

The geometric classification of Leibniz algebras ¹

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Abstract. We describe all rigid algebras and all irreducible components in the variety of four dimensional Leibniz algebras \mathfrak{Leib}_4 over \mathbb{C} . In particular, we prove that the Grunewald–O’Halloran conjecture is not valid and the Vergne conjecture is valid for \mathfrak{Leib}_4 .

Keywords: Leibniz algebra, Grunewald–O’Halloran conjecture, Vergne conjecture, orbit closure, degeneration, rigid algebra

1. INTRODUCTION

Degenerations of algebras is an interesting subject, which was studied in various papers (see, for example, [3, 4, 7, 9, 10, 14–16, 19, 23]). In particular, there are many results concerning degenerations of algebras of low dimensions in a variety defined by a set of identities. One of important problems in this direction is the description of so-called rigid algebras. These algebras are of big interest, since the closures of their orbits under the action of generalized linear group form irreducible components of a variety under consideration (with respect to the Zariski topology). For example, the rigid algebras were classified in the varieties of two dimensional bicommutative algebras in [14], three dimensional Novikov algebras in [3], three dimensional Leibniz in [22], four dimensional Lie algebras in [4], four dimensional Jordan algebras in [17], four dimensional Zinbiel algebras and nilpotent four dimensional Leibniz algebras in [16], unital five dimensional associative algebras in [19], nilpotent five- and six-dimensional Lie algebras in [9, 23], nilpotent five- and six-dimensional Malcev algebras in [15], and some other.

The Leibniz algebras were introduced as a generalization of Lie algebras. The study of the structure theory and other properties of Leibniz algebras was initiated by Loday in [18]. Leibniz algebras were also studied in [6, 8, 20]. An algebra A is called a *Leibniz algebra* if it satisfies the identity

$$(xy)z = (xz)y + x(yz).$$

It is easy to see that any Lie algebra is a Leibniz algebra. At this moment, the algebraic classification of n dimensional Leibniz algebras over \mathbb{C} is known only for $n \leq 4$.

In this paper we describe all rigid algebras and all irreducible components in variety of \mathfrak{Leib}_4 . As a result, we show that \mathfrak{Leib}_4 has 6 rigid algebras and 17 irreducible components.

2. DEFINITIONS AND NOTATION

All spaces in this paper are considered over \mathbb{C} , and we write simply dim , Hom and \otimes instead of $dim_{\mathbb{C}}$, $Hom_{\mathbb{C}}$ and $\otimes_{\mathbb{C}}$. An algebra A is a set with a structure of a vector space and a binary operation that induces a bilinear map from $A \times A$ to A .

Given an n -dimensional vector space V , the set $Hom(V \otimes V, V) \cong V^* \otimes V^* \otimes V$ is a vector space of dimension n^3 . This space has a structure of the affine variety \mathbb{C}^{n^3} . Indeed, let us fix a basis e_1, \dots, e_n of V . Then any $\mu \in Hom(V \otimes V, V)$ is determined by n^3 structure constants $c_{i,j}^k \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^n c_{i,j}^k e_k$. A subset of $Hom(V \otimes V, V)$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables $c_{i,j}^k$ ($1 \leq i, j, k \leq n$).

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All algebra structures on V satisfying Leibniz identity form a Zariski-closed subset of the variety $\text{Hom}(V \otimes V, V)$. We denote this subset by \mathfrak{Leib}_n . The general linear group $GL(V)$ acts on \mathfrak{Leib}_n by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in V$, $\mu \in \mathfrak{Leib}_n \subset \text{Hom}(V \otimes V, V)$ and $g \in GL(V)$. Thus, \mathfrak{Leib}_n is decomposed into $GL(V)$ -orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the orbit of $\mu \in \mathfrak{Leib}_n$ under the action of $GL(V)$ and $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$.

Let A and B be two n -dimensional Leibniz algebras and $\mu, \lambda \in \mathfrak{Leib}_n$ represent A and B respectively. We say that A degenerates to B and write $A \rightarrow B$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ . We write $A \not\rightarrow B$ if $\lambda \notin \overline{O(\mu)}$.

Let A be represented by $\mu \in \mathfrak{Leib}_n$. Then A is *rigid* in \mathfrak{Leib}_n if $O(\mu)$ is an open subset of \mathfrak{Leib}_n . Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called *irreducible component*. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. In particular, A is rigid in \mathfrak{Leib}_n iff $\overline{O(\mu)}$ is an irreducible component of \mathfrak{Leib}_n . This is a general fact about algebraic varieties whose proof can be found, for example, in [16].

We use the following notation:

- (1) $\text{Ann}_L(A) = \{a \in A \mid ax = 0 \text{ for all } x \in A\}$ is the left annihilator of A ;
- (2) $\text{Ann}_R(A) = \{a \in A \mid ax = 0 \text{ for all } x \in A\}$ is the right annihilator of A ;
- (3) $\text{Ann}(A) = \text{Ann}_R(A) \cap \text{Ann}_L(A)$ is the annihilator of A ;
- (4) $A^{(+2)}$ is the space $\{xy + yx \mid x, y \in A\}$.

Given spaces U and W , we write simply $U > W$ instead of $\dim U > \dim W$. We write UW ($U, W \subset V$) for the product of subspaces of V with respect to the multiplication μ . We use the notation $S_i = \langle e_i, \dots, e_4 \rangle$, $i = 1, \dots, 4$.

3. METHODS

In the present work we use the methods applied in our previous works (see [14–16]).

To prove degenerations, we will construct families of matrices parametrized by t . Namely, let A and B be two Leibniz algebras represented by the structures μ and λ from \mathfrak{Leib}_n respectively. Let e_1, \dots, e_n be a basis of V and $c_{i,j}^k$ ($1 \leq i, j, k \leq n$) be the structure constants of λ in this basis. If there exist $a_i^j(t) \in \mathbb{C}$ ($1 \leq i, j \leq n, t \in \mathbb{C}^*$) such that $E_i^t = \sum_{j=1}^n a_i^j(t)e_j$ ($1 \leq i \leq n$) form a basis of V for any $t \in \mathbb{C}^*$, and the structure constants of μ in the basis E_1^t, \dots, E_n^t are such polynomials $c_{i,j}^k(t) \in \mathbb{C}[t]$ that $c_{i,j}^k(0) = c_{i,j}^k$, then $A \rightarrow B$. In this case E_1^t, \dots, E_n^t is called a *parametrized basis* for $A \rightarrow B$. It is easy to see that any algebra degenerates to the algebra with zero multiplication.

Let now $A(*) := \{A(\alpha)\}_{\alpha \in I}$ be a set of algebras, and let B be another algebra. Suppose that, for $\alpha \in I$, $A(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathfrak{Leib}_n$ and $B \in \mathfrak{Leib}_n$ is represented by the structure λ . Then $A(*) \rightarrow B$ means $\lambda \in \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$, and $A(*) \not\rightarrow B$ means $\lambda \notin \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$.

Let $A(*)$, B , $\mu(\alpha)$ ($\alpha \in I$) and λ be as above. To prove $A(*) \rightarrow B$ it is enough to construct a family of pairs $(g(t), f(t))$ parametrized by $t \in \mathbb{C}^*$, where $f(t) \in I$ and $g(t) \in GL(V)$. Namely, let e_1, \dots, e_n be a basis of V and $c_{i,j}^k$ ($1 \leq i, j, k \leq n$) be the structure constants of λ in this basis. If we construct $a_i^j : \mathbb{C}^* \rightarrow \mathbb{C}$ ($1 \leq i, j \leq n$) and $f : \mathbb{C}^* \rightarrow I$ such that $E_i^t = \sum_{j=1}^n a_i^j(t)e_j$ ($1 \leq i \leq n$) form a basis of V for any $t \in \mathbb{C}^*$, and the structure constants of $\mu_{f(t)}$ in the basis E_1^t, \dots, E_n^t are such polynomials $c_{i,j}^k(t) \in \mathbb{C}[t]$ that $c_{i,j}^k(0) = c_{i,j}^k$, then $A(*) \rightarrow B$. In this case E_1^t, \dots, E_n^t and $f(t)$ are called a parametrized basis and a *parametrized index* for $A(*) \rightarrow B$ respectively.

Note also the following fact. Suppose that, for $\alpha \in \mathbb{C}$, the structure $\mu(\alpha) \in \mathfrak{Leib}_n$ has structure constants $c_{i,j}^k(\alpha) \in \mathbb{C}$ in the basis e_1, \dots, e_n , where $c_{i,j}^k(t) \in \mathbb{C}[t]$ for all $1 \leq i, j, k \leq n$. Let X be some subset of \mathfrak{Leib}_n such that $\mu(\alpha) \in$

\overline{X} for $\alpha \in \mathbb{C} \setminus S$, where S is a finite subset of \mathbb{C} . Then $\mu(\alpha) \in \overline{X}$ for all $\alpha \in \mathbb{C}$. Indeed, $\mu(\alpha) \in \overline{\{\mu(\beta)\}_{\beta \in \mathbb{C} \setminus S}} \subset \overline{X}$ for any $\alpha \in \mathbb{C}$. Thus, to prove that $\mu(\alpha) \in \overline{X}$ for all $\alpha \in \mathbb{C}$ we will prove that $\mu(\alpha) \in \overline{X}$ for all but finitely many α .

Note that $A(*) \not\rightarrow B$ if $\dim \text{Der}(A(\alpha)) > \dim \text{Der}(B)$ for all $\alpha \in I$, where $\text{Der}(A)$ is the Lie algebra of derivations of A . Moreover, in the case of one algebra, if $A \rightarrow B$ and $A \not\cong B$, then $\text{Der}(A) < \text{Der}(B)$.

In other cases, the main tool for proving assertions of the form $A(*) \not\rightarrow B$ we will use the following lemma.

Lemma 1 (Lemma 2 of [16]). *Let \mathcal{B} be a Borel subgroup of $GL(V)$ and $\mathcal{R} \subset \mathfrak{Leib}_n$ be a \mathcal{B} -stable closed subset. If $A(*) \rightarrow B$ and for any $\alpha \in I$ the algebra $A(\alpha)$ can be represented by a structure $\mu(\alpha) \in \mathcal{R}$, then there is $\lambda \in \mathcal{R}$ representing B .*

In all our applications of Lemma 1 we will use the group of lower triangular matrices as a Borel subgroup. This lemma can be applied in the case $|I| = 1$ to prove non-degenerations. In particular, if $\text{Ann}_L(A) > \text{Ann}_L(B)$, $A^2 < B^2$ or $A^{(+2)} < B^{(+2)}$, the $A \not\rightarrow B$ by Lemma 1. We will use also the following two criteria that follow from Lemma 1. If $A \rightarrow B$, then the dimension of a maximal trivial subalgebra of B more or equal to the dimension of a maximal trivial subalgebra of A and the dimension of a maximal anticommutative subalgebra D of B such that $\dim(BD) \leq 1$ is more or equal to the dimension of a maximal anticommutative subalgebra of A satisfying the same condition.

In some cases we will construct the set \mathcal{R} for the assertion $A(*) \not\rightarrow B$ explicitly. In this case we will define \mathcal{R} as a set of structures μ satisfying some polynomial equations. In such a description we always denote by c_{ij}^k ($1 \leq i, j, k \leq n$) the structure constants of μ . Note also that in this case a condition of the form $S_i S_j \subset S_k$ is equivalent to a set of polynomial equations. Moreover, the set defined by such a condition is stable under the action of the group of lower triangular matrices.

4. MAIN THEOREM

The goal of this section is to describe irreducible components in \mathfrak{Leib}_4 . The algebraic classification of four dimensional Leibniz algebras is based on the papers [1, 5, 21]. Let us give a brief introduction to this classification. The analogue of Levi-Malcev's theorem about the splitting of the solvable radical for Leibniz algebras was proved in [2]. Note that any semisimple Leibniz algebra is a Lie algebra, and hence a Leibniz algebra is formed by a semisimple Lie subalgebra and a solvable Leibniz subalgebra. The first step in the classification of four dimensional Leibniz algebras was done in [1], where all nilpotent four dimensional Leibniz algebras were classified. Then the description of all four dimensional solvable non-nilpotent Leibniz algebras was obtained in [5]. Finally, it was proved in [21] that there is only one non-solvable indecomposable Leibniz algebra whose dimension is less or equal to four, namely, the simple Lie algebra \mathfrak{sl}_2 .

Based on the just mentioned works, we have constructed Table 1 that describes all four dimensional non-nilpotent non-Lie Leibniz algebras. Let us also introduce the following four dimensional Lie algebra structures:

$$\begin{array}{ll} \mathfrak{R}_2 & e_1 e_2 = -e_2 e_1 = e_2, \quad e_3 e_4 = -e_4 e_3 = e_4; \\ \mathfrak{sl}_2 & e_1 e_2 = -e_2 e_1 = e_2, \quad e_1 e_3 = -e_3 e_1 = -e_3, \quad e_2 e_3 = -e_3 e_2 = e_1; \\ g_4(a, b) & e_1 e_2 = -e_2 e_1 = e_2, \quad e_1 e_3 = -e_3 e_1 = e_2 + a e_3, \quad e_1 e_4 = -e_4 e_1 = e_3 + b e_4; \\ g_5(a) & e_1 e_2 = -e_2 e_1 = e_2, \quad e_1 e_3 = -e_3 e_1 = e_2 + a e_3, \quad e_1 e_4 = -e_4 e_1 = (a + 1) e_4, \quad e_2 e_3 = -e_3 e_2 = e_4. \end{array}$$

Due to the results of [4], the variety of four dimensional Lie algebras contains 4 irreducible components, namely, $O(\mathfrak{sl}_2)$, $O(\mathfrak{R}_2)$, $\{O(g_5(a))\}_{a \in \mathbb{C}}$, and $\{O(g_4(a, b))\}_{a, b \in \mathbb{C}}$. In particular, there are 2 rigid four dimensional Lie algebras, namely, \mathfrak{sl}_2 and \mathfrak{R}_2 .

The main result of the present paper is the following theorem.

Theorem 2. *The variety \mathfrak{Leib}_4 has 17 irreducible components:*

$$\begin{array}{l} \overline{\{O(g_4(a, b))\}_{a, b \in \mathbb{C}}}, \overline{\{O(g_5(a))\}_{a \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_4^a)\}_{a \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_8^a)\}_{a \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_9^a)\}_{a \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_{10}^a)\}_{a \in \mathbb{C}}}, \\ \overline{\{O(\mathfrak{L}_{15}^a)\}_{a \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_{18}^a)\}_{a \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_{21}^{a, b})\}_{a, b \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_{22}^{a, b})\}_{a, b \in \mathbb{C}}}, \overline{\{O(\mathfrak{L}_{23}^{a, b})\}_{a, b \in \mathbb{C}}}, \\ \overline{O(\mathfrak{sl}_2)}, \overline{O(\mathfrak{R}_1)}, \overline{O(\mathfrak{R}_2)}, \overline{O(\mathfrak{R}_3)}, \overline{O(\mathfrak{L}_2)}, \overline{O(\mathfrak{L}_{44})}. \end{array}$$

In particular, there are 6 rigid four dimensional Leibniz algebras:

$$\mathfrak{sl}_2, \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{L}_2, \mathfrak{L}_{44}.$$

Non-Lie algebras mentioned in the theorem are described in Table 1. The rest of the paper is devoted to the proof of Theorem 2.

4.1. Nilpotent Leibniz algebras and conjectures about them. Several conjectures state that nilpotent Lie algebras form a very small subvariety in the variety of Lie algebras. Grunewald and O'Halloran conjectured in [11] that for any n -dimensional nilpotent Lie algebra A there exists an n -dimensional non-nilpotent Lie algebra B such that $B \rightarrow A$. At the same time, Vergne conjectured in [24] that a nilpotent Lie algebra cannot be rigid in the variety of all Lie algebras. Analogous assertions can be conjectured for other varieties. Grunewald–O'Halloran Conjecture was proved for four dimensional Lie and Jordan algebras in [4, 17] and for three dimensional Novikov and Leibniz algebras in [3, 7]. Also, some results concerning Grunewald–O'Halloran Conjecture for Lie algebras were obtained in [12, 13]. Since the current work is devoted to Leibniz algebras, we consider here conjectures about nilpotent Leibniz algebras.

Grunewald–O'Halloran Conjecture. For any n -dimensional nilpotent Leibniz algebra A ($n \geq 2$) there exists an n -dimensional non-nilpotent Leibniz algebra B such that $B \rightarrow A$.

Vergne–Grunewald–O'Halloran Conjecture. Any irreducible component in the variety of n -dimensional Leibniz algebras ($n \geq 2$) contains a non-nilpotent algebra.

Vergne Conjecture. Any n -dimensional nilpotent Leibniz algebra ($n \geq 2$) is not rigid in the variety of n -dimensional Leibniz algebras.

It is clear that Vergne–Grunewald–O'Halloran Conjecture follows from Grunewald–O'Halloran Conjecture and Vergne Conjecture follows from Vergne–Grunewald–O'Halloran Conjecture. As a part of the proof of Theorem 2, we will prove that Vergne–Grunewald–O'Halloran Conjecture is valid in \mathfrak{Leib}_4 . On the other hand, we will show that Grunewald–O'Halloran Conjecture is not satisfied in the same variety. Thus, we will disprove Grunewald–O'Halloran Conjecture for Leibniz algebras.

Let us also introduce the following four dimensional nilpotent Leibniz algebra structures:

$$\begin{array}{ll} \mathfrak{N}_3^a & e_1e_1 = e_4, \quad e_1e_2 = ae_4, \quad e_2e_1 = -ae_4, \quad e_2e_2 = e_4, \quad e_3e_3 = e_4; \\ \mathfrak{L}_2^n & e_1e_1 = e_2, \quad e_2e_1 = e_3, \quad e_3e_1 = e_4; \\ \mathfrak{L}_5^n & e_1e_1 = e_3, \quad e_2e_1 = e_3, \quad e_2e_2 = e_4, \quad e_3e_1 = e_4; \\ \mathfrak{L}_{11}^n & e_1e_1 = e_4, \quad e_1e_2 = -e_3, \quad e_1e_3 = -e_4, \quad e_2e_1 = e_3, \quad e_2e_2 = e_4, \quad e_3e_1 = e_4. \end{array}$$

It was proved in [16] that the variety of four dimensional nilpotent Leibniz algebras is formed by four irreducible components, namely, $O(\mathfrak{L}_2^n)$, $O(\mathfrak{L}_5^n)$, $O(\mathfrak{L}_{11}^n)$ and $\{O(\mathfrak{N}_3^a)\}_{a \in \mathbb{C}}$.

Lemma 3. Any irreducible component in \mathfrak{Leib}_4 contains a non-nilpotent algebra.

Proof. It is enough to prove that \mathfrak{L}_2^n , \mathfrak{L}_5^n , \mathfrak{L}_{11}^n and \mathfrak{N}_3^a belong to the closure of the union of orbits of non-nilpotent Leibniz algebras for any $a \in \mathbb{C}$. Let us prove this assertion case by case.

- The parametrized basis $E_1^t = te_1 + te_4$, $E_2^t = t^2e_2 + t^2e_4$, $E_3^t = t^3e_3 + t^3e_4$, $E_4^t = -t^4e_3$ gives the degeneration $\mathfrak{L}_{40} \rightarrow \mathfrak{L}_2^n$.
- The parametrized basis $E_1^t = te_1 - te_2 + e_3$, $E_2^t = t^2e_2 + e_3$, $E_3^t = te_3 - t^3e_4$, $E_4^t = t^4e_4$ with parametrized index $\epsilon(t) = -t$ give the assertion $\mathfrak{L}_{18}^* \rightarrow \mathfrak{L}_5^n$.
- The parametrized basis $E_1^t = te_1 - \frac{t}{2}e_2$, $E_2^t = \frac{t^2}{2}e_2 + t^2e_3$, $E_3^t = t^3e_3 - \frac{t^3}{4}e_4$, $E_4^t = \frac{t^4}{4}e_4$ with parametrized index $\epsilon(t) = \frac{t^2+1}{4}$ give the assertion $\mathfrak{L}_{15}^* \rightarrow \mathfrak{L}_{11}^n$.
- The parametrized basis $E_1^t = te_1 + \frac{t}{2}e_4$, $E_2^t = ite_1 - ie_3 - \frac{it}{2}e_4$, $E_3^t = te_2$, $E_4^t = t^2e_4$ gives the degeneration $\mathfrak{L}_9^{1+ia} \rightarrow \mathfrak{N}_3^a$ for any $a \in \mathbb{C}$.

Let us consider the degeneration $\mathfrak{L}_{40} \rightarrow \mathfrak{L}_2^n$ to clarify our proof. Writing nonzero products of \mathfrak{L}_{40} in the basis E_i^t , we get

$$E_1^t E_1^t = t^2 e_2 + t^2 e_4 = E_2^t, \quad E_2^t E_1^t = t^3 e_3 + t^3 e_4 = E_3^t, \quad E_3^t E_1^t = t^4 e_4 = t E_3^t + E_4^t.$$

It is easy to see now that for $t = 0$ we obtain the multiplication table of \mathfrak{L}_2^n . The remaining assertions can be considered in the same way. \square

Though we have proved that $\overline{O(\mathfrak{L}_5^n)}$ and $\overline{O(\mathfrak{L}_{11}^n)}$ are not irreducible components of $\mathfrak{L}\text{eib}_4$, we have not found an algebra that degenerates either to \mathfrak{L}_5^n or to \mathfrak{L}_{11}^n in the proof of Lemma 3. In the proof of the next result we show that in fact there is no algebra that degenerates to \mathfrak{L}_5^n .

Theorem 4. *Grunewald–O’Halloran Conjecture is not valid in $\mathfrak{L}\text{eib}_4$.*

Proof. To prove the theorem, we will show that there is no four dimensional Leibniz algebra that degenerates to \mathfrak{L}_5^n . Since \mathfrak{L}_5^n is not a Lie algebra, only algebras from Table 1 can degenerate to it. Direct calculations show that the dimension of the Lie algebra of derivations is more or equal to 3 for all algebras from Table 1 except \mathfrak{R}_1 and \mathfrak{L}_{44} . Since $\dim \text{Der}(\mathfrak{L}_5^n) = 3$ (see, for example [16]) it suffices to prove that $\mathfrak{R}_1 \not\rightarrow \mathfrak{L}_5^n$ and $\mathfrak{L}_{44} \not\rightarrow \mathfrak{L}_5^n$.

- $\mathfrak{R}_1 \not\rightarrow \mathfrak{L}_5^n$ follows from the fact that $\text{Ann}_L(\mathfrak{R}_1) > \text{Ann}_L(\mathfrak{L}_5^n)$.
- To prove the assertion $\mathfrak{L}_{44} \not\rightarrow \mathfrak{L}_5^n$ let us consider the set

$$\mathcal{R} = \left\{ \mu \in \mathfrak{L}\text{eib}_4 \mid \begin{array}{l} S_1 S_3 + S_4 S_2 = 0, S_3 S_2 + S_4 S_1 \subset S_4, S_2 S_2 + S_3 S_1 \subset S_3, S_2 S_1 + S_1 S_2 \subset S_2, \\ c_{12}^2 + c_{21}^2 = 0, c_{31}^3 = 2c_{21}^2, c_{12}^3 = c_{21}^3 \end{array} \right\}.$$

It is not difficult to show that \mathcal{R} is a closed subset of $\mathfrak{L}\text{eib}_4$ that is stable under the action of the subgroup of lower triangular matrices and contains the structure \mathfrak{L}_{44} . It is also not difficult to show that $\mathcal{R} \cap O(\mathfrak{L}_5^n) = \emptyset$. \square

4.2. The proof of the main theorem. Now we are ready to prove Theorem 2. As a first step, we are going to prove that the irreducible components of the variety of four dimensional Lie algebras remain irreducible in $\mathfrak{L}\text{eib}_4$. This fact follows from the next general lemma.

Lemma 5. *Suppose that $A(*) = \{A(\alpha)\}_{\alpha \in T}$ is a set of n -dimensional non-Lie Leibniz algebras. If B is an n -dimensional Lie algebra such that $\text{Ann}(B) = 0$, then $A(*) \not\rightarrow B$.*

Proof. Since $A(\alpha)$ is non-Lie, the ideal $I(\alpha) = \{xy + yx \mid x, y \in A(\alpha)\}$ is nonzero for any $\alpha \in T$. Since $I(\alpha) \subset \text{Ann}_L(A(\alpha))$, we have $\text{Ann}_L(A(\alpha)) > 0 = \text{Ann}(B) = \text{Ann}_L(B)$ for any $\alpha \in T$, and hence $A(*) \not\rightarrow B$. \square

It easily follows from Lemma 5 that if \mathcal{C} is an irreducible component in the variety of n -dimensional Lie algebras containing an algebra with the zero annihilator, then \mathcal{C} is an irreducible component in $\mathfrak{L}\text{eib}_n$. Really, suppose that \mathcal{C}_0 is an irreducible component of $\mathfrak{L}\text{eib}_n$ containing \mathcal{C} . Let \mathcal{L} be a set of all Lie algebras in \mathcal{C}_0 . We have $\mathcal{C} \not\subset \mathcal{C}_0 \setminus \mathcal{L}$ by Lemma 5. Since $\mathcal{C}_0 = \overline{\mathcal{C}_0 \setminus \mathcal{L}} \cup \mathcal{L}$ and \mathcal{C}_0 is irreducible, we have $\mathcal{C}_0 = \mathcal{L}$, and hence $\mathcal{C}_0 = \mathcal{C}$.

Corollary 6. *$\overline{O(\mathfrak{sl}_2)}$, $\overline{O(\mathfrak{R}_2)}$, $\{\overline{O(g_5(a))}\}_{a \in \mathbb{C}}$ and $\{\overline{O(g_4(a, b))}\}_{a, b \in \mathbb{C}}$ are irreducible components of $\mathfrak{L}\text{eib}_4$.*

Proof. The structure \mathfrak{sl}_2 is rigid in $\mathfrak{L}\text{eib}_4$ as a unique non-solvable structure in the variety. The remaining part of the corollary follows from the fact that $\text{Ann}(\mathfrak{R}_2) = 0$, $\text{Ann}(g_5(a)) = 0$ for $a \neq -1$, and $\text{Ann}(g_4(a, b)) = 0$ for all $a, b \in \mathbb{C}^*$. \square

Proof of Theorem 2. Let \mathcal{W} be the union of closed sets listed in the theorem. The assertions proved in Table 2 and the classification given in Table 1 show that all four dimensional non-nilpotent non-Lie Leibniz algebras belong to \mathcal{W} . All Lie algebras belong to \mathcal{W} by the results of the paper [4]. Then it follows from Lemma 3 that all nilpotent Leibniz algebras belong to \mathcal{W} too.

Thus, it remains to show that for any two different sets \mathcal{X} and \mathcal{Y} listed in the theorem $\mathcal{X} \not\subset \mathcal{Y}$. It follows from Corollary 6 and the fact that the set of four dimensional Lie algebras is a closed subset of $\mathcal{L}\text{eib}_4$ that the required assertion is true if \mathcal{X} or \mathcal{Y} is formed by Lie algebras.

Let us start with the algebras \mathfrak{R}_1 and \mathfrak{R}_3 . These are the only two algebras from Table 1 that have two dimensional nilpotent radical. Since all other algebras have a three dimensional nilpotent radical, \mathfrak{R}_1 and \mathfrak{R}_3 do not belong to the closure of orbits all the remaining structures of $\mathcal{L}\text{eib}_4$. Since $\overline{\text{Ann}_L(\mathfrak{R}_1)} > \text{Ann}_L(\mathfrak{R}_3)$ and $(\mathfrak{R}_3)^{(+2)} < (\mathfrak{R}_1)^{(+2)}$, we have $\mathfrak{R}_1 \not\rightarrow \mathfrak{R}_3$ and $\mathfrak{R}_3 \not\rightarrow \mathfrak{R}_1$. Thus, $O(\mathfrak{R}_1)$ and $O(\mathfrak{R}_3)$ are irreducible components. Since $\dim(\mathfrak{R}_1)^2 = \dim(\mathfrak{R}_3)^2 = 2$, these two components do not contain $\{\mathcal{L}_4^a\}_{a \in \mathbb{C}}$, $\{\mathcal{L}_8^a\}_{a \in \mathbb{C}}$, $\{\mathcal{L}_9^a\}_{a \in \mathbb{C}}$, $\{\mathcal{L}_{10}^a\}_{a \in \mathbb{C}}$, $\{\mathcal{L}_{21}^{a,b}\}_{a,b \in \mathbb{C}}$, $\{\mathcal{L}_{22}^{a,b}\}_{a,b \in \mathbb{C}}$, $\{\mathcal{L}_{23}^{a,b}\}_{a,b \in \mathbb{C}}$, \mathcal{L}_2 , and \mathcal{L}_{44} . Since $\text{Ann}_L(\mathfrak{R}_1) > \text{Ann}_L(\mathcal{L}_{15}^a) = \text{Ann}_L(\mathcal{L}_{18}^a)$ for any $a \neq 0$, we have $\mathfrak{R}_1 \not\rightarrow \mathcal{L}_{15}^a, \mathcal{L}_{18}^a$ for $a \neq 0$. Note that \mathfrak{R}_3 contains a three dimensional anticommutative subalgebra $D = \langle e_2, e_3, e_4 \rangle$ such that $\dim(\mathfrak{R}_3 D) = 1$. Since both \mathcal{L}_{15}^a and \mathcal{L}_{18}^a do not have such a subalgebra for any $a \neq 0$, we have $\mathfrak{R}_3 \not\rightarrow \mathcal{L}_{15}^a, \mathcal{L}_{18}^a$ for $a \neq 0$.

All the remaining algebras are solvable non-nilpotent Leibniz algebras with a three dimensional nilpotent radical. Moreover, one can check that each of them is represented by a structure μ such that $\langle e_2, e_3, e_4 \rangle$ is the nilpotent radical and, moreover, the structure constants c_{ij}^k ($1 \leq i, j, k \leq 4$) satisfy the conditions $c_{ij}^k = 0$ if $i, j \geq 2$ and $k \leq \max(i, j)$ and $c_{1i}^j = c_{i1}^j = 0$ for any $2 \leq i, j \leq 4$ such that $j > i$. During this proof we will call a structure with three dimensional nilpotent radical that satisfies the described conditions a *standard structure*. Let us put in the correspondence to a standard structure μ the 6-tuple $S_\mu = (c_{21}^2, c_{12}^2, c_{31}^3, c_{13}^3, c_{41}^4, c_{14}^4) \in \mathbf{k}^6$. It is not difficult to show that if $S_\mu = (a_1, b_1, a_2, b_2, a_3, b_3)$ and $\lambda \in O(\mu)$ is a standard structure, then there is some permutation $\sigma : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ and some $c \in \mathbf{k}^*$ such that $S_\lambda = (ca_{\sigma(1)}, cb_{\sigma(1)}, ca_{\sigma(2)}, cb_{\sigma(2)}, ca_{\sigma(3)}, cb_{\sigma(3)})$. Suppose now that $\{\mu_s\}_{s \in T}$ is a set of standard structures, $S_{\mu_s} = (a_{1,s}, b_{1,s}, a_{2,s}, b_{2,s}, a_{3,s}, b_{3,s})$, and the linear polynomials $f_1, \dots, f_l \in \mathbf{k}[x_1, x_2, x_3, x_4, x_5, x_6]$ are such that $f_r(a_{1,s}, b_{1,s}, a_{2,s}, b_{2,s}, a_{3,s}, b_{3,s}) = 0$ for all $s \in T$ and $1 \leq r \leq l$. If λ is a standard structure with $S_\lambda = (a_1, b_1, a_2, b_2, a_3, b_3)$, then it easily follows from Lemma 1 that there is some permutation $\sigma : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ and some $c \in \mathbf{k}^*$ such that $f_r(ca_{\sigma(1)}, cb_{\sigma(1)}, ca_{\sigma(2)}, cb_{\sigma(2)}, ca_{\sigma(3)}, cb_{\sigma(3)}) = 0$ for all $1 \leq r \leq l$.

Computation of 6-tuples for the structures under consideration gives the following results:

$$\begin{aligned} S_{\mathcal{L}_2} &= (1, -1, 0, -1, 1, 0), & S_{\mathcal{L}_4^a} &= (1, a, a+1, -1, 0, 0), & S_{\mathcal{L}_8^a} &= (1, a, a+1, -1, -a, 0), \\ S_{\mathcal{L}_9^a} &= (1, a, 2, -1, -a, 0), & S_{\mathcal{L}_{10}^a} &= (1, a, 2, -1, 0, 0), & S_{\mathcal{L}_{15}^a} &= (0, 1, 0, 0, -1, 0), \\ S_{\mathcal{L}_{18}^a} &= (0, 1, 0, 0, 0, 0), & S_{\mathcal{L}_{21}^{a,b}} &= (1, a, b, -1, -a, 0), & S_{\mathcal{L}_{22}^{a,b}} &= (1, a, b, -1, 0, 0), \\ S_{\mathcal{L}_{23}^{a,b}} &= (1, a, b, 0, 0, 0), & S_{\mathcal{L}_{44}} &= (1, 2, 3, -1, 0, 0). \end{aligned}$$

Almost all the required assertions follow now from our 6-tuple argument. Also the assertions of the form $A(*) \not\rightarrow B$, where $A(*) \in \{\mathcal{L}_{21}^{*,*}, \mathcal{L}_{22}^{*,*}, \mathcal{L}_{23}^{*,*}\}$ and $B \in \{\mathcal{L}_2, \mathcal{L}_4^a, \mathcal{L}_8^a, \mathcal{L}_9^a, \mathcal{L}_{10}^a, \mathcal{L}_{15}^a, \mathcal{L}_{18}^a, \mathcal{L}_{44}\}_{a \in \mathbb{C}}$ follow from the fact that $A(a, b)$ has three dimensional trivial subalgebra for any $a, b \in \mathbb{C}$ and B does not have three dimensional trivial subalgebra. Let us consider the remaining assertions case by case.

- $\mathcal{L}_4^* \not\rightarrow \mathcal{L}_{44}$. The required assertion follows from the fact that $\text{Der}(\mathcal{L}_{44}) > \text{Der}(\mathcal{L}_4^a)$ for any $a \in \mathbb{C}$.
- $\mathcal{L}_8^* \not\rightarrow \mathcal{L}_2$. The required assertion follows from the fact that $\text{Der}(\mathcal{L}_2) > \text{Der}(\mathcal{L}_8^a)$ for any $a \in \mathbb{C}$.
- To prove the assertions $\mathcal{L}_9^* \not\rightarrow \mathcal{L}_{15}^a$ and $\mathcal{L}_{10}^* \not\rightarrow \mathcal{L}_{18}^a$ let us consider the set

$$\mathcal{R} = \left\{ \mu \in \mathcal{L}\text{eib}_4 \left| \begin{array}{l} S_1 S_1 \subset S_2, S_2 S_2 \subset S_4, S_3 S_1 + S_1 S_3 \subset S_3, S_4 S_1 \subset S_4, S_1 S_4 + S_4 S_2 = 0, \\ c_{13}^3 + c_{31}^3 = 0, c_{41}^4 = 2c_{31}^3, c_{13}^4 = c_{31}^4, c_{23}^4 = c_{32}^4 \end{array} \right. \right\}.$$

It is not difficult to show that \mathcal{R} is a closed subset of $\mathcal{L}\text{eib}_4$ that is stable under the action of the subgroup of lower triangular matrices and contains the structures \mathcal{L}_9^b and \mathcal{L}_{10}^b for any $b \in \mathbb{C}$. To see this it is enough to consider the basis e_1, e_3, e_2, e_4 . It is also not difficult to show that $\mathcal{R} \cap O(\mathcal{L}_{15}^a) = \mathcal{R} \cap O(\mathcal{L}_{18}^a) = \emptyset$ for any $a \in \mathbb{C}$.

□

5. APPENDIX: TABLES

Table 1. Four dimensional non-nilpotent non-Lie Leibniz algebras.

A	multiplication tables
\mathfrak{R}_1	$e_3e_1 = e_3, e_4e_2 = e_4$
\mathfrak{R}_3	$e_2e_4 = -e_4, e_3e_1 = e_3, e_4e_2 = e_4$
\mathfrak{L}_2	$e_1e_1 = e_4, e_1e_2 = -e_2, e_1e_3 = e_3, e_2e_1 = e_2, e_2e_3 = e_4, e_3e_1 = -e_3, e_3e_2 = -e_4$
\mathfrak{L}_4^a	$e_1e_2 = -e_2, e_2e_1 = e_2, e_3e_1 = ae_3, e_3e_2 = e_4, e_4e_1 = (1+a)e_4$
\mathfrak{L}_5	$e_1e_1 = e_4, e_1e_2 = -e_2, e_2e_1 = e_2, e_3e_1 = -e_3, e_3e_2 = e_4$
\mathfrak{L}_6	$e_1e_1 = -e_3, e_1e_2 = -e_2, e_2e_1 = e_2 + e_4, e_3e_2 = e_4, e_4e_1 = e_4$
\mathfrak{L}_7	$e_3e_1 = e_3, e_3e_2 = e_4, e_4e_1 = e_4$
$\mathfrak{L}_8^{a \neq -1, 0}$	$e_1e_2 = -e_2, e_1e_3 = -ae_3, e_2e_1 = e_2, e_2e_3 = ae_4, e_3e_1 = ae_3, e_3e_2 = e_4, e_4e_1 = (a+1)e_4$
\mathfrak{L}_9^a	$e_1e_2 = -e_2, e_1e_3 = -ae_3, e_2e_1 = e_2, e_2e_2 = e_4, e_3e_1 = ae_3, e_4e_1 = 2e_4$
\mathfrak{L}_{10}^a	$e_1e_2 = -e_2, e_2e_1 = e_2, e_2e_2 = e_4, e_3e_1 = ae_3, e_4e_1 = 2e_4$
\mathfrak{L}_{11}	$e_1e_1 = e_3, e_1e_2 = -e_2, e_2e_1 = e_2, e_2e_2 = e_4, e_4e_1 = 2e_4$
\mathfrak{L}_{12}	$e_1e_2 = -e_2, e_2e_1 = e_2, e_2e_2 = e_4, e_3e_1 = 2e_3 + e_4, e_4e_1 = 2e_4$
\mathfrak{L}_{13}	$e_1e_2 = -e_2 - e_3, e_1e_3 = -e_3, e_2e_1 = e_2 + e_3, e_2e_2 = e_4, e_3e_1 = e_3, e_4e_1 = 2e_4$
\mathfrak{L}_{14}	$e_1e_3 = -e_3, e_2e_2 = e_4, e_3e_1 = e_3$
\mathfrak{L}_{15}^a	$e_1e_1 = ae_4, e_1e_2 = e_4, e_1e_3 = -e_3, e_2e_2 = e_4, e_3e_1 = e_3$
\mathfrak{L}_{16}	$e_1e_1 = -2e_4, e_1e_3 = -e_3, e_2e_2 = e_4, e_3e_1 = e_3$
\mathfrak{L}_{17}	$e_2e_2 = e_4, e_3e_1 = e_3$
\mathfrak{L}_{18}^a	$e_1e_1 = ae_4, e_1e_2 = e_4, e_2e_2 = e_4, e_3e_1 = e_3$
\mathfrak{L}_{19}	$e_1e_2 = e_4, e_2e_1 = e_4, e_2e_2 = e_4, e_3e_1 = e_3$
$\mathfrak{L}_{21}^{a, b \neq 0}$	$e_1e_2 = -e_2, e_1e_3 = -ae_3, e_2e_1 = e_2, e_3e_1 = ae_3, e_4e_1 = be_4$
$\mathfrak{L}_{22}^{a \neq 0, b \neq 0}$	$e_1e_2 = -e_2, e_2e_1 = e_2, e_3e_1 = ae_3, e_4e_1 = be_4$
$\mathfrak{L}_{23}^{a, b}$	$e_2e_1 = e_2, e_3e_1 = ae_3, e_4e_1 = be_4$
\mathfrak{L}_{24}^a	$e_1e_1 = e_4, e_1e_2 = -e_2, e_1e_3 = -ae_3, e_2e_1 = e_2, e_3e_1 = ae_3$
$\mathfrak{L}_{25}^{a \neq 0}$	$e_1e_1 = e_4, e_1e_2 = -e_2, e_2e_1 = e_2, e_3e_1 = ae_3$
\mathfrak{L}_{26}^a	$e_1e_1 = e_4, e_2e_1 = e_2, e_3e_1 = ae_3$
\mathfrak{L}_{27}	$e_1e_2 = -e_2, e_1e_3 = e_4, e_2e_1 = e_2$
\mathfrak{L}_{28}	$e_1e_3 = e_4, e_2e_1 = e_2$
$\mathfrak{L}_{29}^{a \neq 0}$	$e_1e_2 = -e_2 - e_3, e_1e_3 = -e_3, e_2e_1 = e_2 + e_3, e_3e_1 = e_3, e_4e_1 = ae_4$
\mathfrak{L}_{30}	$e_1e_1 = e_4, e_1e_2 = -e_2 - e_3, e_1e_3 = -e_3, e_2e_1 = e_2 + e_3, e_3e_1 = e_3$
\mathfrak{L}_{32}^a	$e_2e_1 = e_2 + e_3, e_3e_1 = e_3, e_4e_1 = ae_4$
\mathfrak{L}_{33}	$e_1e_1 = e_4, e_2e_1 = e_2 + e_3, e_3e_1 = e_3$
$\mathfrak{L}_{34}^{a \neq 0}$	$e_1e_4 = -ae_4, e_2e_1 = e_2 + e_3, e_3e_1 = e_3, e_4e_1 = ae_4$
$\mathfrak{L}_{35}^{a \neq -1}$	$e_1e_2 = ae_3, e_1e_4 = -e_4, e_2e_1 = e_3, e_4e_1 = e_4$
\mathfrak{L}_{36}	$e_1e_1 = e_3, e_1e_2 = -e_3, e_1e_4 = -e_4, e_2e_1 = e_3, e_4e_1 = e_4$
\mathfrak{L}_{37}	$e_1e_1 = e_2, e_1e_4 = -e_4, e_2e_1 = e_3, e_4e_1 = e_4$
\mathfrak{L}_{38}^a	$e_1e_2 = ae_3, e_2e_1 = e_3, e_4e_1 = e_4$
\mathfrak{L}_{39}	$e_1e_1 = e_3, e_1e_2 = -e_3, e_2e_1 = e_3, e_4e_1 = e_4$
\mathfrak{L}_{40}	$e_1e_1 = e_2, e_2e_1 = e_3, e_4e_1 = e_4$
\mathfrak{L}_{41}	$e_2e_1 = e_2 + e_3, e_3e_1 = e_3 + e_4, e_4e_1 = e_4$
\mathfrak{L}_{44}	$e_1e_2 = -e_2, e_2e_1 = e_2, e_2e_2 = e_3, e_3e_1 = 2e_3, e_3e_2 = e_4, e_4e_1 = 3e_4$

Table 2. Orbit closures for some families and degenerations of four dimensional Leibniz algebras.

Assertions	Parametrized bases				Parametrized indices
$\mathfrak{L}_4^* \rightarrow \mathfrak{L}_5$	$E_1^t = e_1 + e_4,$	$E_2^t = e_2,$	$E_3^t = te_3,$	$E_4^t = te_4$	$\epsilon(t) = t - 1$
$\mathfrak{L}_4^* \rightarrow \mathfrak{L}_6$	$E_1^t = e_1 - e_3,$	$E_2^t = e_2 + e_4,$	$E_3^t = te_3,$	$E_4^t = te_4$	$\epsilon(t) = t$
$\mathfrak{R}_1 \rightarrow \mathfrak{L}_7$	$E_1^t = e_1 + e_2,$	$E_2^t = te_2,$	$E_3^t = e_3 + e_4,$	$E_4^t = te_4$	
$\mathfrak{L}_{10}^* \rightarrow \mathfrak{L}_{11}$	$E_1^t = e_1 + e_3,$	$E_2^t = e_2,$	$E_3^t = te_3,$	$E_4^t = e_4$	$\epsilon(t) = t$
$\mathfrak{L}_{10}^* \rightarrow \mathfrak{L}_{12}$	$E_1^t = e_1,$	$E_2^t = te_2,$	$E_3^t = e_3 + e_4,$	$E_4^t = t^2 e_4$	$\epsilon(t) = 2 - t^2$
$\mathfrak{L}_9 \rightarrow \mathfrak{L}_{13}$	$E_1^t = e_1,$	$E_2^t = e_2 + e_3,$	$E_3^t = te_3,$	$E_4^t = e_4$	$\epsilon(t) = t + 1$
$\mathfrak{L}_{16} \rightarrow \mathfrak{L}_{14}$	$E_1^t = e_1,$	$E_2^t = \frac{1}{t}e_2,$	$E_3^t = e_3,$	$E_4^t = \frac{1}{t^2}e_4$	
$\mathfrak{L}_{15}^* \rightarrow \mathfrak{L}_{16}$	$E_1^t = e_1,$	$E_2^t = \frac{1}{t}e_2,$	$E_3^t = e_3,$	$E_4^t = \frac{1}{t^2}e_4$	$\epsilon(t) = -\frac{2}{t^2}$
$\mathfrak{R}_1 \rightarrow \mathfrak{L}_{17}$	$E_1^t = e_1,$	$E_2^t = te_2 + e_4,$	$E_3^t = e_3,$	$E_4^t = te_4$	
$\mathfrak{L}_{18}^* \rightarrow \mathfrak{L}_{19}$	$E_1^t = e_1 + \frac{1}{t}e_2,$	$E_2^t = \frac{1}{t}e_2,$	$E_3^t = e_3,$	$E_4^t = \frac{1}{t^2}e_4$	$\epsilon(t) = -\frac{1}{t^2}$
$\mathfrak{L}_{21}^{*,*} \rightarrow \mathfrak{L}_{24}^a$	$E_1^t = e_1 + e_4,$	$E_2^t = e_2,$	$E_3^t = e_3,$	$E_4^t = te_4$	$\epsilon(t) = (a, t)$
$\mathfrak{L}_{22}^{*,*} \rightarrow \mathfrak{L}_{25}^a$	$E_1^t = e_1 + e_4,$	$E_2^t = e_2,$	$E_3^t = e_3,$	$E_4^t = te_4$	$\epsilon(t) = (a, t)$
$\mathfrak{L}_{23}^{*,*} \rightarrow \mathfrak{L}_{26}^a$	$E_1^t = e_1 + e_4,$	$E_2^t = e_2,$	$E_3^t = e_3,$	$E_4^t = te_4$	$\epsilon(t) = (a, t)$
$\mathfrak{R}_3 \rightarrow \mathfrak{L}_{27}$	$E_1^t = e_2 + e_3,$	$E_2^t = e_4,$	$E_3^t = te_1,$	$E_4^t = te_3$	
$\mathfrak{R}_1 \rightarrow \mathfrak{L}_{28}$	$E_1^t = e_1 + e_4,$	$E_2^t = e_3,$	$E_3^t = te_2,$	$E_4^t = te_4$	
$\mathfrak{L}_{21}^{*,*} \rightarrow \mathfrak{L}_{29}^a$	$E_1^t = e_1,$	$E_2^t = e_2 + e_3,$	$E_3^t = te_3,$	$E_4^t = e_4$	$\epsilon(t) = (t + 1, a)$
$\mathfrak{L}_{29}^* \rightarrow \mathfrak{L}_{30}$	$E_1^t = e_1 + e_4,$	$E_2^t = e_2,$	$E_3^t = e_3,$	$E_4^t = te_4$	$\epsilon(t) = t$
$\mathfrak{L}_{23}^{*,*} \rightarrow \mathfrak{L}_{32}^a$	$E_1^t = e_1,$	$E_2^t = e_2 + e_3,$	$E_3^t = te_3,$	$E_4^t = e_4$	$\epsilon(t) = (t + 1, a)$
$\mathfrak{L}_{26}^* \rightarrow \mathfrak{L}_{33}$	$E_1^t = e_1,$	$E_2^t = e_2 + e_3,$	$E_3^t = te_3,$	$E_4^t = e_4$	$\epsilon(t) = t + 1$
$\mathfrak{L}_{22}^{*,*} \rightarrow \mathfrak{L}_{34}^a$	$E_1^t = ae_1,$	$E_2^t = e_3 + e_4,$	$E_3^t = te_4,$	$E_4^t = e_2$	$\epsilon(t) = \left(\frac{1}{a}, \frac{t+1}{a}\right)$
$\mathfrak{L}_{15}^{\frac{(1-a)^2}{-a}} \rightarrow \mathfrak{L}_{35}^{a \neq 1}$	$E_1^t = e_1 + \frac{1}{a-1}e_2,$	$E_2^t = t(a-1)e_2,$	$E_3^t = te_4,$	$E_4^t = e_3$	
$\mathfrak{L}_{24}^* \rightarrow \mathfrak{L}_{36}$	$E_1^t = te_1,$	$E_2^t = e_2 - te_4,$	$E_3^t = t^2 e_4,$	$E_4^t = e_3$	$\epsilon(t) = \frac{1}{t}$
$\mathfrak{L}_{34}^* \rightarrow \mathfrak{L}_{37}$	$E_1^t = te_1 + e_2,$	$E_2^t = t(e_2 + e_3),$	$E_3^t = t^2 e_3,$	$E_4^t = e_4$	$\epsilon(t) = \frac{1}{t}$
$\mathfrak{L}_{18}^{\frac{(1-a)^2}{-a}} \rightarrow \mathfrak{L}_{38}^{a \neq 1}$	$E_1^t = e_1 + \frac{1}{a-1}e_2,$	$E_2^t = t(a-1)e_2,$	$E_3^t = te_4,$	$E_4^t = e_3$	
$\mathfrak{L}_{22}^{*,*} \rightarrow \mathfrak{L}_{39}$	$E_1^t = te_1 + \frac{1}{t}e_3,$	$E_2^t = e_2 - e_3,$	$E_3^t = te_3,$	$E_4^t = e_4$	$\epsilon(t) = \left(t, \frac{1}{t}\right)$
$\mathfrak{L}_{23}^{*,*} \rightarrow \mathfrak{L}_{40}$	$E_1^t = te_1 + e_2 + e_3,$	$E_2^t = te_2 + 2te_3,$	$E_3^t = 2t^2 e_3,$	$E_4^t = e_4$	$\epsilon(t) = \left(2, \frac{1}{t}\right)$
$\mathfrak{L}_{23}^{*,*} \rightarrow \mathfrak{L}_{41}$	$E_1^t = e_1,$	$E_2^t = e_2 + 2e_3 + e_4,$	$E_3^t = 2t(e_3 + e_4),$	$E_4^t = 2t^2 e_4$	$\epsilon(t) = (t + 1, 2t + 1)$

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