 implies that $a = 0$ and $\lambda = \omega = 0$. This means that $D = 0$. The proof is complete. □

Now we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. Let Δ be a 2-local derivation on $\mathcal{L}$. Take a derivation D on $\mathcal{L}$ such that

$$\Delta(h_0 + i_0) = D(h_0 + i_0).$$

Consider the 2-local derivation $\Delta - D$. Let $x \in \mathcal{L}$. Take a derivation δ on $\mathcal{L}$ such that

$$(\Delta - D)(h_0 + i_0) = \delta(h_0 + i_0),\ (\Delta - D)(x) = \delta(x).$$

Since $\delta(h_0+i_0) = 0$, by Lemmata 3.3 and 3.5 we obtain that $\delta(x) = 0$, i.e., $\Delta(x) = D(x)$. This means that $\Delta = D$ is a derivation. The proof is complete. □

Remark 3.6. *The analogues of Lemmata 3.3 and 3.5 are not true for simple Lie algebras.*

Let $\mathcal{G}$ be a simple Lie algebra and suppose that there exists a non zero element $x_0 \in \mathcal{G}$ such that

$$D(x_0) = 0,\ D \in Der(\mathcal{G}) \ \Rightarrow\ D = 0.$$

Take the inner derivation $D = R_{x_0}$ generated by the element x_0. Then $D(x_0) = 0$, but D is a non trivial derivation.

3.2. 2-local derivations of nilpotent Leibniz algebras.

In this subsection under a certain assumption we give a general construction of 2-local derivations which are not derivations for an arbitrary variety (not necessarily associative, Lie or Leibniz) of algebras. This construction then applied to show that nilpotent Leibniz algebras always admit 2-local derivations which are not derivations.

For an arbitrary algebra $\mathcal{L}$ with multiplication denoted as xy let

$$\mathcal{L}^2 = \operatorname{span}\{xy : x, y \in \mathcal{L}\}$$

and

$$\operatorname{Ann}(\mathcal{L}) = \{x \in \mathcal{L} : xy = yx = 0 \text{ for all } y \in \mathcal{L}\}.$$

Note that a linear operator δ on $\mathcal{L}$ such that $\delta|_{\mathcal{L}^2} \equiv 0$ and $\delta(\mathcal{L}) \subseteq \operatorname{Ann}(\mathcal{L})$ is a derivation. Indeed, for every $x, y \in \mathcal{L}$ we have

$$\delta(xy) = 0 = \delta(x)y + x\delta(y).$$

Theorem 3.7. *Let $\mathcal{L}$ be a n-dimensional algebra with $n \geq 2$. Suppose that*

(i) $\dim \mathcal{L}^2 \leq n - 2$;

(ii) *the annihilator $Ann(\mathcal{L})$ of $\mathcal{L}$ is non trivial.*

Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.

Proof. Let us consider a decomposition of $\mathcal{L}$ in the following form

$$\mathcal{L} = \mathcal{L}^2 \oplus V.$$

Due to $\dim \mathcal{L}^2 \leq n - 2$, we have $\dim V = k \geq 2$. Let $\{e_1, \dots, e_k\}$ be a basis of V.

Let us define a homogeneous non additive function f on $\mathbb{C}^2$ as follows

$$f(y_1, y_2) = \begin{cases} \frac{y_1^2}{y_2}, & \text{if } \ y_2 \neq 0 \\ 0, & \text{if } \ y_2 = 0, \end{cases}$$

where $(y_1, y_2) \in \mathbb{C}$.

Let us fix a non zero element $z \in \operatorname{Ann}(\mathcal{L})$. Define an operator Δ on $\mathcal{L}$ by

$$\Delta(x) = f(\lambda_1, \lambda_2)z,\ \ x = x_1 + \sum_{i=1}^{k} \lambda_i e_i,$$

where $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, k$, $x_1 \in \mathcal{L}^2$. The operator Δ is not a derivation since it is not linear.

Let us now show that Δ is a 2-local derivation. Define a linear operator δ on $\mathcal{L}$ by

$$\delta(x) = (a\lambda_1 + b\lambda_2)z, \ \ x = x_1 + \sum_{i=1}^{k} \lambda_i e_i,$$

where $a, b \in \mathbb{C}$. Since $\delta|_{\mathcal{L}^2} \equiv 0$ and $\delta(\mathcal{L}) \subseteq \mathrm{Ann}(\mathcal{L})$ the operator δ is a derivation.

Let $x = x_1 + \sum\limits_{i=1}^{k} \lambda_i e_i$ and $y = y_1 + \sum\limits_{i=1}^{k} \mu_i e_i$ be elements of $\mathcal{L}$. We choose a and b such that

$$\Delta(x) = \delta(x), \ \ \Delta(y) = \delta(y).$$

Let us rewrite the above equalities as system of linear equations with respect to unknowns a, b as follows

$$\begin{cases} \lambda_1 a + \lambda_2 b & = f(\lambda_1, \lambda_2) \\ \mu_1 a + \mu_2 b & = f(\mu_1, \mu_2) \end{cases}$$

Since the function f is homogeneous the system has non trivial solution. Therefore, Δ is a 2-local derivation, as required. The proof is complete. $\square$

Given a Leibniz algebra $\mathcal{L}$, we define the *lower central sequence* defined recursively as

$$\mathcal{L}^1 = \mathcal{L}, \ \ \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}], \quad k \geq 1.$$

Definition 3.8. *A Leibniz algebra $\mathcal{L}$ is said to be nilpotent, if there exists $t \in \mathbb{N}$ such that $\mathcal{L}^t = \{0\}$. The minimal number t with such property is said to be the index of nilpotency of the algebra $\mathcal{L}$.*

Since for an nilpotent algebra we have $\dim \mathcal{L}^2 \leq n - 1$, the index of nilpotency of an n-dimensional nilpotent Leibniz algebra is not greater than $n + 1$.

A Leibniz algebra $\mathcal{L}$ is called *null-filiform* if $\dim \mathcal{L}^k = n + 1 - k$ for $1 \leq k \leq n + 1$.

Clearly, a null-filiform Leibniz algebra has maximal index of nilpotency. Moreover, it is easy to show that a nilpotent Leibniz algebra is null-filiform if and only if it is a one-generated algebra (see [9]) . Note that this notion has no sense in the Lie algebras case, because Lie algebras are at least two-generated.

For every nilpotent Leibniz algebra with nilindex equal to t we have that $\{0\} \neq \mathcal{L}^{t-1} \subseteq \mathrm{Ann}(\mathcal{L})$. It is known [9] that up to isomorphism there exists a unique n-dimensional nilpotent Leibniz which satisfies the condition $\dim \mathcal{L}^2 = n - 1$ which is the null-filiform algebra, i.e. $\dim \mathcal{L}^2 \leq n - 2$ for all nilpotent Leibniz algebras except the null-filiform. Therefore Theorem 3.7 implies the following result.

Corollary 3.9. *Let $\mathcal{L}$ be a finite-dimensional non null-filiform nilpotent Leibniz algebra with $\dim \mathcal{L} \geq 2$. Then $\mathcal{L}$ admits a 2-local derivation which is not a derivation.*

Remark 3.10. *Let us show that every 2-local derivation of the algebra NF_n is a derivation.*

It is know that the unique null-filiform algebra, denoted by NF_n, admits a basis $\{e_1, e_2, \ldots, e_n\}$ in which its the table of multiplications is the following (see [9]):

$$[e_i, e_1] = e_{i+1}, \ \ 1 \leq i \leq n.$$

Let $D \in Der(NF_n)$ and $D(e_1) = \sum\limits_{i=1}^{n} \alpha_i e_i$ for some $\alpha_i \in \mathbb{C}$, then it is easy to check that

$$D(e_j) = j\alpha_1 e_j + \sum_{i=2}^{n-j+1} \alpha_i e_{j+i-1},\ 2 \leq j \leq n. \tag{3.2}$$

Let Δ be a 2-local derivation. For the elements $e_1, x, y \in NF_n$ there exist derivations $D_{e_1,x}$ and $D_{e_1,y}$ such that

$$\Delta(e_1) = D_{e_1,x}(e_1) = D_{e_1,y}(e_1) = \sum_{i=1}^{n} \alpha_i e_i.$$

From (3.2) we conclude that each derivation on NF_n is uniquely defined by its value on the element e_1. Therefore, $D_{e_1,x}(z) = D_{e_1,y}(z)$ for any $z \in NF_n$. Thus, we obtain that if $\Delta(e_1) = D_{e_1,x}(e_1)$ for some $D_{e_1,x}$, then $\Delta(z) = D_{e_1,x}(z)$ for any $z \in NF_n$, i.e., Δ is a derivation.

4. Local derivations on Leibniz algebras

4.1. **Local derivations on simple Leibniz algebras.**

Now we shall give the main result concerning local derivations on simple Leibniz algebras.

Theorem 4.1. *Let $\mathcal{L}$ be a simple complex Leibniz algebra. Then any local derivation on $\mathcal{L}$ is a derivation.*

Let us first consider a simple Leibniz algebra $\mathcal{L} = \mathcal{G} \dot{+} \mathcal{I}$ such that $\mathcal{G}$ and $\mathcal{I}$ are isomorphic $\mathcal{G}$-modules.

Lemma 4.2. *Let Δ be a local derivation on $\mathcal{L}$ such that Δ maps $\mathcal{L}$ into $\mathcal{I}$. Then Δ is a derivation.*

Proof. Fix a basis $\{x_1, \ldots, x_m\}$ in $\mathcal{G}$. In this case the system of vectors $\{y_i : y_i = \theta(x_i), i \in \overline{1,m}\}$ is a basis in $\mathcal{I}$, where θ is a module isomorphism of $\mathcal{G}$-modules $\mathcal{G}$ and $\mathcal{I}$, in particular, $\theta([x, y]) = [\theta(x), y]$ for all $x, y \in \mathcal{G}$.

For an element $x = x_i\, (i \in \overline{1,m})$ take an element $a_i \in \mathcal{G}$ and a number $\omega_i \in \mathbb{C}$ such that

$$\Delta(x_i) = [x_i, a_i] + \omega_i \theta(x_i).$$

Since $\Delta(x_i) \in \mathcal{I}$ and $[x_i, a_i] \in \mathcal{G}$, it follows that $[x_i, a_i] = 0$. Thus

$$\Delta(x_i) = \omega_i \theta(x_i) = \omega_i y_i.$$

Now for the element $x = x_i + x_j$, where $i \neq j$, take an element $a_{i,j} \in \mathcal{G}$ and a number $\omega_{i,j} \in \mathbb{C}$ such that

$$\Delta(x_i + x_j) = [x_i + x_j, a_{i,j}] + \omega_{i,j} \theta(x_i + x_j) \in \mathcal{I}.$$

Then $[x_i + x_j, a_{i,j}] = 0$. Thus

$$\Delta(x_i + x_j) = \omega_{i,j} \theta(x_i + x_j) = \omega_{i,j}(y_i + y_j).$$

On the other hand

$$\Delta(x_i + x_j) = \omega_i x_i + \omega_j y_j.$$

Comparing the last two equalities we obtain $\omega_i = \omega_j$ for all i, j. This means that there exists a number $\omega \in \mathbb{C}$ such that

$$\Delta(x_i) = \omega y_i \tag{4.1}$$

for all $i = 1, \dots, m$.

Now for $x = x_i + y_i \in \mathcal{G} + \mathcal{I}$ take an element $a_x \in \mathcal{G}$ and numbers $\omega_x, \lambda_x \in \mathbb{C}$ such that

$$\Delta(x_i + y_i) = [x_i + y_i, a_x] + \omega_x \theta(x_i) + \lambda_x y_i \in \mathcal{I}.$$

Then $[x_i, a_x] = 0$, and

$$0 = \theta(0) = \theta([x_i, a_x]) = [\theta(x_i), a_x] = [y_i, a_x].$$

Thus

$$\Delta(x_i + y_i) = \omega_x \theta(x_i) + \lambda_x y_i = (\omega_x + \lambda_x) y_i.$$

Taking into account (4.1) we obtain that

$$\Delta(y_i) = \Delta(x_i + y_i) - \Delta(x_i) = (\omega_x + \lambda_x) y_i - \omega y_i = (\omega_x - \omega + \lambda_x) y_i.$$

This means that for every $i \in \{1, \dots, m\}$ there exists a number $\lambda_i \in \mathbb{C}$ such that

$$\Delta(y_i) = \lambda_i y_i$$

for all $i = 1, \dots, m$.

Now take an element $x = x_i + x_j + y_i + y_j \in \mathcal{G} + \mathcal{I}$, where $i \neq j$. Since

$$\Delta(x_i + x_j + y_i + y_j) = [x_i + x_j + y_i + y_j, a_x] + \omega_x \theta(x_i + x_j) + \lambda_x (y_i + y_j) \in \mathcal{I},$$

we get that $[x_i + x_j, a_x] = 0$, and therefore $[y_i + y_j, a_x] = 0$. Thus

$$\Delta(x_i + x_j + y_i + y_j) = \omega_x \theta(x_i + x_j) + \lambda_x (y_i + y_j) = (\omega_x + \lambda_x)(y_i + y_j).$$

Taking into account (4.1) we obtain that

$$\begin{aligned} \Delta(y_i + y_j) &= \Delta(x_i + x_j + y_i + y_j) - \Delta(x_i + x_j) = \\ &= (\omega_x + \lambda_x)(y_i + y_j) - \omega(y_i + y_j) = (\omega_x - \omega + \lambda_x)(y_i + y_j). \end{aligned}$$

On the other hand

$$\Delta(y_i + y_j) = \Delta(y_i) + \Delta(y_j) = \lambda_i y_i + \lambda_j y_j.$$

Comparing the last two equalities we obtain that $\lambda_i = \lambda_j$ for all i and j. This means that there exists a number $\lambda \in \mathbb{C}$ such that

$$\Delta(y_i) = \lambda y_i \tag{4.2}$$

for all $i = 1, \dots, m$. Combining (4.1) and (4.2) we obtain that $\Delta = \omega\theta + \lambda \mathrm{pr}_{\mathcal{I}}$. This means that Δ is a derivation. The proof is complete. □

Let now Δ be an arbitrary local derivation on $\mathcal{L}$. For an arbitrary element $x \in \mathcal{L}$ take an element $a_x \in \mathcal{G}$ and a number $\omega_x \in \mathbb{C}$ such that

$$\Delta(x) = [x, a_x] + \omega_x \theta(x).$$

Then the mapping

$$x \in \mathcal{G} \to [x, a_x] \in \mathcal{G}$$

is a well-defined local derivation on $\mathcal{G}$, and therefore by [5, Theorem 3.1] it is an inner derivation generated by an element $a \in \mathcal{G}$. Then the local derivation $\Delta - R_a$ maps $\mathcal{L}$ into $\mathcal{I}$. By Lemma 4.2 we get that $\Delta - R_a$ is a derivation and therefore Δ is also a derivation.

In the next Lemma we consider a simple Leibniz algebra $\mathcal{L} = \mathcal{G} \dot{+} \mathcal{I}$ such that $\mathcal{G}$ and $\mathcal{I}$ are not isomorphic $\mathcal{G}$-modules.

Lemma 4.3. *Let Δ be a local derivation on $\mathcal{L}$ such that Δ maps $\mathcal{L}$ into $\mathcal{I}$. Then Δ is a derivation.*

Proof. Fix a basis $\{y_1, \ldots, y_k\}$ in $\mathcal{I}$. We can assume that for any y_i there exists a weight β_i such that $y_i \in \mathcal{I}_{\beta_i}$.

Let h_0 be a strongly regular element in $\mathcal{H}$. For $y = h_0 + y_i \in \mathcal{G} + \mathcal{I}$ take an element $a_y \in \mathcal{G}$ and number $\lambda_y \in \mathbb{C}$ such that

$$\Delta(h_0 + y_i) = [h_0, a_y] + [y_i, a_y] + \lambda_y y_i \in \mathcal{I}.$$

Then $[h_0, a_y] = 0$, and therefore $a_y \in \mathcal{H}$. Further

$$\Delta(y_i) = \Delta(h_0 + y_i) = [y_i, a_y] + \lambda_y y_i = (\beta_i(a_y) + \lambda_y) y_i.$$

This means that there exist numbers $\lambda_1, \ldots, \lambda_m$ such that

$$\Delta(y_i) = \lambda_i y_i$$

for all $i = 1, \ldots, m$.

Now we will show that $\lambda_1 = \ldots = \lambda_m$. Take y_{i_1}, y_{i_2}, $i_1 \neq i_2$. Denote $i_{\beta_1} = y_{i_1}$, $i_{\beta_2} = y_{i_2}$. We have

$$\Delta(i_{\beta_1} + i_{\beta_2}) = \lambda_{i_1} i_{\beta_1} + \lambda_{i_2} i_{\beta_2}. \tag{4.3}$$

Without lost of generality we can assume that β_1 is a fixed highest weight of $\mathcal{I}$. It is known [14, Page 108] that difference of two weights represented as

$$\beta_1 - \beta_2 = n_1\alpha_1 + \ldots + n_l\alpha_l,$$

where $\alpha_1, \ldots, \alpha_l$ are simple roots of $\mathcal{G}$, $n_1, \ldots, n_l$ are non negative integers.

Case 1. $\alpha_0 = n_1\alpha_1 + \ldots + n_l\alpha_l$ is not a root. Consider an element

$$x = n_1 e_{\alpha_1} + \ldots + n_l e_{\alpha_l} + i_{\beta_1} + i_{\beta_2}.$$

Take an element $a_x = h + \sum\limits_{\alpha\in\Phi} c_\alpha e_\alpha \in \mathcal{G}$ and number λ_x such that

$$\Delta(x) = [x, a_x] + \lambda_x (i_{\beta_1} + i_{\beta_2}).$$

Since $\Delta(x) \in \mathcal{I}$, we obtain that

$$\left[\sum_{s=1}^{l} n_s e_{\alpha_s}, h + \sum_{\alpha\in\Phi} c_\alpha e_\alpha\right] = 0.$$

Let us rewrite the last equality as

$$\sum_{s=1}^{l} n_s \alpha_s(h) e_{\alpha_s} + \sum_{t=1}^{l}\sum_{\alpha\in\Phi} * e_{\alpha+\alpha_t} = 0,$$

where the symbols $*$ denote appropriate coefficients. The second summand does not contain any element of the form e_{α_s}. Indeed, if we assume that $\alpha_s = \alpha + \alpha_t$, we have that $\alpha = \alpha_s - \alpha_t$. But $\alpha_s - \alpha_t$ is not a root, because α_s, α_t are simple roots. Hence all coefficients of the first summand are zero, i.e.,

$$n_1\alpha_1(h) = \ldots = n_l\alpha_l(h) = 0.$$

Further

$$\Delta(i_{\beta_1} + i_{\beta_2}) = \Delta(x) = [i_{\beta_1} + i_{\beta_2}, a_x] + \lambda_x (i_{\beta_1} + i_{\beta_2}).$$

Let us calculate the commutator $[i_{\beta_1}+i_{\beta_2},a_x]$. We have

$$\begin{aligned}[i_{\beta_1}+i_{\beta_2},a_x] &= \left[i_{\beta_1}+i_{\beta_2},h+\sum_{\alpha\in\Phi}c_\alpha e_\alpha\right]= \\ &= \beta_1(h)i_{\beta_1}+\beta_2(h)i_{\beta_2}+\sum_{t=1}^{2}\sum_{\alpha\in\Phi}*i_{\beta_t+\alpha}.\end{aligned}$$

The last summand does not contain i_{β_1} and i_{β_2}, because $\beta_1-\beta_2$ is not a root by the assumption. This means that

$$\Delta(i_{\beta_1}+i_{\beta_2})=(\beta_1(h)+\lambda_x)i_{\beta_1}+(\beta_2(h)+\lambda_x)i_{\beta_2}. \tag{4.4}$$

The difference of the coefficients of the right side is

$$\beta_1(h)-\beta_2(h)=\sum_{s=1}^{l}n_s\alpha_s(h)=0,$$

because $n_1\alpha_1(h)=\ldots=n_l\alpha_l(h)=0$. Finally, comparing coefficients in (4.3) and (4.4) we get

$$\lambda_{i_1}=\beta_1(h)+\lambda_x=\beta_2(h)+\lambda_x=\lambda_{i_2}.$$

Case 2. $\alpha_0=n_1\alpha_1+\ldots+n_l\alpha_l$ is a root. Since β_1 is a highest weight, we get $\dim\mathcal{I}_{\beta_1}=1$. Further since $\beta_1-\beta_2$ is a root, [15, Lemma 3.2.9] implies that $\dim\mathcal{I}_{\beta_2}=\dim\mathcal{I}_{\beta_1}$, and therefore there exist numbers $t_{-\alpha_0}\neq 0$ and t_{α_0} such that $[i_{\beta_1},e_{-\alpha_0}]=t_{-\alpha_0}i_{\beta_2}$, $[i_{\beta_2},e_{\alpha_0}]=t_{\alpha_0}i_{\beta_1}$. Consider an element

$$x=t_{-\alpha_0}e_{\alpha_0}+t_{\alpha_0}e_{-\alpha_0}+i_{\beta_1}+i_{\beta_2}.$$

Take an element $a_x=h+\sum\limits_{\alpha\in\Phi}c_\alpha e_\alpha\in\mathcal{G}$ and number λ_x such that

$$\Delta(x)=[x,a_x]+\lambda_x(i_{\beta_1}+i_{\beta_2}).$$

Since $\Delta(x)\in\mathcal{I}$, we obtain that

$$\left[t_{-\alpha_0}e_{\alpha_0}+t_{\alpha_0}e_{-\alpha_0},h+\sum_{\alpha\in\Phi}c_\alpha e_\alpha\right]=0.$$

Let us rewrite the last equality as

$$\alpha_0(h)t_{-\alpha_0}e_{\alpha_0}-\alpha_0(h)t_{\alpha_0}e_{-\alpha_0}+(t_{-\alpha_0}c_{-\alpha_0}-t_{\alpha_0}c_{\alpha_0})h_{\alpha_0}+\sum_{\alpha\neq\pm\alpha_0}*e_{\alpha\pm\alpha_0}=0,$$

where $h_{\alpha_0}=[e_{\alpha_0},e_{-\alpha_0}]\in\mathcal{H}$. The last summand in the sum does not contain elements e_{α_0} and $e_{-\alpha_0}$. Indeed, if we assume that $\alpha_0=\alpha-\alpha_0$, we have that $\alpha=2\alpha_0$. But $2\alpha_0$ is not a root. Hence the first three coefficients of this sum are zero, i.e.,

$$\alpha_0(h)=0,\ t_{\alpha_0}c_{\alpha_0}=t_{-\alpha_0}c_{-\alpha_0}. \tag{4.5}$$

Further

$$\Delta(i_{\beta_1}+i_{\beta_2})=\Delta(x)=[i_{\beta_1}+i_{\beta_2},a_x]+\lambda_x(i_{\beta_1}+i_{\beta_2}).$$

Let us consider the product $[i_{\beta_1}+i_{\beta_2}, a_x]$. We have

$$\begin{aligned}
[i_{\beta_1}+i_{\beta_2}, a_x] &= \left[i_{\beta_1}+i_{\beta_2}, h+\sum_{\alpha\in\Phi} c_\alpha e_\alpha\right] = \\
&= [i_{\beta_1}, h] + c_{\alpha_0}[i_{\beta_2}, e_{\alpha_0}] + [i_{\beta_2}, h] + c_{-\alpha_0}[i_{\beta_1}, e_{-\alpha_0}] + \\
&+ c_{\alpha_0}[i_{\beta_1}, e_{\alpha_0}] + c_{-\alpha_0}[i_{\beta_2}, e_{-\alpha_0}] + \sum_{t=1}^{2}\sum_{\alpha\neq\pm\alpha_0} c_\alpha [i_{\beta_t}, e_\alpha] = \\
&= (\beta_1(h) + t_{\alpha_0}c_{\alpha_0}) i_{\beta_1} + (\beta_2(h) + t_{-\alpha_0}c_{-\alpha_0}) i_{\beta_2} + \\
&+ *i_{2\beta_1-\beta_2} + *i_{2\beta_2-\beta_1} + \sum_{t=1}^{2}\sum_{\alpha\neq\pm\alpha_0} *i_{\beta_t+\alpha}.
\end{aligned}$$

The last three summands do not contain i_{β_1} and i_{β_2}, because $\beta_1-\beta_2=\alpha_0$ and $\alpha\neq\pm\alpha_0$. This means that

$$\Delta(i_{\beta_1}+i_{\beta_2}) = (\beta_1(h) + t_{\alpha_0}c_{\alpha_0} + \lambda_x) i_{\beta_1} + (\beta_2(h) + t_{-\alpha_0}c_{-\alpha_0} + \lambda_x) i_{\beta_2}. \tag{4.6}$$

Taking into account (4.5) we find the difference of coefficients in the right side:

$$(\beta_1(h) + t_{\alpha_0}c_{\alpha_0}) - (\beta_2(h) + t_{-\alpha_0}c_{-\alpha_0}) = \alpha_0(h) + t_{\alpha_0}c_{\alpha_0} - t_{-\alpha_0}c_{-\alpha_0} = 0.$$

Combining (4.3) and (4.6) we obtain that

$$\lambda_{i_1} = \beta_1(h) + t_{\alpha_0}c_{\alpha_0} + \lambda_x = \beta_2(h) + t_{-\alpha_0}c_{-\alpha_0} + \lambda_x = \lambda_{i_2}.$$

So, we have proved that

$$\Delta(x_i+y_i) = \lambda y_i,$$

where $\lambda\in\mathbb{C}$. This means that $\Delta = \lambda \mathrm{pr}_{\mathcal{I}}$. The proof is complete. □

Let Δ be an arbitrary local derivation and let $x\in\mathcal{G}$ be an arbitrary element. Take an element $a_x\in\mathcal{G}$ such that

$$\Delta(x) = [x, a_x].$$

As in the case $\dim\mathcal{G} = \dim\mathcal{I}$, the mapping

$$x\in\mathcal{G}\to[x, a_x]\in\mathcal{G}$$

is an inner derivation generated by an element $a\in\mathcal{G}$, and $\Delta - R_a$ maps $\mathcal{L}$ into $\mathcal{I}$. This means that Δ is a derivation, that completes the proof of Theorem 4.1.

4.2. **Local derivations on filiform Leibniz algebras.**

In this subsection we consider a special class of nilpotent Leibniz algebras, so-called filiform Leibniz algebras, and show that they admit local derivations which are not derivations.

A nilpotent Leibniz algebra $\mathcal{L}$ is called *filiform* if $\dim\mathcal{L}^k = n-k$ for $2\leq k\leq n$, where $\mathcal{L}^1=\mathcal{L}$, $\mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}]$, $k\geq 1$.

Theorem 4.4. *Let $\mathcal{L}$ be a finite-dimensional filiform Leibniz algebra with $\dim\mathcal{L}\geq 3$. Then $\mathcal{L}$ admits a local derivation which is not a derivation.*

For filiform Lie algebras this result was proved in [5, Theorem 4.1]. Hence it is suffices to consider filiform non-Lie Leibniz algebras.

Recall [20] that each complex n-dimensional filiform Leibniz algebra admits a basis $\{e_1, e_2, \ldots, e_n\}$ such that the table of multiplication of the algebra has one of the following forms:

$$F_1(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta): \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1, e_2] = \sum\limits_{k=4}^{n-1} \alpha_k e_k + \theta e_n, & \\ [e_i, e_2] = \sum\limits_{k=i+2}^{n} \alpha_{k+2-i} e_k, & 2 \le i \le n-2; \end{cases}$$

$$F_2(\beta_3, \beta_4, \ldots, \beta_n, \gamma): \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1, \\ [e_1, e_2] = \sum\limits_{k=4}^{n} \beta_k e_k, & \\ [e_2, e_2] = \gamma e_n, & \\ [e_i, e_2] = \sum\limits_{k=i+2}^{n} \beta_{k+2-i} e_k, & 3 \le i \le n-2; \end{cases}$$

$$F_3(\theta_1, \theta_2, \theta_3): \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n-1, \\ [e_1, e_1] = \theta_1 e_n, & \\ [e_1, e_2] = -e_3 + \theta_2 e_n, & \\ [e_2, e_2] = \theta_3 e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in \operatorname{span}\{e_{i+j+1}, \ldots, e_n\}, & \begin{cases} 1 \le i \le n-2, \\ 2 \le j \le n-i, \ i < j \end{cases} \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^i e_n, & 2 \le i \le n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ for odd n and $\alpha = 0$ for even n. Moreover, the structure constants of an algebra from $F_3(\theta_1, \theta_2, \theta_3)$ should satisfy the Leibniz identity.

It is easy to see that algebras of the first and the second families are non-Lie algebras. Moreover, an algebra of the third family is a Lie algebra if and only if $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$.

Firstly we consider the family of algebras $\mathcal{L} = F_1(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$. Let us define a linear operator D on $\mathcal{L}$ by

$$D\left(\sum_{k=1}^{n} x_k e_k\right) = \alpha(x_1 + x_2)e_{n-1} + \beta x_3 e_n, \tag{4.7}$$

where $\alpha, \beta \in \mathbb{C}$.

Lemma 4.5. *The linear operator D on $\mathcal{L}$ defined by* (4.7) *is a derivation if and only if $\alpha = \beta$.*

Proof. Suppose that the linear operator D defined by (4.7) is a derivation. Since $[e_1, e_1] = e_3$, we have that

$$D([e_1, e_1]) = D(e_3) = \beta e_n$$

and

$$[D(e_1), e_1] + [e_1, D(e_1)] = [\alpha e_{n-1}, e_1] + [e_1, \alpha e_{n-1}] = \alpha e_n.$$

Thus $\alpha = \beta$.

Conversely, let D be a linear operator defined by (4.7) with $\alpha = \beta$. We may assume that $\alpha = \beta = 1$.

In order to prove that D is a derivation it is sufficient to show that

$$D([e_i, e_j]) = [D(e_i), e_j] + [e_i, D(e_j)]$$

for all $1 \leq i, j \leq n$.

Case 1. $i + j = 2$. Then $i = j = 1$ and in this case we can check as above.

Case 2. $i + j = 3$. Then

$$[D(e_2), e_1] + [e_2, D(e_1)] = [e_{n-1}, e_1] + [e_2, e_{n-1}] = [e_{n-1}, e_1] = e_n = D(e_3) = D([e_2, e_1])$$

and

$$[D(e_1), e_2] + [e_1, D(e_2)] = [e_{n-1}, e_2] + [e_1, e_{n-1}] = 0 = D\left(\sum_{k=4}^{n-1} \alpha_k e_k + \theta e_n\right) = D([e_1, e_2]).$$

Case 3. $i + j \geq 4$. Then

$$D([e_i, e_j]) = 0 = [D(e_i), e_j] + [e_i, D(e_j)].$$

The proof is complete. $\square$

Now we consider the linear operator Δ defined by (4.7) with $\alpha = 1, \beta = 0$.

Lemma 4.6. *The linear operator Δ is a local derivation which is not a derivation.*

Proof. By Lemma 4.5, Δ is not a derivation.

Let us show that Δ is a local derivation. Denote by D_1 the derivation defined by (4.7) with $\alpha = \beta = 1$. Let D_2 be a linear operator on $\mathcal{L}$ defined by

$$D_2\left(\sum_{k=1}^{n} x_k e_k\right) = (x_1 + x_2)e_n.$$

Since $D_2|_{[\mathcal{L},\mathcal{L}]} \equiv 0$ and $D_2(\mathcal{L}) \subseteq Z(\mathcal{L})$, it follows that

$$D_2([x, y]) = 0 = [D_2(x), y] + [x, D_2(y)]$$

for all $x, y \in \mathcal{L}$. So, D_2 is a derivation.

Finally, for any $x = \sum\limits_{k=1}^{n} x_k e_k$ we find a derivation D such that $\Delta(x) = D(x)$.

Case 1. $x_1 + x_2 = 0$. Then

$$\Delta(x) = 0 = D_2(x).$$

Case 2. $x_1 + x_2 \neq 0$. Set

$$D = D_1 + tD_2,$$

where $t = -\dfrac{x_3}{x_1 + x_2}$. Then

$$\begin{aligned} D(x) &= D_1(x) + tD_2(x) = (x_1 + x_2)e_{n-1} + x_3 e_n + t(x_1 + x_2)e_n = \\ &= (x_1 + x_2)e_{n-1} + x_3 e_n - x_3 e_n = \Delta(x). \end{aligned}$$

The proof is complete. $\square$

Now let us consider the second and third classes.

For an algebra $\mathcal{L} = F_2(\beta_3, \beta_4, \ldots, \beta_n, \gamma)$ from the second class define a linear operator D on $\mathcal{L}$ by

$$D\left(\sum_{k=1}^{n} x_k e_k\right) = \alpha x_1 e_{n-1} + \beta x_3 e_n, \tag{4.8}$$

where $\alpha, \beta \in \mathbb{C}$.

For an algebra $\mathcal{L} = F_3(\theta_1, \theta_2, \theta_3)$ define a linear operator D on $\mathcal{L}$ by

$$D\left(\sum_{k=1}^{n} x_k e_k\right) = \alpha x_2 e_{n-1} + \beta x_3 e_n, \tag{4.9}$$

where $\alpha, \beta \in \mathbb{C}$.

As in the proof of Lemma 4.5, we can check that the operator D defined by (4.8) or (4.9) is a derivation if and only if $\alpha = \beta$. Therefore the operator D defined by (4.8) or (4.9) gives is the example of a local derivations which is not a derivation.

5. Automorphisms of simple Leibniz algebras

Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a simple Leibniz algebra, where $\mathcal{G}$ is a simple Lie algebra and $\mathcal{I}$ is its right irreducible module.

We consider an automorphism φ of $\mathcal{L}$. The ideal generated by squares of elements of $\mathcal{L}$ coincides with the span$\{[x, x] : x \in \mathcal{L}\}$. Then for $x = \sum\limits_{k=1}^{n} \lambda_i [x_i, x_i] \in \mathcal{I}$ we have

$$\begin{aligned} \varphi(x) &= \varphi\left(\sum_{k=1}^{n} \lambda_i [x_i, x_i]\right) = \sum_{k=1}^{n} \lambda_i \varphi([x_i, x_i]) = \\ &= \sum_{k=1}^{n} \lambda_i [\varphi(x_i), \varphi(x_i)] = \sum_{k=1}^{n} \lambda_i [y_i, y_i], \end{aligned}$$

i.e., $\varphi(\mathcal{I}) \subseteq \mathcal{I}$.

Now let $y = \sum\limits_{k=1}^{n} \lambda_i [y_i, y_i]$ be an arbitrary element of the ideal $\mathcal{I}$. Since φ is an automorphism, for every $y_i \in \mathcal{L}$ there exists $x_i \in \mathcal{L}$ such that $\varphi(x_i) = y_i$. Then we have

$$\begin{aligned} y &= \sum_{k=1}^{n} \lambda_i [y_i, y_i] = \sum_{k=1}^{n} \lambda_i [\varphi(x_i), \varphi(x_i)] = \\ &= \sum_{k=1}^{n} \lambda_i \varphi([x_i, x_i]) = \varphi\left(\sum_{k=1}^{n} \lambda_i [x_i, x_i]\right). \end{aligned}$$

This implies that for the element $z = \sum\limits_{k=1}^{n} \lambda_i [x_i, x_i] \in \mathcal{I}$ we have $\varphi(z) = y$. So we have proved that $\mathcal{I} \subseteq \varphi(\mathcal{I})$, and therefore $\varphi(\mathcal{I}) = \mathcal{I}$.

Now we shall show that any $\varphi \in \mathrm{Aut}(\mathcal{L})$ can be represented as the sum $\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{G},\mathcal{I}} + \varphi_{\mathcal{I},\mathcal{I}}$, where $\varphi_{\mathcal{G},\mathcal{G}} : \mathcal{G} \to \mathcal{G}$ is an automorphism on $\mathcal{G}$, $\varphi_{\mathcal{G},\mathcal{I}} : \mathcal{G} \to \mathcal{I}$ is a $\mathcal{G}$-module homomorphism from $\mathcal{G}$ into $\mathcal{I}$, and $\varphi_{\mathcal{I},\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ is a $\mathcal{G}$-module isomorphism of $\mathcal{I}$. In particular,

$$\varphi(x + i) = \varphi_{\mathcal{G},\mathcal{G}}(x) + \varphi_{\mathcal{G},\mathcal{I}}(x) + \varphi_{\mathcal{I},\mathcal{I}}(i),\ x + i \in \mathcal{G} + \mathcal{I}.$$

Lemma 5.1. *Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a simple complex Leibniz algebra and let $\varphi \in Aut(\mathcal{L})$ be an automorphism. Then*

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}},$$

if $\dim \mathcal{G} \neq \dim \mathcal{I}$, *and*

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}},$$

if $\dim \mathcal{G} = \dim \mathcal{I}$.

Proof. Let $x, y \in \mathcal{G}$, then

$$\begin{aligned}\varphi_{\mathcal{G},\mathcal{G}}([x,y]) + \varphi_{\mathcal{G},\mathcal{I}}([x,y]) &= \varphi([x,y]) = [\varphi(x),\varphi(y)] = \\ &= [\varphi_{\mathcal{G},\mathcal{G}}(x) + \varphi_{\mathcal{G},\mathcal{I}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y) + \varphi_{\mathcal{G},\mathcal{I}}(y)] = \\ &= [\varphi_{\mathcal{G},\mathcal{G}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)] + [\varphi_{\mathcal{G},\mathcal{I}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)].\end{aligned}$$

This implies

$$\varphi_{\mathcal{G},\mathcal{G}}([x,y]) = [\varphi_{\mathcal{G},\mathcal{G}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)], \tag{5.1}$$

$$\varphi_{\mathcal{G},\mathcal{I}}([x,y]) = [\varphi_{\mathcal{G},\mathcal{I}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)]. \tag{5.2}$$

Set

$$\psi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}}.$$

Let us show that ψ is also an automorphism. Indeed,

$$\begin{aligned}\psi([x+i, y+j]) &= \psi([x,y] + [i,y]) = \varphi_{\mathcal{G},\mathcal{G}}([x,y]) + \varphi_{\mathcal{I},\mathcal{I}}([i,y]) = \\ &= [\varphi_{\mathcal{G},\mathcal{G}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)] + \varphi([i,y]) = [\varphi_{\mathcal{G},\mathcal{G}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)] + [\varphi(i), \varphi(y)] = \\ &= [\varphi_{\mathcal{G},\mathcal{G}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)] + [\varphi_{\mathcal{I},\mathcal{I}}(i), \varphi_{\mathcal{G},\mathcal{G}}(y) + \varphi_{\mathcal{G},\mathcal{I}}(y)] = \\ &= [\varphi_{\mathcal{G},\mathcal{G}}(x), \varphi_{\mathcal{G},\mathcal{G}}(y)] + [\varphi_{\mathcal{I},\mathcal{I}}(i), \varphi_{\mathcal{G},\mathcal{G}}(y)] = \\ &= [\varphi_{\mathcal{G},\mathcal{G}}(x) + \varphi_{\mathcal{I},\mathcal{I}}(i), \varphi_{\mathcal{G},\mathcal{G}}(y) + \varphi_{\mathcal{I},\mathcal{I}}(j)] = [\psi(x+i), \psi(y+j)].\end{aligned}$$

Now consider an automorphism

$$\eta = \varphi \circ \psi^{-1}.$$

Then

$$\eta(x+i) = x + \eta_{\mathcal{G},\mathcal{I}}(x) + i,$$

where $\eta_{\mathcal{G},\mathcal{I}} = \varphi_{\mathcal{G},\mathcal{I}} \circ \varphi^{-1}_{\mathcal{G},\mathcal{G}}$. Applying (5.1) and (5.2) to η we obtain that

$$\eta_{\mathcal{G},\mathcal{I}}([x,y]) = [\eta_{\mathcal{G},\mathcal{I}}(x), y].$$

This means that $\eta_{\mathcal{G},\mathcal{I}}$ is a $\mathcal{G}$-module homomorphism from $\mathcal{G}$ into $\mathcal{I}$.

Case 1. Let $\dim \mathcal{G} \neq \dim \mathcal{I}$. In this case by Schur's Lemma we obtain that $\eta_{\mathcal{G},\mathcal{I}} = 0$. Now the equality $\eta_{\mathcal{G},\mathcal{I}} = \varphi_{\mathcal{G},\mathcal{I}} \circ \varphi^{-1}_{\mathcal{G},\mathcal{G}}$ implies that $\varphi_{\mathcal{G},\mathcal{I}} = 0$. Thus

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}}.$$

Case 2. Let $\dim \mathcal{G} = \dim \mathcal{I}$. In this case again by Schur's Lemma we obtain that $\eta_{\mathcal{G},\mathcal{I}} = \omega\theta$, where $\omega \in \mathbb{C}$. Thus $\varphi_{\mathcal{G},\mathcal{I}} = \eta_{\mathcal{G},\mathcal{I}} \circ \varphi_{\mathcal{G},\mathcal{G}} = \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}}$, and therefore

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}}.$$

The proof is complete. □

Further we shall use the following lemma.

Lemma 5.2. *Let $\varphi \in Aut(\mathcal{L})$ be an automorphism. Then*

a) *if $\varphi_{\mathcal{G},\mathcal{G}} = id_{\mathcal{G}}$, then $\varphi_{\mathcal{I},\mathcal{I}} = \lambda id_{\mathcal{I}}$;*
b) *if $\varphi_{\mathcal{I},\mathcal{I}} = id_{\mathcal{I}}$, then $\varphi_{\mathcal{G},\mathcal{G}} = id_{\mathcal{G}}$.*

Proof. a) Similar to the proof of (5.2) we obtain that

$$\varphi_{\mathcal{I},\mathcal{I}}([i,x]) = [\varphi_{\mathcal{I},\mathcal{I}}(i), \varphi_{\mathcal{G},\mathcal{G}}(x)] = [\varphi_{\mathcal{I},\mathcal{I}}(i), x]$$

for all $i \in \mathcal{I}, x \in \mathcal{G}$. By Schur's Lemma we obtain that $\varphi_{\mathcal{I},\mathcal{I}} = \lambda id_{\mathcal{I}}$.

b) Since

$$[i,x] = \varphi_{\mathcal{I},\mathcal{I}}([i,x]) = [\varphi_{\mathcal{I},\mathcal{I}}(i), \varphi_{\mathcal{G},\mathcal{G}}(x)] = [i, \varphi_{\mathcal{G},\mathcal{G}}(x)],$$

we obtain that $[i, \varphi_{\mathcal{G},\mathcal{G}}(x) - x] = 0$ for all $i \in \mathcal{I}, x \in \mathcal{G}$. As in the proof of Lemma 3.3 we have $[\mathcal{I}, \mathcal{G}_{\varphi_{\mathcal{G},\mathcal{G}}(x)-x}] = 0$, and $\varphi_{\mathcal{G},\mathcal{G}}(x) - x = 0$. The proof is complete. □

Lemma 5.3. *Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a simple complex Leibniz algebra with $\dim \mathcal{G} = \dim \mathcal{I}$. Then any automorphism $\varphi \in Aut(\mathcal{L})$ can be represented as*

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}} + \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1},$$

where $\omega \in \mathbb{C}$ and $0 \neq \lambda \in \mathbb{C}$.

Proof. Let $\varphi_{\mathcal{G},\mathcal{G}}$ be an automorphism of $\mathcal{G}$. Let us show that $\phi = \varphi_{\mathcal{G},\mathcal{G}} + \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}$ is an automorphism. It suffices to check that

$$\phi([i, x]) = [\phi(i), \phi(x)]$$

for all $x \in \mathcal{G}$, $i \in \mathcal{I}$. Since θ is a $\mathcal{G}$-module isomorphism, it follows that $\theta([\theta^{-1}(i), x]) = [i, x]$. We have

$$\begin{aligned}
\phi([i, x]) &= \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}([i, x]) = \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}(\theta([\theta^{-1}(i), x]) = \\
&= \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}}([\theta^{-1}(i), x]) = \lambda\theta([\varphi_{\mathcal{G},\mathcal{G}}(\theta^{-1}(i)), \varphi_{\mathcal{G},\mathcal{G}}(x)]) = \\
&= [(\lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1})(i), \varphi_{\mathcal{G},\mathcal{G}}(x)] = [\phi(i), \phi(x)].
\end{aligned}$$

Let us consider

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}}.$$

Set

$$\psi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}},\ \phi = \varphi_{\mathcal{G},\mathcal{G}} + \theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1} \text{ and } \eta = \psi \circ \phi^{-1}.$$

Then $\eta = \mathrm{id}_{\mathcal{G}} + \eta_{\mathcal{I},\mathcal{I}}$, and therefore by Lemma 5.2, it follows that $\eta_{\mathcal{I},\mathcal{I}} = \lambda \mathrm{id}_{\mathcal{I}}$. Thus

$$\psi = \eta \circ \phi = (\mathrm{id}_{\mathcal{G}} + \lambda \mathrm{id}_{\mathcal{I}}) \circ (\varphi_{\mathcal{G},\mathcal{G}} + \theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}) = \varphi_{\mathcal{G},\mathcal{G}} + \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}.$$

Hence

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}} + \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}.$$

The proof is complete. □

6. Local and 2-Local automorphisms on simple Leibniz algebras

6.1. **2-Local automorphisms of simple Leibniz algebras.**

Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a complex simple Leibniz algebra. Then any $\varphi \in Aut(\mathcal{L})$ decomposes into

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{G},\mathcal{I}} + \varphi_{\mathcal{I},\mathcal{I}},$$

$$\varphi_{\mathcal{G},\mathcal{I}} = \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}}$$

where $\omega \in \mathbb{C}$ and $\varphi_{\mathcal{G},\mathcal{I}}$ is assumed to be zero when $\dim \mathcal{G} \neq \dim \mathcal{I}$.

Lemma 6.1. *Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a simple Leibniz algebra and let $\varphi \in Aut(\mathcal{L})$ be such that $\varphi(h_0) = h_0$, where h_0 is a strongly regular element from $\mathcal{H}$. Then*

a) *$\varphi(e_\alpha) = t_\alpha e_\alpha$ and $\varphi(e_{-\alpha}) = t_\alpha^{-1} e_{-\alpha}$, where $t_\alpha \in \mathbb{C}$ for all $\alpha \in \Phi$,*
b) *$\varphi(h) = h$ for all $h \in \mathcal{H}$.*

Proof. Let $\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{G},\mathcal{I}} + \varphi_{\mathcal{I},\mathcal{I}}$. Since

$$h_0 = \varphi(h_0) = \varphi_{\mathcal{G},\mathcal{G}}(h_0) + \varphi_{\mathcal{G},\mathcal{I}}(h_0),$$

it follows that $\varphi_{\mathcal{G},\mathcal{G}}(h_0) = h_0$ and $\varphi_{\mathcal{G},\mathcal{I}}(h_0) = 0$ (that is $\theta(h_0) = 0$). Thus $\varphi_{\mathcal{G},\mathcal{I}} \equiv 0$. Now assertions a) and b) follows from [6, Lemma 2.2]. The proof is complete. □

Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$ and let

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{\alpha\in\Phi} \mathcal{G}_\alpha, \quad \mathcal{I} = \bigoplus_{\beta\in\Gamma} \mathcal{I}_\beta.$$

It is known [14, P. 108] that the $\mathcal{G}$-module $\mathcal{I}$ is generated by the elements of the form

$$[\ldots[[i^+, e_{-\alpha_1}], e_{-\alpha_2}], \ldots, e_{-\alpha_k}], \tag{6.1}$$

where i^+ is a highest weight vector of $\mathcal{G}$-module $\mathcal{I}$, $e_{-\alpha_i} \in \mathcal{G}_{-\alpha_i}$ and $\alpha_i \in \Phi$ is a positive root for all $i = 1, \ldots, k$, $k \in \mathbb{N}$.

Let $\dim \mathcal{I}_\beta = s_\beta,\ \beta \in \Gamma$ and let $\left\{i_\beta^{(1)}, \ldots, i_\beta^{(s_\beta)}\right\}$ be a basis of $\mathcal{I}_\beta$, consisting of elements of the form (6.1). Set

$$i_0 = \sum_{\beta\in\Gamma}\sum_{k=1}^{s_\beta} i_{l_\beta}^{(k)}. \tag{6.2}$$

Lemma 6.2. *Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a simple Leibniz algebra with $\dim \mathcal{G} \neq \dim \mathcal{I}$ and let $\varphi \in Aut(\mathcal{L})$ be such that $\varphi(h_0 + i_0) = h_0 + i_0$. Then φ is an identity automorphism of $\mathcal{L}$.*

Proof. Since

$$h_0 + i_0 = \varphi(h_0 + i_0) = \varphi_{\mathcal{G},\mathcal{G}}(h_0) + \varphi_{\mathcal{I},\mathcal{I}}(i_0),$$

it follows that $\varphi_{\mathcal{G},\mathcal{G}}(h_0) = h_0$ and $\varphi_{\mathcal{I},\mathcal{I}}(i_0) = i_0$. By Lemma 6.1 $\varphi_{\mathcal{G},\mathcal{G}}$ acts diagonally on $\mathcal{G}$, i.e., $\varphi_{\mathcal{G},\mathcal{G}}(e_\alpha) = t_\alpha e_\alpha$, $t_\alpha \in \mathbb{C}$, $\alpha \in \Phi$.

Let i_+ be the highest weight vector of $\mathcal{I}$ and let α be a positive root. Then

$$\begin{aligned} [\varphi(i_+), e_\alpha] &= t_\alpha^{-1}[\varphi(i_+), \varphi(e_\alpha)] = t_\alpha^{-1}\varphi([i_+, e_\alpha]) = \varphi(0) = 0 \\ [\varphi(i_+), h] &= [\varphi(i_+), \varphi(h)] = \alpha_+(h)\varphi(i_+), \end{aligned}$$

where $\alpha_+ \in \Gamma$ is a highest weight. This means that $\varphi(i_+)$ is also a highest weight vector. Since the highest weight subspace is one-dimensional, it follows that $\varphi(i_+) = \lambda_+ i_+$, where $\lambda_+ \in \mathbb{C}$. Now taking into account that $\varphi(i_+) = \lambda_+ i_+$ from previous lemma we conclude that

$$\begin{aligned} \varphi\left(i_\beta^{(k)}\right) &= \lambda_+[\ldots[[i_+, \varphi(e_{-\alpha_1})], \varphi(e_{-\alpha_2})], \ldots, \varphi(e_{-\alpha_k})] = \\ &= \left(\prod_{p=1}^{k} t_{-\alpha_p}\lambda_+\right)[\ldots[[i_+, e_{-\alpha_1}], e_{-\alpha_2}], \ldots, e_{-\alpha_k}] = c_\beta^{(k)}\lambda_+ i_\beta^{(k)}, \end{aligned}$$

i.e.,

$$\varphi\left(i_\beta^{(k)}\right) = c_\beta^{(k)}\lambda_+ i_\beta^{(k)},$$

for $1 \leq k \leq s_\beta, \beta \in \Gamma$. Taking into account these equalities, from $i_0 = \varphi_{\mathcal{I},\mathcal{I}}(i_0)$, we obtain $\varphi(i_\beta^{(k)}) = i_\beta^{(k)}$ for all for $1 \leq k \leq s_\beta, \beta \in \Gamma$. This imply $\varphi(i) = i$ for all $i \in \mathcal{I}$. By Lemma 5.2, it follows that $\varphi = id_{\mathcal{L}}$. The proof is complete. $\square$

Let $\dim \mathcal{H} = k \geq 2$ and let $\alpha_1, \ldots, \alpha_k$ be simple roots. Set $i_s = \theta(h_{\alpha_s})$, $s = 1, \ldots, k$, $i_\alpha = \theta(e_\alpha)$, $\alpha \in \Phi$. Take

$$i_0 = \sum_{s=1}^{k} i_s + \sum_{\alpha\in\Phi} i_\alpha.$$

Lemma 6.3. *Let $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$ be a complex simple Leibniz algebra with* $\dim \mathcal{G} = \dim \mathcal{I}$ *and* $\dim \mathcal{H} = k \geq 2$. *Suppose that φ is an automorphism on $\mathcal{L}$ such that $\varphi(h_0+i_0) = h_0+i_0$. Then $\varphi = id_{\mathcal{L}}$.*

Proof. Let $\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{G},\mathcal{I}} + \varphi_{\mathcal{I},\mathcal{I}}$. Since

$$\varphi_{\mathcal{G},\mathcal{G}}(h_0) = h_0,\ \varphi_{\mathcal{G},\mathcal{I}}(h_0) + \varphi_{\mathcal{I},\mathcal{I}}(i_0) = i_0,$$

by Lemma 6.1 we have that $\varphi_{\mathcal{G},\mathcal{G}}(e_\alpha) = t_\alpha e_\alpha$, $\alpha \in \Phi$ and $\varphi_{\mathcal{G},\mathcal{G}}(h) = h$ for all $h \in \mathcal{H}$. Then

$$\varphi_{\mathcal{I},\mathcal{I}}(i_s) = \lambda\theta(\varphi_{\mathcal{G},\mathcal{G}}(\theta^{-1}(i_s))) = \lambda\theta(\varphi_{\mathcal{G},\mathcal{G}}(h_{\alpha_s})) = \lambda\theta(h_{\alpha_s}) = \lambda i_s$$

and

$$\varphi_{\mathcal{I},\mathcal{I}}(i_\alpha) = \lambda\theta(\varphi_{\mathcal{G},\mathcal{G}}(\theta^{-1}(i_\alpha))) = \lambda\theta(\varphi_{\mathcal{G},\mathcal{G}}(e_\alpha)) = \lambda\theta(e_\alpha) = \lambda t_\alpha i_\alpha.$$

Further

$$\begin{aligned} i_0 &= \varphi_{\mathcal{G},\mathcal{I}}(h_0) + \varphi_{\mathcal{I},\mathcal{I}}(i_0) = \omega\theta(h_0) + \varphi_{\mathcal{I},\mathcal{I}}(i_0) = \\ &= \omega\sum_{s=1}^{k} a_s i_s + \sum_{s=1}^{k}\lambda i_s + \sum_{\alpha\in\Phi}\lambda t_\alpha i_\alpha. \end{aligned}$$

i.e.,

$$\sum_{\alpha\in\Phi}(1-\lambda t_\alpha)i_\alpha = \sum_{s=1}^{k}(\omega a_s + \lambda - 1)i_s.$$

Since the right side of this equality belongs to $\mathcal{I}_0$ and the left side does not belong to $\mathcal{I}_0$, it follows that $\lambda t_\alpha = 1$ for all $\alpha \in \Phi$. Hence $\varphi_{\mathcal{I},\mathcal{I}}(i_\alpha) = i_\alpha$ for all $\alpha \in \Phi$.

Since $\dim \mathcal{H} \geq 2$, any row of the Cartan matrix of a simple Lie algebra $\mathcal{G}$ contains negative number (see [14, page 59]). This means that for any simple root $\alpha_i \in \Phi$ there exists a simple root $\alpha_j \in \Phi$ such that

$$a_{i,j} = (\alpha_i, \alpha_j) < 0,$$

where $(\cdot,\cdot)$ is a bilinear form on $\mathcal{H}^*$ induced by the Killing form on $\mathcal{G}$. Then by [14, Page 45, Lemma 9.4], we obtain that $\alpha_i + \alpha_j$ is also a root and $[e_{\alpha_j}, e_{\alpha_i}] = n_{\alpha_j,\alpha_i} e_{\alpha_i+\alpha_j}$, where n_{α_j,α_i} is a non zero integer.

Further

$$\begin{aligned} [i_{\alpha_j}, e_{\alpha_i}] &= [\theta(e_{\alpha_j}), e_{\alpha_i}] = \theta\left([e_{\alpha_j}, e_{\alpha_i}]\right) = \theta(n_{\alpha_j,\alpha_i} e_{\alpha_i+\alpha_j}) = \\ &= n_{\alpha_j,\alpha_i} i_{\alpha_i+\alpha_j}. \end{aligned}$$

Applying to this equality φ, we obtain that

$$\begin{aligned} n_{\alpha_j,\alpha_i} i_{\alpha_i+\alpha_j} &= n_{\alpha_j,\alpha_i}\varphi(i_{\alpha_i+\alpha_j}) = \varphi([i_{\alpha_j}, e_{\alpha_i}]) = [\varphi(i_{\alpha_j}), \varphi(e_{\alpha_i})] = \\ &= [i_{\alpha_j}, t_{\alpha_i} e_{\alpha_i}] = n_{\alpha_j,\alpha_i} t_{\alpha_i} i_{\alpha_i+\alpha_j}. \end{aligned}$$

Thus $t_{\alpha_i} = 1$ for all $i = 1, \ldots, k$, i.e. $\varphi_{\mathcal{G},\mathcal{G}}$ acts identically on the subset of all simple roots $\{h_{\alpha_i}, e_{\alpha_i}, e_{-\alpha_i} : 1 \leq i \leq k\}$. Since this subset generates the algebra $\mathcal{G}$, it follows that $\varphi_{\mathcal{G},\mathcal{G}}$ acts identically on $\mathcal{G}$, i.e., $\varphi_{\mathcal{G},\mathcal{G}} = \mathrm{id}_{\mathcal{G}}$. By Lemma 5.2 there exists a number λ such that $\varphi_{\mathcal{I},\mathcal{I}} = \lambda\mathrm{id}_{\mathcal{I}}$. Since $\varphi_{\mathcal{I},\mathcal{I}}(i_\alpha) = i_\alpha$, it follows that $\lambda = 1$, i.e., $\varphi_{\mathcal{I},\mathcal{I}} = \mathrm{id}_{\mathcal{I}}$.

Finally

$$i_0 = \varphi_{\mathcal{G},\mathcal{I}}(h_0) + \varphi_{\mathcal{I},\mathcal{I}}(i_0) = \varphi_{\mathcal{G},\mathcal{I}}(h_0) + i_0,$$

implies that $\varphi_{\mathcal{G},\mathcal{I}}(h_0) = 0$, and therefore $\varphi_{\mathcal{G},\mathcal{I}} \equiv 0$. So,

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \varphi_{\mathcal{I},\mathcal{I}} = \mathrm{id}_{\mathcal{L}}.$$

The proof is complete. $\square$

Example 6.4. *Lemma 6.3 is not true for algebras with* $\dim \mathcal{H} = 1$.

There is a unique complex simple Leibniz algebra with one-dimensional Cartan subalgebra and $\dim \mathcal{G} = \dim \mathcal{I}$. *This is the 6-dimensional simple Leibniz algebra*

$$\mathcal{L} = \mathfrak{sl}_2 \dot{+} span\{x_0, x_1, x_2\} = span\{h, e, f, x_0, x_1, x_2\},$$

and non zero products of the basis vectors in $\mathcal{L}$ *are represented as follows [22]:*

$$\begin{array}{lll}
[x_k, e] = -k(3-k)x_{k-1}, & k \in \{1, 2\}, & \\
[x_k, f] = x_{k+1}, & k \in \{0, 1\}, & \\
[x_k, h] = (2-2k)x_k, & k \in \{0, 1, 2\}, & \\
[e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\
[h, e] = -2e, & [f, h] = -2f, & [f, e] = -h.
\end{array}$$

Note that the $\mathfrak{sl}_2$*-module isomorphism* $\theta : \mathfrak{sl}_2 \to \mathcal{I}$ *is defined by*

$$\theta(h) = 2x_1,\ \theta(e) = 2x_0,\ \theta(f) = x_2.$$

Let $\varphi \in Aut(\mathcal{L})$ *be an automorphism such that* $\varphi(h_0 + i_0) = h_0 + i_0$, *where* $h_0 = h$, $i_0 = x_1 + x_0 + x_2$. *Then either* $\varphi = id_{\mathcal{L}}$, *or either*

$$\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \theta \circ \varphi_{\mathcal{G},\mathcal{G}} - \theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}, \tag{6.3}$$

where

$$\varphi_{\mathcal{G},\mathcal{G}}(h) = h,\ \varphi_{\mathcal{G},\mathcal{G}}(e) = -e,\ \varphi_{\mathcal{G},\mathcal{G}}(f) = -f.$$

Let $\varphi = \varphi_{\mathcal{G},\mathcal{G}} + \omega\theta \circ \varphi_{\mathcal{G},\mathcal{G}} + \lambda\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1}$. *Then* $\varphi(h + i_0) = h + i_0$ *implies that* $\varphi_{\mathcal{G},\mathcal{G}}(h) = h$ *and* $\omega\theta(h) + \lambda\theta(\varphi_{\mathcal{G},\mathcal{G}}(\theta^{-1}(i_0))) = i_0$. *Using Lemma 6.1 we obtain that*

$$\varphi_{\mathcal{G},\mathcal{G}}(h) = h,\ \varphi_{\mathcal{G},\mathcal{G}}(e) = te,\ \varphi_{\mathcal{G},\mathcal{G}}(f) = t^{-1}f.$$

Since

$$x_1 + x_0 + x_2 = i_0 = \omega\theta(h) + \lambda\theta(\varphi_{\mathcal{G},\mathcal{G}}(\theta^{-1}(i_0))) = (2\omega + \lambda)x_1 + \lambda t x_0 + \lambda t^{-1} x_2,$$

it follows that $2\omega + \lambda = 1$, $\lambda t = \lambda t^{-1} = 1$.

Case 1. $\lambda = t = 1$. *In this case* $\omega = 0$. *Thus* $\varphi_{\mathcal{G},\mathcal{G}} = id_{\mathcal{G}}$, *and therefore* $\theta \circ \varphi_{\mathcal{G},\mathcal{G}} \circ \theta^{-1} = id_{\mathcal{I}}$. *Hence* $\varphi = id_{\mathcal{L}}$.

Case 2. $\lambda = t = -1$. *In this case* $\omega = 1$, *and we obtain an automorphism of the form* (6.3).

Lemma 6.5. *Let* ∇ *be a 2-local automorphism of the complex simple Leibniz algebra* $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$, *where* $\mathcal{I} = span\{x_0, x_1, x_2\}$, *such that* $\nabla(h) = h$ *and* $\nabla(h + i_0) = h + i_0$, *where* $i_0 = x_0 + x_1 + x_2$. *Then* $\nabla = id_{\mathcal{L}}$.

Proof. Let $x \in \mathfrak{sl}_2$. Take an automorphism $\varphi^{x,h} = \varphi^{x,h}_{\mathfrak{sl}_2,\mathfrak{sl}_2} + \varphi^{x,h}_{\mathfrak{sl}_2,\mathcal{I}} + \varphi^{x,h}_{\mathcal{I},\mathcal{I}} \in \mathrm{Aut}(\mathcal{L})$ such that $\nabla(x) = \varphi^{x,h}(x)$ and $\nabla(h) = \varphi^{x,h}(h)$. Since

$$h = \nabla(h) = \varphi^{x,h}(h) = \varphi^{x,h}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(h) + \varphi^{x,h}_{\mathfrak{sl}_2,\mathcal{I}}(h),$$

it follows that $\varphi^{x,h}_{\mathfrak{sl}_2,\mathcal{I}} \equiv 0$. Then

$$\nabla(x) = \varphi^{x,h}(x) = \varphi^{x,h}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(x) \in \mathfrak{sl}_2,\ \forall\, x \in \mathfrak{sl}_2.$$

Let now $x \in \mathfrak{sl}_2$ be a non zero element. Take an automorphism $\varphi^{x,h+i_0} = \varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2} + \varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathcal{I}} + \varphi^{x,h+i_0}_{\mathcal{I},\mathcal{I}}$ such that $\nabla(x) = \varphi^{x,h+i_0}(x)$ and $\nabla(h + i_0) = \varphi^{x,h+i_0}(h + i_0)$. Since $\nabla(x) \in \mathfrak{sl}_2$ and

$$\nabla(x) = \varphi^{x,h+i_0}(x) = \varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(x) + \varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathcal{I}}(x),$$

we have that $\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathcal{I}}\equiv 0$. Then

$$h+i_0=\nabla(h+i_0)=\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(h)+\varphi^{x,h+i_0}_{\mathcal{I},\mathcal{I}}(i_0),$$

implies that $\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(h)=h$ and $\varphi^{x,h+i_0}_{\mathcal{I},\mathcal{I}}(i_0)=i_0$. By Example 6.4 we have that $\varphi^{x,h+i_0}\equiv \mathrm{id}_{\mathcal{L}}$, and therefore

$$\nabla(x)=\varphi^{x,h+i_0}(x)=x.$$

Let $x\in\mathfrak{sl}_2$ be a non zero element and let $i\in\mathcal{I}$. Take an automorphism $\varphi^{x,x+i}=\varphi^{x,x+i}_{\mathfrak{sl}_2,\mathfrak{sl}_2}+\varphi^{x,x+i}_{\mathfrak{sl}_2,\mathcal{I}}+\varphi^{x,x+i}_{\mathcal{I},\mathcal{I}}$ such that $\nabla(x)=\varphi^{x,x+i}(x)$ and $\nabla(x+i)=\varphi^{x,x+i}(x+i)$. Since

$$x=\nabla(x)=\varphi^{x,x+i}(x)=\varphi^{x,x+i}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(x)+\varphi^{x,x+i}_{\mathfrak{sl}_2,\mathcal{I}}(x),$$

it follows that $\varphi^{x,x+i}_{\mathfrak{sl}_2,\mathcal{I}}\equiv 0$. Then

$$\nabla(x+i)=\varphi^{x,x+i}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(x)+\varphi^{x,x+i}_{\mathcal{I},\mathcal{I}}(i)=x+i',$$

where $i'\in\mathcal{I}$.

Now we shall show that $\nabla(x+i)=x+i$ for all $x\in\mathfrak{sl}_2$, $i\in\mathcal{I}$.

Case 1. Let $x=c_1h+c_0e+c_2f$, where $|c_0|+|c_2|\neq 0$, and let $i\in\mathcal{I}$. Take an automorphism $\varphi^{x,h+i_0}=\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}+\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathcal{I}}+\varphi^{x,h+i_0}_{\mathcal{I},\mathcal{I}}$ such that $\nabla(x)=\varphi^{x,h+i_0}(x)$ and $\nabla(h+i_0)=\varphi^{x,h+i_0}(h+i_0)$. Since $\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(h)=h$, by Lemma 6.1 there exists a non zero number t such that $\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(e)=te$ and $\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(f)=t^{-1}f$. Then

$$x=\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(x)=c_1h+tc_0e+t^{-1}c_2f.$$

Since $|c_0|+|c_2|\neq 0$, it follows that $t=1$. Thus $\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}=\mathrm{id}_{\mathfrak{sl}_2}$ and $\varphi^{x,h+i_0}_{\mathcal{I},\mathcal{I}}=\lambda\,\mathrm{id}_{\mathcal{I}}$ (see Lemma 5.2). Further

$$i_0=\varphi^{x,h+i_0}_{\mathfrak{sl}_2,\mathcal{I}}(h)+\varphi^{x,h+i_0}_{\mathcal{I},\mathcal{I}}(i_0)=\omega\theta(h)+\lambda i_0=(2\omega+\lambda)x_1+\lambda(x_0+x_2).$$

Thus $\lambda=1$ and $\omega=0$, and therefore $\varphi^{x,h+i_0}=\mathrm{id}_{\mathfrak{sl}_2}$. So,

$$\nabla(x+i)=x+i.$$

Case 2. Let $x=c_1h+c_0e+c_2f$, where $c_1\neq 0$, and let $i\in\mathcal{I}$. Take an automorphism $\varphi^{x,e+i_0}=\varphi^{x,e+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}+\varphi^{x,e+i_0}_{\mathfrak{sl}_2,\mathcal{I}}+\varphi^{x,e+i_0}_{\mathcal{I},\mathcal{I}}$ such that $\nabla(x)=\varphi^{x,e+i_0}(x)$ and $\nabla(e+i_0)=\varphi^{x,e+i_0}(e+i_0)$. From Case 1, it follows that $\nabla(e+i_0)=e+i_0$. Then

$$\varphi^{x,e+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(h)=h \text{ and } \varphi^{x,e+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(e)=e,$$

because $c_1\neq 0$. Thus Lemma 6.1 implies that $\varphi^{x,e+i_0}_{\mathfrak{sl}_2,\mathfrak{sl}_2}\equiv\mathrm{id}_{\mathfrak{sl}_2}$. By Lemma 5.2 we have $\varphi^{x,e+i_0}_{\mathcal{I},\mathcal{I}}=\lambda\mathrm{id}_{\mathcal{I}}$. Further

$$i_0=\varphi^{x,e+i_0}_{\mathfrak{sl}_2,\mathcal{I}}(e)+\varphi^{x,e+i_0}_{\mathcal{I},\mathcal{I}}(i_0)=\omega\theta(e)+\lambda i_0=(2\omega+\lambda)x_0+\lambda(x_1+x_2).$$

Thus $\lambda=1$ and $\omega=0$, and therefore $\varphi^{x,e+i_0}=\mathrm{id}_{\mathfrak{sl}_2}$. So,

$$\nabla(x+i)=x+i.$$

Case 3. Let $i\in\mathcal{I}$. Take an automorphism $\varphi^{i,h+i}$ such that $\nabla(i)=\varphi^{i,h+i}(i)$ and $\nabla(h+i)=\varphi^{i,h+i}(h+i)$. From Case 2, it follows that

$$h+i=\nabla(h+i)=\varphi^{i,h+i}_{\mathfrak{sl}_2,\mathfrak{sl}_2}(h)+\varphi^{i,h+i}_{\mathfrak{sl}_2,\mathcal{I}}(h)+\varphi^{i,h+i}_{\mathcal{I},\mathcal{I}}(i),$$

and therefore

$$i=\varphi^{i,h+i}_{\mathfrak{sl}_2,\mathcal{I}}(h)+\varphi^{i,h+i}_{\mathcal{I},\mathcal{I}}(i).$$

Then

$$i-\nabla(i)=(\varphi^{i,h+i}_{\mathfrak{sl}_2,\mathcal{I}}(h)+\varphi^{i,h+i}_{\mathcal{I},\mathcal{I}}(i))-\varphi^{i,h+i}_{\mathcal{I},\mathcal{I}}(i)=\varphi^{i,h+i}_{\mathfrak{sl}_2,\mathcal{I}}(h)=*x_1. \tag{6.4}$$

Now take an automorphism $\varphi^{i,e+i}$ such that

$$\nabla(i)=\varphi^{i,e+i}(i) \text{ and } \nabla(e+i)=\varphi^{i,e+i}(e+i).$$

From Case 1, it follows that $\nabla(e+i)=e+i$. Then

$$i-\nabla(i)=(\varphi^{i,e+i}_{\mathfrak{sl}_2,\mathcal{I}}(e)+\varphi^{i,e+i}_{\mathcal{I},\mathcal{I}}(i))-\varphi^{i,e+i}_{\mathcal{I},\mathcal{I}}(i)=\varphi^{i,e+i}_{\mathfrak{sl}_2,\mathcal{I}}(e)=*x_0. \tag{6.5}$$

Combining (6.4) and (6.5), we obtain that $\nabla(i)=i$. The proof is complete. □

Theorem 6.6. *Any 2-local automorphism of complex simple Leibniz algebra $\mathcal{L}=\mathcal{G}\dot{+}\mathcal{I}$ is an automorphism.*

Proof. Case 1. Let $\dim\mathcal{G}\neq\dim\mathcal{I}$ or $\dim\mathcal{H}\geq 2$. Let ∇ be a 2-local automorphism and $\nabla(h_0+i_0)=\varphi_{h_0+i_0}(h_0+i_0)$ for some $\varphi_{h_0+i_0}\in Aut(\mathcal{L})$. Denote $\widetilde{\nabla}=\varphi^{-1}_{h_0+i_0}\circ\nabla$. Then for a 2-local automorphism $\widetilde{\nabla}$ we have $\widetilde{\nabla}(h_0+i_0)=h_0+i_0$. For an element $x\in\mathcal{L}$ there exists $\widetilde{\varphi}_{x,h_0+i_0}\in Aut(\mathcal{L})$ such that

$$\widetilde{\varphi}_{x,h_0+i_0}(h_0+i_0)=\widetilde{\nabla}(h_0+i_0)=h_0+i_0 \text{ and } \widetilde{\nabla}(x)=\widetilde{\varphi}_{x,h_0+i_0}(x).$$

Using the Lemmata 6.2 and 6.3 we conclude that $\widetilde{\varphi}_{x,h_0+i_0}=\mathrm{id}_{\mathcal{L}}$. Thus $\widetilde{\nabla}(x)=\widetilde{\varphi}_{x,h_0+i_0}(x)=x$ for each $x\in\mathcal{L}$, and therefore $\varphi^{-1}_{h_0+i_0}\circ\nabla=\mathrm{id}_{\mathcal{L}}$. Hence $\nabla=\varphi_{h_0+i_0}$ is an automorphism.

Case 2. Let $\mathcal{L}$ be an algebra from Example 6.4 and let ∇ be a 2-local automorphism on $\mathcal{L}$. Take a 2-local automorphism $\varphi_{h,h+i_0}$ such that $\nabla(h)=\varphi_{h,h+i_0}(h)$ and $\nabla(h+i_0)=\varphi_{h,h+i_0}(h+i_0)$. Then h and $h+i_0$ both are fixed points of 2-local automorphism $\varphi^{-1}_{h,h+i_0}\circ\nabla$, and therefore by Lemma 6.5, it is an identical automorphism. Thus $\nabla=\varphi_{h,h+i_0}$ is an automorphism. The proof is complete. □

6.2. **2-Local automorphisms on filiform Leibniz algebras.**

The following theorems are similar to the corresponding theorems for the Lie algebras case and their the proofs are obtained by replacing the words "Lie algebra" by "Leibniz algebra" (see [6]).

Theorem 6.7. *Let $\mathcal{L}$ be an n-dimensional Leibniz algebra with $n\geq 2$. Suppose that*

(i) $\dim[\mathcal{L},\mathcal{L}]\leq n-2$;
(ii) $Ann(\mathcal{L})\cap[\mathcal{L},\mathcal{L}]\neq\{0\}$.

Then $\mathcal{L}$ admits a 2-local automorphism which is not an automorphism.

Theorem 6.8. *Let $\mathcal{L}$ be a finite-dimensional non null-filiform nilpotent Leibniz algebra with $\dim\mathcal{L}\geq 2$. Then $\mathcal{L}$ admits a 2-local automorphism which is not an automorphism.*

Let NF_n be the unique n-dimensional nilpotent Leibniz with condition $\dim[\mathcal{L},\mathcal{L}]=n-1$ (see the end of Section 3).

Let $\varphi\in Aut(NF_n)$ and $\varphi(e_1)=\sum\limits_{i=1}^{n}\alpha_ie_i$ for some $\alpha_i\in\mathbb{C},\ \alpha_1\neq 0$, then it is easy to check that

$$\varphi(e_j)=\alpha_1^{j-1}\sum_{i=1}^{n+1-j}\alpha_ie_{i+j-1},\ 2\leq j\leq n.$$

Using this property, as in the case of derivations, we conclude that an automorphism of NF_n is uniquely defined by its value on the element e_1 and any 2-local automorphism of this algebra is an automorphism.

6.3. **Local automorphisms on simple Leibniz algebras.**

The following result shows that the problem concerning local automorphism of simple Leibniz algebras is reduced to the similar problem for simple Lie algebras.

Theorem 6.9. *Let ∇ be a local automorphism of complex simple Leibniz algebra $\mathcal{L} = \mathcal{G}\dot{+}\mathcal{I}$. Then ∇ is an automorphism if and only if its $\nabla_{\mathcal{G},\mathcal{G}}$ part is an automorphism of the Lie algebra $\mathcal{G}$.*

Proof. The necessity part is evident and we shall consider the sufficient part.

Case 1. $\dim\mathcal{G} = \dim\mathcal{I}$. Take basis's $\{x_1,\ldots,x_m\}$ and $\{y_i : y_i = \theta(x_i), i \in \overline{1,m}\}$ on $\mathcal{G}$ and $\mathcal{I}$, respectively, as in the proof of Lemma 4.2.

Suppose that ∇ is a local automorphism of $\mathcal{L}$ such that its $\nabla_{\mathcal{G},\mathcal{G}}$ part is an automorphism. Consider an automorphism $\psi = \nabla_{\mathcal{G},\mathcal{G}} + \theta\circ\nabla_{\mathcal{G},\mathcal{G}}\circ\theta^{-1}$. Then $\psi^{-1}\circ\nabla$ is a local automorphism of $\mathcal{L}$ such that $(\psi^{-1}\circ\nabla)_{\mathcal{G},\mathcal{G}} = \mathrm{id}_{\mathcal{G}}$. So, below it suffices to consider a local automorphism ∇ such that $\nabla_{\mathcal{G},\mathcal{G}} = \mathrm{id}_{\mathcal{G}}$.

Let $x_k \in \mathcal{G}$. Then

$$\nabla(x_k) = \nabla_{\mathcal{G},\mathcal{G}}(x_k) + \nabla_{\mathcal{G},\mathcal{I}}(x_k) = x_k + \nabla_{\mathcal{G},\mathcal{I}}(x_k).$$

Take an automorphism φ^{x_k} such that $\nabla(x_k) = \varphi^{x_k}(x_k)$. Then

$$\nabla(x_k) = \varphi^{x_k}(x_k) = \varphi^{x_k}_{\mathcal{G},\mathcal{G}}(x_k) + \omega_{x_k}\theta(\varphi^{x_k}_{\mathcal{G},\mathcal{G}}(x_k)).$$

Comparing the last two equalities we obtain that $\varphi^{x_k}_{\mathcal{G},\mathcal{G}}(x_k) = x_k$, and therefore

$$\begin{aligned}\nabla(x_k) &= \varphi^{x_k}_{\mathcal{G},\mathcal{G}}(x_k) + \omega_{x_k}\theta(\varphi^{x_k}_{\mathcal{G},\mathcal{G}}(x_k)) = \\ &= x_k + \omega_{x_k}\theta(x_k) = x_k + \omega_{x_k}y_k.\end{aligned}$$

Likewise for an element $x = x_k + x_s \in \mathcal{G}$ we have that

$$\begin{aligned}\nabla(x) &= \varphi^{x}(x) = \varphi^{x}_{\mathcal{G},\mathcal{G}}(x_k+x_s) + \omega_{x_k+x_s}\theta(\varphi^{x}_{\mathcal{G},\mathcal{G}}(x_k+x_s)) = \\ &= x_k + x_s + \omega_{x_k+x_s}\theta(x_k+x_s) = x_k + x_s + \omega_{x_k+x_s}(y_k+y_s).\end{aligned}$$

Since

$$\nabla(x) = \nabla(x_k+x_s) = \nabla(x_k) + \nabla(x_s) = x_k + x_s + \omega_{x_k}y_k + \omega_{x_s}y_s,$$

we have that $\omega_{x_k+x_s} = \omega_{x_k} = \omega_{x_s}$. This means that there exists $\omega \in \mathbb{C}$ such that $\nabla(x) = x + \omega\theta(x)$, i.e., $\nabla_{\mathcal{G},\mathcal{I}} = \omega\theta$.

Now take an element $x = x_k + y_k \in \mathcal{G} + \mathcal{I}$ and an automorphism φ^x such that $\nabla(x) = \varphi^x(x)$. Then

$$\begin{aligned}\nabla(x) &= \varphi^{x}(x) = \varphi^{x}_{\mathcal{G},\mathcal{G}}(x_k) + \omega_{x_k}\theta(\varphi^{x}_{\mathcal{G},\mathcal{G}}(x_k)) + \lambda_x\theta(\varphi^{x}_{\mathcal{G},\mathcal{G}}(\theta^{-1}(y_k))) = \\ &= x_k + \omega_x\theta(x_k) + \lambda_x y_k = x_k + (\omega_x+\lambda_x)y_k.\end{aligned}$$

Further for an element $x = x_k + x_s + y_k + y_s \in \mathcal{G} + \mathcal{I}$ take an automorphism φ^x such that $\nabla(x) = \varphi^x(x)$. Then

$$\begin{aligned}\nabla(x) &= \varphi^{x}(x) = \varphi^{x}_{\mathcal{G},\mathcal{G}}(x_k+x_s) + \omega_{x_k}\theta(\varphi^{x}_{\mathcal{G},\mathcal{G}}(x_k+x_s)) + \lambda_x\theta(\varphi^{x}_{\mathcal{G},\mathcal{G}}(\theta^{-1}(y_k+y_s))) = \\ &= x_k + x_s + \omega_x\theta(x_k+x_s) + \lambda_x(y_k+y_s) = x_k + x_s + (\omega_x+\lambda_x)(y_k+y_s).\end{aligned}$$

Combing the last two equalities we have that there exists $t \in \mathbb{C}$ such that $\nabla(x_k + y_s) = x_k + ty_k$. Then

$$\begin{aligned}\nabla(y_k) &= \nabla(x_k + y_k) - \nabla(x_k) = \\ &= x_k + ty_k - (x_k + \omega y_k) = (t-\omega) y_k.\end{aligned}$$

This means that $\nabla_{\mathcal{I},\mathcal{I}} = (t-\omega)\mathrm{id}_{\mathcal{I}}$. Hence $\nabla = \mathrm{id}_{\mathcal{G}} + \omega\theta + (t-\omega)\mathrm{id}_{\mathcal{I}}$.

Case 2. $\dim \mathcal{G} \neq \dim \mathcal{I}$. Take an element of the form (6.2), i.e.,

$$i_0 = \sum_{\beta\in\Gamma} \sum_{k=1}^{s_\beta} i_{l_\beta}^{(k)}.$$

Let $\varphi_{h_0+i_0}$ be an automorphism such that $\nabla(h_0 + i_0) = \varphi_{h_0+i_0}(h_0 + i_0)$. If necessary, we can replace ∇ by $\varphi_{h_0+i_0}^{-1} \circ \nabla$, and suppose that $\nabla(h_0 + i_0) = h_0 + i_0$.

Since

$$h_0 + i_0 = \nabla(h_0 + i_0) = \nabla_{\mathcal{G},\mathcal{G}}(h_0) + \nabla_{\mathcal{I},\mathcal{I}}(i_0),$$

it follows that $\nabla_{\mathcal{G},\mathcal{G}}(h_0) = h_0$ and $\nabla_{\mathcal{I},\mathcal{I}}(i_0) = i_0$. Since $\nabla_{\mathcal{G},\mathcal{G}}$ is an automorphism, by Lemma 6.1, for every $\alpha \in \Phi$ there exists a non zero $t_\alpha \in \mathbb{C}$ such that $\nabla_{\mathcal{G},\mathcal{G}}(e_\alpha) = t_\alpha e_\alpha$, $\nabla_{\mathcal{G},\mathcal{G}}(e_{-\alpha}) = t_\alpha^{-1} e_{-\alpha}$ and $\nabla_{\mathcal{G},\mathcal{G}}(h) = h$ for all $h \in \mathcal{H}$.

Let $i_\beta \in \mathcal{I}_\beta$. Then

$$\nabla(h_0 + i_\beta) = \nabla(h_0) + \nabla(i_\beta) = h_0 + \nabla(i_\beta).$$

Take an automorphism $\varphi^{h_0+i_\beta}$ such that $\nabla(h_0 + i_\beta) = \varphi^{h_0+i_\beta}(h_0 + i_\beta)$. Then

$$\nabla(h_0 + i_\beta) = \varphi^{h_0+i_\beta}(h_0 + i_\beta) = \varphi_{\mathcal{G},\mathcal{G}}^{h_0}(h_0) + \varphi_{\mathcal{I},\mathcal{I}}^{h_0+i_\beta}(i_\beta).$$

Comparing the last two equalities we obtain that $\varphi_{\mathcal{G},\mathcal{G}}^{h_0+i_\beta}(h_0) = h_0$, and therefore $\varphi^{h_0+i_\beta}$ acts as diagonal matrix on $\mathcal{L}$. Thus

$$\nabla(i_\beta) = \varphi_{\mathcal{I},\mathcal{I}}^{h_0+i_\beta}(i_\beta) = c_\beta i_\beta.$$

Since $\nabla(i_0) = i_0$, it follows that $\nabla(i_\beta) = i_\beta$ for all β. So, $\nabla_{\mathcal{I},\mathcal{I}} = \mathrm{id}_{\mathcal{I}}$.

Let $\alpha \in \Phi$. Considering $\mathcal{I}$ as $(\mathfrak{sl}_2)_\alpha$-module, where $(\mathfrak{sl}_2)_\alpha \equiv \mathrm{span}\{e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]\}$, we can find a non trivial irreducible submodule $\mathcal{J}_\alpha$ of $\mathcal{I}$. Then $\mathcal{J}_\alpha$ admits a basis $\{x_0^\alpha, \ldots, x_n^\alpha\}$ such that [22]:

$$\begin{array}{ll} [x_k^\alpha, e_\alpha] = x_{k+1}^\alpha, & k \in \{0, \ldots, n-1\}, \\ [x_k^\alpha, e_{-\alpha}] = -k(n+1-k) x_{k-1}^\alpha, & k \in \{1, \ldots, n\}, \\ [x_k^\alpha, h_\alpha] = (n-2k) x_k^\alpha, & k \in \{0, \ldots, n\}. \end{array}$$

The matrix of the right multiplication operator $R_{h_\alpha+e_\alpha}$ on $\mathcal{J}_\alpha$ has the following form

$$\begin{pmatrix} n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & n-2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & n-4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -(n-4) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -(n-2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -n \end{pmatrix}.$$

Direct computations show that n is a eigenvalue of this matrix and we take a non zero eigenvector $i = \sum\limits_{s=0}^{n} t_s x_s^\alpha \in \mathcal{J}_\alpha$ corresponding to this eigenvalue, i.e., $[i, h_\alpha + e_\alpha] = ni$, with $t_0 \neq 0$. For an element $x = h_\alpha + e_\alpha + i$ choose an automorphism φ^x such that $\nabla(x) = \varphi^x(x)$. Then

$$[\nabla(x), \nabla(x)] = [h_\alpha + t_\alpha e_\alpha + i, h + t_\alpha e_\alpha + i] = [i, h_\alpha + t_\alpha e_\alpha]$$

and

$$\begin{aligned}[\nabla(x), \nabla(x)] &= [\varphi^x(h_\alpha + e_\alpha + i), \varphi^x(h_\alpha + e_\alpha + i)] = \\ &= \varphi^x([h_\alpha + e_\alpha + i, h_\alpha + e_\alpha + i]) = \\ &= \varphi^x([i, h_\alpha + e_\alpha]) = n\varphi^x(i) = n\nabla(i) = ni.\end{aligned}$$

The last two equalities implies that

$$[i, h_\alpha + t_\alpha e_\alpha] = ni = [i, h_\alpha + e_\alpha].$$

Comparing coefficients at the basis element x_1^α in the above equality we conclude that

$$t_\alpha t_0 + (n-2)t_1 = t_0 + (n-2)t_1.$$

Thus $t_\alpha = 1$, and therefore $\nabla(e_\alpha) = e_\alpha$. So, $\nabla = \mathrm{id}_{\mathcal{L}}$. The proof is complete. $\square$

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