BiHom-pre-Lie algebras, BiHom-Leibniz algebras and Rota-Baxter operators on BiHom-Lie algebras

Ling Liu
College of Mathematics, Physics and Information Engineering,
Zhejiang Normal University,
Jinhua 321004, China
e-mail: ntliulin@zjnu.cn

Abdenacer Makhlouf
Université de Haute Alsace,
Laboratoire de Mathématiques, Informatique et Applications,
4, rue des frères Lumière, F-68093 Mulhouse, France
e-mail: Abdenacer.Makhlouf@uha.fr

Claudia Menini
University of Ferrara,
Department of Mathematics,
Via Machiavelli 35, Ferrara, I-44121, Italy
e-mail: men@unife.it

Florin Panaite
Institute of Mathematics of the Romanian Academy,
PO-Box 1-764, RO-014700 Bucharest, Romania
e-mail: florin.panaite@imar.ro

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Abstract

We investigate some properties of Rota-Baxter operators on BiHom-Lie algebras. Along the way, we introduce BiHom analogues of pre-Lie and Leibniz algebras.
Keywords: Rota-Baxter operator; BiHom-Lie algebra

Introduction

Algebras of Hom-type appeared in the Physics literature of the 1990's, in the context of quantum deformations of some algebras of vector fields, such as the Witt and Virasoro algebras, in connection with oscillator algebras ([2, 10]). A quantum deformation consists of replacing the usual derivation by a σ-derivation. It turns out that the algebras obtained in this way do not satisfy the Jacobi identity anymore, but instead they satisfy a modified version involving a homomorphism. This kind of algebras were called Hom-Lie algebras and studied by Hartwig, Larsson and Silvestrov in [9, 11]. The Hom analogue of associative algebras was introduced in [15], where it is shown that the commutator bracket defined by the multiplication in a Hom-associative algebra leads to a Hom-Lie algebra. A categorical approach to Hom-type algebras was considered in [5]. A generalization has been given in [7], where a construction of a Hom-category including a group action led to concepts of BiHom-type algebras. Hence, BiHom-associative algebras and BiHom-Lie algebras, involving two linear maps (called structure maps), were introduced. The main axioms for these types of algebras (BiHom-associativity, BiHom-skew-symmetry and BiHom-Jacobi condition) were dictated by categorical considerations.

Rota-Baxter operators first appeared in G. Baxter's work in probability and the study of fluctuation theory ([3]). Afterwards, Rota-Baxter algebras were intensively studied by Rota ([16, 17]) in connection with combinatorics. Rota-Baxter operators have appeared in a wide range of areas in pure and applied mathematics, for example in the work of Connes and Kreimer ([6]) about their Hopf algebra approach to renormalization of quantum field theory. This seminal work was followed by an important development of the theory of Rota-Baxter algebras and their connections to other algebraic structures, see for example the book [8] and its references. In the context of Lie algebras, Rota-Baxter operators were introduced independently by Belavin and Drinfeld ([4]) and Semenov-Tian-Shansky ([18]), in connection with solutions of the (modified) classical Yang-Baxter equation.

The first aim of this paper is to obtain a BiHom analogue of the classical result, due to Aguiar ([1]), saying that, if $(L, [\cdot, \cdot])$ is a Lie algebra and $R : L \to L$ is a Rota-Baxter operator of weight 0, and one defines a new operation on L by $x \cdot y = [R(x), y]$, then $(L, \cdot)$ is a left pre-Lie algebra. The Hom analogue of this result was obtained in [14]. So, we first define, in Section 2, the concepts of (left and right) BiHom-pre-Lie algebras and present some properties of these objects. Then, our aim was to prove that, if $(L, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra and $R : L \to L$ is a Rota-Baxter operator of weight 0 such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$, then $(L, \cdot, \alpha, \beta)$ is a left BiHom-pre-Lie algebra, where $x \cdot y = [R(x), y]$. It turns out that this does not work (unless the structure maps α and β are bijective). We were thus led to realize that, apart from the concept of BiHom-Lie algebra (introduced in [7]), there exist other natural BiHom analogues of Lie and Hom-Lie algebras, that we called left and right BiHom-Lie algebras. We arrived at these concepts as follows (we concentrate on the left case): we introduced first the concept of left BiHom-Leibniz algebra (the natural BiHom analogue of Loday's concept of left Leibniz algebra) and then a left BiHom-Lie algebra is a left BiHom-Leibniz algebra satisfying the BiHom-skew-symmetry condition. It turns out (Proposition 3.9) that the concepts of BiHom-Lie algebra and left BiHom-Lie algebra coincide if the structure maps are bijective, and that the BiHom analogue of Aguiar's result mentioned above holds indeed for left BiHom-Lie algebras (Proposition 3.11), although it does not hold in general for BiHom-Lie algebras.

In a previous paper ([12]), where we studied Rota-Baxter operators on BiHom-associative algebras, we proved the following result. Let $(A, \cdot, \alpha, \beta)$ be a BiHom-associative algebra, $R : A \to A$ a Rota-Baxter operator of weight λ commuting with α and β, and define a new multiplication on A by $a*b = R(a)\cdot b + a\cdot R(b) + \lambda a\cdot b$, for all $a, b \in A$; then $(A, *, \alpha, \beta)$ is also a BiHom-associative algebra. The second aim of the current paper is to prove a similar result for BiHom-Lie algebras; this is achieved in Proposition 3.13. We also prove similar results for left or right BiHom-Leibniz algebras and for left or right BiHom-Lie algebras.

1 Preliminaries

We work over a base field $\Bbbk$. All algebras, linear spaces etc. will be over $\Bbbk$; unadorned $\otimes$ means $\otimes_{\Bbbk}$. We denote by ${}_{\Bbbk}\mathcal{M}$ the category of linear spaces over $\Bbbk$. Unless otherwise specified, the algebras that will appear in what follows are *not* supposed to be unital, and a multiplication

$\mu : A \otimes A \to A$ on a linear space A is denoted by juxtaposition: $\mu(v \otimes v') = vv'$. For the composition of two maps f and g, we write either $g \circ f$ or simply gf. For the identity map on a linear space A we use the notation id_A. We denote by $\circlearrowleft_{x,y,z}$ summation over the cyclic permutations of some elements x, y, z.

Definition 1.1 *A left (respectively right) pre-Lie algebra is a linear space A endowed with a linear map $\cdot : A \otimes A \to A$ satisfying (for all $x, y, z \in A$)*

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z, \tag{1.1}$$

respectively

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = x \cdot (z \cdot y) - (x \cdot z) \cdot y. \tag{1.2}$$

Any associative algebra is a left and a right pre-Lie algebra. If $(A, \cdot)$ is a left or a right pre-Lie algebra, then $(A, [x, y] = x \cdot y - y \cdot x)$ is a Lie algebra.

Non-skew-symmetric Lie algebras are called Leibniz algebras; they were introduced by Loday.

Definition 1.2 *([13]) A left (respectively right) Leibniz algebra is a linear space L endowed with a bilinear map $[\cdot, \cdot] : L \times L \to L$ satisfying (for all $x, y, z \in L$):*

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]], \tag{1.3}$$

respectively

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]. \tag{1.4}$$

A morphism of (left or right) Leibniz algebras L and L' is a linear map $f : L \to L'$ such that $f([x, y]) = [f(x), f(y)]$, for all $x, y \in L$.

A Rota-Baxter structure on an algebra of a given type is defined as follows.

Definition 1.3 *Let A be a linear space and $\mu : A \otimes A \to A$, $\mu(x \otimes y) = xy$, for all $x, y \in A$, a linear multiplication on A and let $\lambda \in \Bbbk$. A Rota-Baxter operator of weight λ for (A, μ) is a linear map $R : A \to A$ satisfying the so-called Rota-Baxter condition*

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy), \quad \forall\, x, y \in A. \tag{1.5}$$

In this case, if we define on A a new multiplication by $x * y = xR(y) + R(x)y + \lambda xy$, for all $x, y \in A$, then $R(x * y) = R(x)R(y)$, for all $x, y \in A$, and R is a Rota-Baxter operator of weight λ for $(A, *)$. If (A, μ) is associative then $(A, *)$ is also associative.

We recall now from [7] several facts about BiHom-type algebras.

Definition 1.4 *A BiHom-associative algebra over $\Bbbk$ is a 4-tuple (A, μ, α, β), where A is a $\Bbbk$-linear space, $\alpha : A \to A$, $\beta : A \to A$ and $\mu : A \otimes A \to A$ are linear maps, with notation $\mu(x \otimes y) = xy$, for all $x, y \in A$, satisfying the following conditions, for all $x, y, z \in A$:*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{1.6}$$

$$\alpha(xy) = \alpha(x)\alpha(y) \text{ and } \beta(xy) = \beta(x)\beta(y), \quad \text{(multiplicativity)} \tag{1.7}$$

$$\alpha(x)(yz) = (xy)\beta(z). \quad \text{(BiHom-associativity)} \tag{1.8}$$

We call α and β (in this order) the structure maps of A.

Definition 1.5 *A BiHom-Lie algebra over* $\Bbbk$ *is a 4-tuple* $(L, [\cdot,\cdot], \alpha, \beta)$*, where* L *is a* $\Bbbk$*-linear space,* $\alpha, \beta : L \to L$ *are linear maps and* $[\cdot,\cdot] : L \times L \to L$ *is a bilinear map, satisfying the following conditions, for all* $x, y, z \in L$ *:*

$$\alpha \circ \beta = \beta \circ \alpha,$$
$$\alpha([x,y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta([x,y]) = [\beta(x), \beta(y)],$$
$$[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \qquad \text{(BiHom-skew-symmetry)}$$
$$\left[\beta^2(x), [\beta(y), \alpha(z)]\right] + \left[\beta^2(y), [\beta(z), \alpha(x)]\right] + \left[\beta^2(z), [\beta(x), \alpha(y)]\right] = 0.$$
(BiHom-Jacobi condition)

We call α *and* β *(in this order) the structure maps of* L*.*

A morphism $f : (L, [\cdot,\cdot], \alpha, \beta) \to (L', [\cdot,\cdot]', \alpha', \beta')$ *of BiHom-Lie algebras is a linear map* $f : L \to L'$ *such that* $\alpha' \circ f = f \circ \alpha$*,* $\beta' \circ f = f \circ \beta$ *and* $f([x,y]) = [f(x), f(y)]'$*, for all* $x, y \in L$*.*

Let $(L, [\cdot,\cdot])$ be an ordinary Lie algebra over $\Bbbk$ and let $\alpha, \beta : L \to L$ be two commuting linear maps such that $\alpha([x,y]) = [\alpha(x), \alpha(y)]$ and $\beta([x,y]) = [\beta(x), \beta(y)]$, for all $x, y \in L$. Define the bilinear map $\{\cdot,\cdot\} : L \times L \to L$, $\{x,y\} = [\alpha(x), \beta(y)]$, for all $x, y \in L$. Then $L_{(\alpha,\beta)} := (L, \{\cdot,\cdot\}, \alpha, \beta)$ is a BiHom-Lie algebra, called the Yau twist of $(L, [\cdot,\cdot])$.

More generally, if $(L, [\cdot,\cdot], \alpha, \beta)$ is a BiHom-Lie algebra, $\alpha', \beta' : L \to L$ linear maps such that $\alpha'([x,y]) = [\alpha'(x), \alpha'(y)]$, $\beta'([x,y]) = [\beta'(x), \beta'(y)]$ for $x, y \in L$, and any two of the maps $\alpha, \beta, \alpha', \beta'$ commute, then $\left(L, [\cdot,\cdot]_{(\alpha',\beta')} := [\cdot,\cdot] \circ (\alpha' \otimes \beta'), \alpha \circ \alpha', \beta \circ \beta'\right)$ is a BiHom-Lie algebra.

2 BiHom-pre-Lie algebras

Definition 2.1 *A left (respectively right) BiHom-pre-Lie algebra is a 4-tuple* $(A, \cdot, \alpha, \beta)$*, where* A *is a linear space and* $\cdot : A \otimes A \to A$ *and* $\alpha, \beta : A \to A$ *are linear maps satifying* $\alpha \circ \beta = \beta \circ \alpha$*,* $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$*,* $\beta(x \cdot y) = \beta(x) \cdot \beta(y)$ *and*

$$\alpha\beta(x) \cdot (\alpha(y) \cdot z) - (\beta(x) \cdot \alpha(y)) \cdot \beta(z) = \alpha\beta(y) \cdot (\alpha(x) \cdot z) - (\beta(y) \cdot \alpha(x)) \cdot \beta(z), \tag{2.1}$$

respectively

$$\alpha(x) \cdot (\beta(y) \cdot \alpha(z)) - (x \cdot \beta(y)) \cdot \alpha\beta(z) = \alpha(x) \cdot (\beta(z) \cdot \alpha(y)) - (x \cdot \beta(z)) \cdot \alpha\beta(y), \tag{2.2}$$

for all $x, y, z \in A$*. We call* α *and* β *(in this order) the structure maps of* A*.*

A morphism $f : (A, \cdot, \alpha, \beta) \to (A', \cdot', \alpha', \beta')$ *of left or right BiHom-pre-Lie algebras is a linear map* $f : A \to A'$ *satisfying* $f(x \cdot y) = f(x) \cdot' f(y)$*, for all* $x, y \in A$*, as well as* $f \circ \alpha = \alpha' \circ f$ *and* $f \circ \beta = \beta' \circ f$*.*

Obviously, if $(A, \cdot, \alpha, \beta)$ is a BiHom-associative algebra then it is also a left and a right BiHom-pre-Lie algebra.

If $(A, \cdot, \alpha, \beta)$ is a left BiHom-pre-Lie algebra and we define a new multiplication $*$ on A by $x * y = y \cdot x$, then $(A, *, \beta, \alpha)$ is a right BiHom-pre-Lie algebra.

Proposition 2.2 *If* $(A, \cdot)$ *is a left (respectively right) pre-Lie algebra and* $\alpha, \beta : A \to A$ *are linear maps satifying* $\alpha \circ \beta = \beta \circ \alpha$*,* $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$ *and* $\beta(x \cdot y) = \beta(x) \cdot \beta(y)$*, for all* $x, y \in A$*, and we define a new multiplication on* A *by* $x * y = \alpha(x) \cdot \beta(y)$*, then* $(A, *, \alpha, \beta)$ *is a left (respectively right) BiHom-pre-Lie algebra, called the Yau twist of* $(A, \cdot)$*.*

Proof. We only prove (2.1) and leave the rest to the reader:

$$\begin{aligned}
&\alpha\beta(x) * (\alpha(y) * z) - (\beta(x) * \alpha(y)) * \beta(z) \\
&\qquad = \alpha^2\beta(x) \cdot (\alpha^2\beta(y) \cdot \beta^2(z)) - (\alpha^2\beta(x) \cdot \alpha^2\beta(y)) \cdot \beta^2(z) \\
&\qquad \overset{(1.1)}{=} \alpha^2\beta(y) \cdot (\alpha^2\beta(x) \cdot \beta^2(z)) - (\alpha^2\beta(y) \cdot \alpha^2\beta(x)) \cdot \beta^2(z) \\
&\qquad = \alpha\beta(y) * (\alpha(x) * z) - (\beta(y) * \alpha(x)) * \beta(z),
\end{aligned}$$

finishing the proof. □

Remark 2.3 *More generally, one can prove that, if $(A, \cdot, \alpha, \beta)$ is a left (respectively right) BiHom-pre-Lie algebra and $\tilde{\alpha}, \tilde{\beta} : A \to A$ are two morphisms of BiHom-pre-Lie algebras such that any two of the maps $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ commute, and we define a new multiplication on A by $x * y = \tilde{\alpha}(x) \cdot \tilde{\beta}(y)$, for all $x, y \in A$, then $(A, *, \alpha \circ \tilde{\alpha}, \beta \circ \tilde{\beta})$ is also a left (respectively right) BiHom-pre-Lie algebra.*

Proposition 2.4 *Let $(A, \cdot, \alpha, \beta)$ be a left or a right BiHom-pre-Lie algebra such that α and β are bijective. Define $[\cdot, \cdot] : A \otimes A \to A$ by $[x, y] = x \cdot y - (\alpha^{-1}\beta(y)) \cdot (\alpha\beta^{-1}(x))$, for all $x, y \in A$. Then $(A, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra.*

Proof. We only give the proof in the case A is a left BiHom-pre-Lie algebra, the other case is similar and left to the reader. The fact that α and β are multiplicative with respect to $[\cdot, \cdot]$ is obvious, and so is the BiHom-skew-symmetry relation. We only have to prove the BiHom-Jacobi condition. We compute, for $x, y, z \in A$:

$$\begin{aligned}
\circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] &= \circlearrowleft_{x,y,z} [\beta^2(x), \beta(y) \cdot \alpha(z) - \beta(z) \cdot \alpha(y)] \\
&= \circlearrowleft_{x,y,z} ([\beta^2(x), \beta(y) \cdot \alpha(z)] - [\beta^2(x), \beta(z) \cdot \alpha(y)]) \\
&= \circlearrowleft_{x,y,z} (\beta^2(x) \cdot (\beta(y) \cdot \alpha(z)) - \alpha^{-1}\beta(\beta(y) \cdot \alpha(z)) \cdot \alpha\beta(x) \\
&\quad -\beta^2(x) \cdot (\beta(z) \cdot \alpha(y)) + \alpha^{-1}\beta(\beta(z) \cdot \alpha(y)) \cdot \alpha\beta(x)) \\
&= \circlearrowleft_{x,y,z} (\beta^2(x) \cdot (\beta(y) \cdot \alpha(z)) - (\alpha^{-1}\beta^2(y) \cdot \beta(z)) \cdot \alpha\beta(x) \\
&\quad -\beta^2(x) \cdot (\beta(z) \cdot \alpha(y)) + (\alpha^{-1}\beta^2(z) \cdot \beta(y)) \cdot \alpha\beta(x)) \\
&= \beta^2(x) \cdot (\beta(y) \cdot \alpha(z)) - (\alpha^{-1}\beta^2(y) \cdot \beta(z)) \cdot \alpha\beta(x) \\
&\quad -\beta^2(x) \cdot (\beta(z) \cdot \alpha(y)) + (\alpha^{-1}\beta^2(z) \cdot \beta(y)) \cdot \alpha\beta(x) \\
&\quad +\beta^2(y) \cdot (\beta(z) \cdot \alpha(x)) - (\alpha^{-1}\beta^2(z) \cdot \beta(x)) \cdot \alpha\beta(y) \\
&\quad -\beta^2(y) \cdot (\beta(x) \cdot \alpha(z)) + (\alpha^{-1}\beta^2(x) \cdot \beta(z)) \cdot \alpha\beta(y) \\
&\quad +\beta^2(z) \cdot (\beta(x) \cdot \alpha(y)) - (\alpha^{-1}\beta^2(x) \cdot \beta(y)) \cdot \alpha\beta(z) \\
&\quad -\beta^2(z) \cdot (\beta(y) \cdot \alpha(x)) + (\alpha^{-1}\beta^2(y) \cdot \beta(x)) \cdot \alpha\beta(z) \\
&\overset{(2.1)\ 3\ times}{=} 0 + 0 + 0 = 0,
\end{aligned}$$

finishing the proof. □

Definition 2.5 *([12] A BiHom-dendriform algebra is a 5-tuple $(A, \prec, \succ, \alpha, \beta)$ consisting of a linear space A and linear maps $\prec, \succ : A \otimes A \to A$ and $\alpha, \beta : A \to A$ satisfying the following conditions (for all $x, y, z \in A$):*

$$\alpha \circ \beta = \beta \circ \alpha, \tag{2.3}$$

$$\alpha(x \prec y) = \alpha(x) \prec \alpha(y),\ \ \alpha(x \succ y) = \alpha(x) \succ \alpha(y), \tag{2.4}$$
$$\beta(x \prec y) = \beta(x) \prec \beta(y),\ \ \beta(x \succ y) = \beta(x) \succ \beta(y), \tag{2.5}$$
$$(x \prec y) \prec \beta(z) = \alpha(x) \prec (y \prec z + y \succ z), \tag{2.6}$$
$$(x \succ y) \prec \beta(z) = \alpha(x) \succ (y \prec z), \tag{2.7}$$
$$\alpha(x) \succ (y \succ z) = (x \prec y + x \succ y) \succ \beta(z). \tag{2.8}$$

We call α and β (in this order) the structure maps of A.

Proposition 2.6 *Let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra such that α and β are bijective. Let $\rhd, \lhd : A \otimes A \to A$ be linear maps defined for all $x, y \in A$ by*

$$x \rhd y = x \succ y - (\alpha^{-1}\beta(y)) \prec (\alpha\beta^{-1}(x)), \qquad x \lhd y = x \prec y - (\alpha^{-1}\beta(y)) \succ (\alpha\beta^{-1}(x)).$$

Then $(A, \rhd, \alpha, \beta)$ (respectively $(A, \lhd, \alpha, \beta)$) is a left (respectively right) BiHom-pre-Lie algebra.

Proof. We only prove the identity (2.1) and leave the rest to the reader. We compute:

$$\begin{aligned}
&\alpha\beta(x) \rhd (\alpha(y) \rhd z) - (\beta(x) \rhd \alpha(y)) \rhd \beta(z) \\
&= \quad \alpha\beta(x) \rhd (\alpha(y) \succ z - \alpha^{-1}\beta(z) \prec \alpha^2\beta^{-1}(y)) \\
&\quad\quad -(\beta(x) \succ \alpha(y) - \beta(y) \prec \alpha(x)) \rhd \beta(z) \\
&= \quad \alpha\beta(x) \succ (\alpha(y) \succ z) - (\beta(y) \succ \alpha^{-1}\beta(z)) \prec \alpha^2(x) \\
&\quad\quad -\alpha\beta(x) \succ (\alpha^{-1}\beta(z) \prec \alpha^2\beta^{-1}(y)) + (\alpha^{-2}\beta^2(z) \prec \alpha(y)) \prec \alpha^2(x) \\
&\quad\quad -(\beta(x) \succ \alpha(y)) \succ \beta(z) + \alpha^{-1}\beta^2(z) \prec (\alpha(x) \succ \alpha^2\beta^{-1}(y)) \\
&\quad\quad +(\beta(y) \prec \alpha(x)) \succ \beta(z) - \alpha^{-1}\beta^2(z) \prec (\alpha(y) \prec \alpha^2\beta^{-1}(x)) \\
&\overset{(2.6),\,(2.8)}{=} (\beta(x) \prec \alpha(y)) \succ \beta(z) + (\beta(x) \succ \alpha(y)) \succ \beta(z) \\
&\quad\quad -(\beta(y) \succ \alpha^{-1}\beta(z)) \prec \alpha^2(x) - \alpha\beta(x) \succ (\alpha^{-1}\beta(z) \prec \alpha^2\beta^{-1}(y)) \\
&\quad\quad +\alpha^{-1}\beta^2(z) \prec (\alpha(y) \prec \alpha^2\beta^{-1}(x)) + \alpha^{-1}\beta^2(z) \prec (\alpha(y) \succ \alpha^2\beta^{-1}(x)) \\
&\quad\quad -(\beta(x) \succ \alpha(y)) \succ \beta(z) + \alpha^{-1}\beta^2(z) \prec (\alpha(x) \succ \alpha^2\beta^{-1}(y)) \\
&\quad\quad +(\beta(y) \prec \alpha(x)) \succ \beta(z) - \alpha^{-1}\beta^2(z) \prec (\alpha(y) \prec \alpha^2\beta^{-1}(x)) \\
&= \quad (\beta(x) \prec \alpha(y)) \succ \beta(z) + (\beta(y) \prec \alpha(x)) \succ \beta(z) \\
&\quad\quad -(\beta(y) \succ \alpha^{-1}\beta(z)) \prec \alpha^2(x) - \alpha\beta(x) \succ (\alpha^{-1}\beta(z) \prec \alpha^2\beta^{-1}(y)) \\
&\quad\quad +\alpha^{-1}\beta^2(z) \prec (\alpha(y) \succ \alpha^2\beta^{-1}(x)) + \alpha^{-1}\beta^2(z) \prec (\alpha(x) \succ \alpha^2\beta^{-1}(y)) \\
&\overset{(2.7)}{=} (\beta(x) \prec \alpha(y)) \succ \beta(z) + (\beta(y) \prec \alpha(x)) \succ \beta(z) \\
&\quad\quad -(\beta(y) \succ \alpha^{-1}\beta(z)) \prec \alpha^2(x) - (\beta(x) \succ \alpha^{-1}\beta(z)) \prec \alpha^2(y) \\
&\quad\quad +\alpha^{-1}\beta^2(z) \prec (\alpha(y) \succ \alpha^2\beta^{-1}(x)) + \alpha^{-1}\beta^2(z) \prec (\alpha(x) \succ \alpha^2\beta^{-1}(y)).
\end{aligned}$$

This expression is obviously symmetric in x and y, so we are done. □

3 BiHom-Leibniz algebras and BiHom-Lie algebras

Definition 3.1 *A left (respectively right) BiHom-Leibniz algebra is a 4-tuple $(L, [\cdot,\cdot], \alpha, \beta)$, where L is a linear space, $[\cdot,\cdot] : L \times L \to L$ is a bilinear map and $\alpha, \beta : L \to L$ are linear maps satisfying $\alpha \circ \beta = \beta \circ \alpha$, $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, $\beta([x, y]) = [\beta(x), \beta(y)]$ and*

$$[\alpha\beta(x), [y, z]] = [[\beta(x), y], \beta(z)] + [\beta(y), [\alpha(x), z]], \tag{3.1}$$

respectively

$$[[x,y],\alpha\beta(z)] = [[x,\beta(z)],\alpha(y)] + [\alpha(x),[y,\alpha(z)]], \tag{3.2}$$

for all $x, y, z \in L$. We call α and β (in this order) the structure maps of L.

A morphism $f : (L, [\cdot,\cdot], \alpha, \beta) \to (L', [\cdot,\cdot]', \alpha', \beta')$ of BiHom-Leibniz algebras is a linear map $f : L \to L'$ such that $\alpha' \circ f = f \circ \alpha$, $\beta' \circ f = f \circ \beta$ and $f([x,y]) = [f(x), f(y)]'$, for all $x, y \in L$.

Proposition 3.2 *If $(L, [\cdot,\cdot])$ is a left (respectively right) Leibniz algebra and $\alpha, \beta : L \to L$ are two commuting morphisms of Leibniz algebras, and we define the map $\{\cdot,\cdot\} : L \times L \to L$, $\{x,y\} = [\alpha(x), \beta(y)]$, for all $x, y \in L$, then $(L, \{\cdot,\cdot\}, \alpha, \beta)$ is a left (respectively right) BiHom-Leibniz algebra, called the Yau twist of L and denoted by $L_{(\alpha,\beta)}$.*

Proof. We prove the case when L is a left Leibniz algebra, the other case is similar and left to the reader. We compute:

$$\{\alpha\beta(x), \{y,z\}\} \;=\; \{\alpha\beta(x), [\alpha(y), \beta(z)]\} = [\alpha^2\beta(x), [\alpha\beta(y), \beta^2(z)]],$$

$$\begin{aligned}
&\{\{\beta(x), y\}, \beta(z)\} + \{\beta(y), \{\alpha(x), z\}\} \\
&\qquad = \{[\alpha\beta(x), \beta(y)], \beta(z)\} + \{\beta(y), [\alpha^2(x), \beta(z)]\} \\
&\qquad = [[\alpha^2\beta(x), \alpha\beta(y)], \beta^2(z)] + [\alpha\beta(y), [\alpha^2\beta(x), \beta^2(z)]],
\end{aligned}$$

and the desired equality follows by applying (1.3) to the elements $\alpha^2\beta(x)$, $\alpha\beta(y)$, $\beta^2(z)$. □

Remark 3.3 *More generally, let $(L, [\cdot,\cdot], \alpha, \beta)$ be a left (respectively right) BiHom-Leibniz algebra and $\alpha', \beta' : L \to L$ two morphisms of BiHom-Leibniz algebras such that any two of the maps $\alpha, \alpha', \beta, \beta'$ commute. If we define the map $\{\cdot,\cdot\} : L \times L \to L$, $\{x,y\} = [\alpha'(x), \beta'(y)]$, for all $x, y \in L$, then $(L, \{\cdot,\cdot\}, \alpha \circ \alpha', \beta \circ \beta')$ is a left (respectively right) BiHom-Leibniz algebra.*

Remark 3.4 *One can easily see that, if $(L, [\cdot,\cdot], \alpha, \beta)$ is a left (respectively right) BiHom-Leibniz algebra and we define $\{\cdot,\cdot\} : L \times L \to L$, $\{x,y\} := [y,x]$, then $(L, \{\cdot,\cdot\}, \beta, \alpha)$ is a right (respectively left) BiHom-Leibniz algebra.*

Definition 3.5 *A left (respectively right) BiHom-Lie algebra is a left (respectively right) BiHom-Leibniz algebra $(L, [\cdot,\cdot], \alpha, \beta)$ satisfying the BiHom-skew-symmetry condition*

$$[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \quad \forall\, x, y \in L. \tag{3.3}$$

A morphism $f : (L, [\cdot,\cdot], \alpha, \beta) \to (L', [\cdot,\cdot]', \alpha', \beta')$ of BiHom-Lie algebras is a linear map $f : L \to L'$ such that $\alpha' \circ f = f \circ \alpha$, $\beta' \circ f = f \circ \beta$ and $f([x,y]) = [f(x), f(y)]'$, for all $x, y \in L$.

Remark 3.6 *In view of Remark 3.4, if $(L, [\cdot,\cdot], \alpha, \beta)$ is a left (respectively right) BiHom-Lie algebra and we define $\{\cdot,\cdot\} : L \times L \to L$, $\{x,y\} := [y,x]$, then $(L, \{\cdot,\cdot\}, \beta, \alpha)$ is a right (respectively left) BiHom-Lie algebra.*

Remark 3.7 *If $(L, [\cdot,\cdot])$ is a Lie algebra and $\alpha, \beta : L \to L$ are two commuting morphisms of Lie algebras, and we define the map $\{\cdot,\cdot\} : L \times L \to L$, $\{x,y\} = [\alpha(x), \beta(y)]$, then $(L, \{\cdot,\cdot\}, \alpha, \beta)$ is a left and right BiHom-Lie algebra, denoted by $L_{(\alpha,\beta)}$ and called the Yau twist of L.*

Remark 3.8 *More generally, let $(L, [\cdot, \cdot], \alpha, \beta)$ be a left (respectively right) BiHom-Lie algebra and $\alpha', \beta' : L \to L$ two morphisms of left (respectively right) BiHom-Lie algebras such that any two of the maps $\alpha, \alpha', \beta, \beta'$ commute. If we define the map $\{\cdot, \cdot\} : L \times L \to L$, $\{x, y\} = [\alpha'(x), \beta'(y)]$, then $(L, \{\cdot, \cdot\}, \alpha \circ \alpha', \beta \circ \beta')$ is a left (respectively right) BiHom-Lie algebra.*

Proposition 3.9 *Let L be a linear space, $[\cdot, \cdot] : L \times L \to L$ a bilinear map, $\alpha, \beta : L \to L$ two commuting linear maps such that $\alpha([x, y]) = [\alpha(x), \alpha(y)]$ and $\beta([x, y]) = [\beta(x), \beta(y)]$ for all $x, y \in L$, and the BiHom-skew-symmetry condition (3.3) is satisfied. Assume that α and β are bijective. Then $(L, [\cdot, \cdot], \alpha, \beta)$ is a BiHom-Lie algebra if and only if it is a left BiHom-Lie algebra if and only if it is a right BiHom-Lie algebra.*

Proof. We prove first that the BiHom-Jacobi condition

$$[\beta^2(x), [\beta(y), \alpha(z)]] + [\beta^2(y), [\beta(z), \alpha(x)]] + [\beta^2(z), [\beta(x), \alpha(y)]] = 0$$

is equivalent to (3.1). The BiHom-Jacobi condition is equivalent to

$$[\beta^2(x), \beta([y, \alpha\beta^{-1}(z)])] + [\beta^2(y), \beta([z, \alpha\beta^{-1}(x)])] + [\beta^2(z), \beta([x, \alpha\beta^{-1}(y)])] = 0,$$

which is equivalent to

$$[\beta(x), [y, \alpha\beta^{-1}(z)]] + [\beta(y), [z, \alpha\beta^{-1}(x)]] + [\beta(z), [x, \alpha\beta^{-1}(y)]] = 0,$$

which is equivalent to

$$[\beta(x), [y, \alpha\beta^{-1}(z)]] + [\beta(y), [z, \alpha\beta^{-1}(x)]] + [\beta(z), \alpha([\alpha^{-1}(x), \beta^{-1}(y)])] = 0,$$

which, by applying the BiHom-skew-symmetry condition to the third term is equivalent to

$$[\beta(x), [y, \alpha\beta^{-1}(z)]] + [\beta(y), [z, \alpha\beta^{-1}(x)]] - [\beta([\alpha^{-1}(x), \beta^{-1}(y)]), \alpha(z)] = 0,$$

which, by replacing x with $\alpha(x)$, is equivalent to

$$[\alpha\beta(x), [y, \alpha\beta^{-1}(z)]] + [\beta(y), [z, \alpha^2\beta^{-1}(x)]] - [[\beta(x), y], \alpha(z)] = 0,$$

which, by replacing z with $\alpha^{-1}\beta(z)$, is equivalent to

$$[\alpha\beta(x), [y, z]] + [\beta(y), [\alpha^{-1}\beta(z), \alpha^2\beta^{-1}(x)]] - [[\beta(x), y], \beta(z)] = 0,$$

which is equivalent to

$$[\alpha\beta(x), [y, z]] + [\beta(y), [\beta(\alpha^{-1}(z)), \alpha(\alpha\beta^{-1}(x))]] - [[\beta(x), y], \beta(z)] = 0,$$

which, by applying the BiHom-skew-symmetry condition to the second bracket in the second term is equivalent to

$$[\alpha\beta(x), [y, z]] - [\beta(y), [\alpha(x), z]] - [[\beta(x), y], \beta(z)] = 0,$$

and this is obviously equivalent to (3.1).

Now we prove that (3.1) is equivalent to (3.2). We begin with (3.1), which is equivalent to

$$[\alpha\beta(x), [y, z]] = [[\beta(x), y], \beta(z)] + [\beta(y), \alpha([x, \alpha^{-1}(z)])],$$

which, by applying the BiHom-skew-symmetry condition to the second term in the right hand side is equivalent to

$$[\alpha\beta(x),[y,z]] = [[\beta(x),y],\beta(z)] - [\beta([x,\alpha^{-1}(z)]),\alpha(y)],$$

which is equivalent to

$$[\alpha\beta(x),[y,z]] = [[\beta(x),y],\beta(z)] - [[\beta(x),\alpha^{-1}\beta(z)]),\alpha(y)],$$

which, by replacing x with $\beta^{-1}(x)$ and z with $\alpha(z)$ is equivalent to

$$[\alpha(x),[y,\alpha(z)]] = [[x,y],\alpha\beta(z)] - [[x,\beta(z)]),\alpha(y)],$$

which is obviously equivalent to (3.2). □

Remark 3.10 *By using one more time the same type of calculations, one can easily prove that, in the hypotheses of Proposition 3.9, $(L,[\cdot,\cdot],\alpha,\beta)$ is a BiHom-Lie algebra if and only if, for all $x,y,z \in L$, the following equation is satisfied:*

$$[[\beta(x),\alpha(y)],\alpha^2(z)] + [[\beta(y),\alpha(z)],\alpha^2(x)] + [[\beta(z),\alpha(x)],\alpha^2(y)] = 0.$$

Proposition 3.11 *Let $(L,[\cdot,\cdot],\alpha,\beta)$ be a left (respectively right) BiHom-Lie algebra and $R: L \to L$ a Rota-Baxter operator of weight 0 such that $R\circ\alpha = \alpha\circ R$ and $R\circ\beta = \beta\circ R$. Define a new operation on L by $x\cdot y := [R(x),y]$ (respectively by $x\cdot y := [x,R(y)]$), for all $x,y \in L$. Then $(L,\cdot,\alpha,\beta)$ is a left (respectively right) BiHom-pre-Lie algebra.*

Proof. We prove the left-handed version and leave the right-handed one to the reader. It is obvious that $\alpha(x\cdot y) = \alpha(x)\cdot\alpha(y)$ and $\beta(x\cdot y) = \beta(x)\cdot\beta(y)$, so we only need to prove the relation (2.1) defining a left BiHom-pre-Lie algebra. We compute:

$$\begin{aligned}
&\alpha\beta(x)\cdot(\alpha(y)\cdot z) - (\beta(x)\cdot\alpha(y))\cdot\beta(z)\\
&= \alpha\beta(x)\cdot[R(\alpha(y)),z] - [R(\beta(x)),\alpha(y)]\cdot\beta(z)\\
&= [R(\alpha\beta(x)),[R(\alpha(y)),z]] - [R([R(\beta(x)),\alpha(y)]),\beta(z)]\\
&\overset{(1.5)}{=} [R(\alpha\beta(x)),[R(\alpha(y)),z]] - [[R(\beta(x)),R(\alpha(y))],\beta(z)] + [R([\beta(x),R(\alpha(y))]),\beta(z)]\\
&= [R(\alpha\beta(x)),[R(\alpha(y)),z]] - [[\beta(R(x)),R(\alpha(y))],\beta(z)] + [R([\beta(x),R(\alpha(y))]),\beta(z)]\\
&\overset{(3.1)}{=} [R(\alpha\beta(x)),[R(\alpha(y)),z]] - [\alpha\beta(R(x)),[R(\alpha(y)),z]] + [\alpha\beta(R(y)),[\alpha(R(x)),z]]\\
&\quad +[R([\beta(x),R(\alpha(y))]),\beta(z)]\\
&= [\alpha\beta(R(y)),[\alpha(R(x)),z]] + [R([\beta(x),\alpha(R(y))]),\beta(z)]\\
&\overset{(3.3)}{=} [\alpha\beta(R(y)),[\alpha(R(x)),z]] - [R([\beta(R(y)),\alpha(x)]),\beta(z)]\\
&= [R(\alpha\beta(y)),[R(\alpha(x)),z]] - [R([R(\beta(y)),\alpha(x)]),\beta(z)]\\
&= \alpha\beta(y)\cdot(\alpha(x)\cdot z) - (\beta(y)\cdot\alpha(x))\cdot\beta(z),
\end{aligned}$$

finishing the proof. □

Lemma 3.12 *We consider a 4-tuple $(L,[\cdot,\cdot],\alpha,\beta)$, where L is a linear space, $\alpha,\beta: L\to L$ are linear maps and $[\cdot,\cdot]: L\times L\to L$ is a bilinear map. Let also $\lambda\in\Bbbk$ be a fixed scalar. Let $R: L\to L$ be a linear map such that*

$$R\circ\alpha = \alpha\circ R \text{ and } R\circ\beta = \beta\circ R. \tag{3.4}$$

Define a new multiplication on L by

$$\{x, y\} = [R(x), y] + [x, R(y)] + \lambda[x, y], \quad \forall\, x, y \in L.$$

Then:

i) If α and β satisfy

$$\alpha([x, y]) = [\alpha(x), \alpha(y)] \quad \text{and} \quad \beta([x, y]) = [\beta(x), \beta(y)], \quad \forall\, x, y \in L, \tag{3.5}$$

then they also satisfy

$$\alpha(\{x, y\}) = \{\alpha(x), \alpha(y)\} \quad \text{and} \quad \beta(\{x, y\}) = \{\beta(x), \beta(y)\}, \quad \forall\, x, y \in L.$$

ii) If α and β satisfy

$$[\beta(x), \alpha(y)] = -[\beta(y), \alpha(x)], \qquad \forall\, x, y \in L, \tag{3.6}$$

then they also satisfy

$$\{\beta(x), \alpha(y)\} = -\{\beta(y), \alpha(x)\}, \quad \forall\, x, y \in L.$$

iii) If R satisfies

$$[R(x), R(y)] \;=\; R([R(x), y] + [x, R(y)] + \lambda[x, y]), \quad \forall\, x, y \in L, \tag{3.7}$$

then

$$R(\{x, y\}) = [R(x), R(y)], \quad \forall\, x, y \in L. \tag{3.8}$$

Proof. i) We compute:

$$\begin{aligned}
\{\alpha(x), \alpha(y)\} &= [R(\alpha(x)), \alpha(y)] + [\alpha(x), R(\alpha(y))] + \lambda[\alpha(x), \alpha(y)] \\
&\overset{(3.4)}{=} [\alpha(R(x)), \alpha(y)] + [\alpha(x), \alpha(R(y))] + \lambda[\alpha(x), \alpha(y)] \\
&\overset{(3.5)}{=} \alpha(\{x, y\}).
\end{aligned}$$

ii) We compute:

$$\begin{aligned}
\{\beta(x), \alpha(y)\} &= [R(\beta(x)), \alpha(y)] + [\beta(x), R(\alpha(y))] + \lambda[\beta(x), \alpha(y)] \\
&\overset{(3.4)}{=} [\beta(R(x)), \alpha(y)] + [\beta(x), \alpha(R(y))] + \lambda[\beta(x), \alpha(y)] \\
&\overset{(3.6)}{=} -[\beta(y), \alpha(R(x))] - [\beta(R(y)), \alpha(x)] - \lambda[\beta(y), \alpha(x)] \\
&\overset{(3.4)}{=} -[\beta(y), R(\alpha(x))] - [R(\beta(y)), \alpha(x)] - \lambda[\beta(y), \alpha(x)] \\
&= -\{\beta(y), \alpha(x)\}.
\end{aligned}$$

iii) It is obvious by (3.7) and the definition of $\{\cdot, \cdot\}$. □

Proposition 3.13 *Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a BiHom-Lie algebra, $\lambda \in \Bbbk$ a fixed scalar and $R : L \to L$ a Rota-Baxter operator of weight λ satisfying $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$ (i.e. (3.4) and (3.7) hold). Define a new multiplication on L by*

$$\{x, y\} = [R(x), y] + [x, R(y)] + \lambda[x, y], \quad \forall\, x, y \in L.$$

Then $(L, \{\cdot,\cdot\}, \alpha, \beta)$ is a BiHom-Lie algebra.

Proof. In view of Lemma 3.12, we only need to prove that $\{\cdot,\cdot\}$ satisfies the BiHom-Jacobi condition. Note also that, by Lemma 3.12 again, (3.8) holds. We compute (for $x, y, z, t \in L$):

$$\begin{aligned}
\{\beta(x), \alpha(z)\} &= [R(\beta(x)), \alpha(z)] + [\beta(x), R(\alpha(z))] + \lambda[\beta(x), \alpha(z)],\\
\{R(\beta(y)), \alpha(z)\} &= [R(R(\beta(y))), \alpha(z)] + [R(\beta(y)), R(\alpha(z))] + \lambda[R(\beta(y)), \alpha(z)],\\
\{\beta^2(x), t\} &= [R(\beta^2(x)), t] + [\beta^2(x), R(t)] + \lambda[\beta^2(x), t].
\end{aligned}$$

So, we obtain

$$\begin{aligned}
&\{\beta^2(x), \{\beta(y), \alpha(z)\}\}\\
&= [R(\beta^2(x)), \{\beta(y), \alpha(z)\}] + [\beta^2(x), R(\{\beta(y), \alpha(z)\})] + \lambda[\beta^2(x), \{\beta(y), \alpha(z)\}]\\
&\overset{(3.8)}{=} [R(\beta^2(x)), \{\beta(y), \alpha(z)\}] + [\beta^2(x), [R(\beta(y)), R(\alpha(z))]] + \lambda[\beta^2(x), \{\beta(y), \alpha(z)\}]\\
&= [R(\beta^2(x)), [R(\beta(y)), \alpha(z)]] + [R(\beta^2(x)), [\beta(y), R(\alpha(z))]]\\
&\quad + [R(\beta^2(x)), \lambda[\beta(y), \alpha(z)]] + [\beta^2(x), [R(\beta(y)), R(\alpha(z))]]\\
&\quad + \lambda[\beta^2(x), [R(\beta(y)), \alpha(z)]] + \lambda[\beta^2(x), [\beta(y), R(\alpha(z))]]\\
&\quad + \lambda[\beta^2(x), \lambda[\beta(y), \alpha(z)]],
\end{aligned}$$

and hence

$$\begin{aligned}
&\{\beta^2(y), \{\beta(z), \alpha(x)\}\}\\
&\qquad = [R(\beta^2(y)), [R(\beta(z)), \alpha(x)]] + [R(\beta^2(y)), [\beta(z), R(\alpha(x))]]\\
&\qquad\quad + [R(\beta^2(y)), \lambda[\beta(z), \alpha(x)]] + [\beta^2(y), [R(\beta(z)), R(\alpha(x))]]\\
&\qquad\quad + \lambda[\beta^2(y), [R(\beta(z)), \alpha(x)]] + \lambda[\beta^2(y), [\beta(z), R(\alpha(x))]]\\
&\qquad\quad + \lambda[\beta^2(y), \lambda[\beta(z), \alpha(x)]],
\end{aligned}$$

$$\begin{aligned}
&\{\beta^2(z), \{\beta(x), \alpha(y)\}\}\\
&\qquad = [R(\beta^2(z)), [R(\beta(x)), \alpha(y)]] + [R(\beta^2(z)), [\beta(x), R(\alpha(y))]]\\
&\qquad\quad + [R(\beta^2(z)), \lambda[\beta(x), \alpha(y)]] + [\beta^2(z), [R(\beta(x)), R(\alpha(y))]] +\\
&\qquad\quad + \lambda[\beta^2(z), [R(\beta(x)), \alpha(y)]] + \lambda[\beta^2(z), [\beta(x), R(\alpha(y))]]\\
&\qquad\quad + \lambda[\beta^2(z), \lambda[\beta(x), \alpha(y)]].
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\{\beta^2(x), \{\beta(y), \alpha(z)\}\} + \{\beta^2(y), \{\beta(z), \alpha(x)\}\} + \{\beta^2(z), \{\beta(x), \alpha(y)\}\}\\
&= [R(\beta^2(x)), [R(\beta(y)), \alpha(z)]] + [R(\beta^2(x)), [\beta(y), R(\alpha(z))]]\\
&\quad + [R(\beta^2(x)), \lambda[\beta(y), \alpha(z)]] + [\beta^2(x), [R(\beta(y)), R(\alpha(z))]]
\end{aligned}$$

$$+\lambda[\beta^2(x), [R(\beta(y)), \alpha(z)]] + \lambda[\beta^2(x), [\beta(y), R(\alpha(z))]] + \lambda[\beta^2(x), \lambda[\beta(y), \alpha(z)]]$$
$$+[R(\beta^2(y)), [R(\beta(z)), \alpha(x)]] + [R(\beta^2(y)), [\beta(z), R(\alpha(x))]]$$
$$+[R(\beta^2(y)), \lambda[\beta(z), \alpha(x)]] + [\beta^2(y), [R(\beta(z)), R(\alpha(x))]]$$
$$+\lambda[\beta^2(y), [R(\beta(z)), \alpha(x)]] + \lambda[\beta^2(y), [\beta(z), R(\alpha(x))]] + \lambda[\beta^2(y), \lambda[\beta(z), \alpha(x)]]$$
$$+[R(\beta^2(z)), [R(\beta(x)), \alpha(y)]] + [R(\beta^2(z)), [\beta(x), R(\alpha(y))]]$$
$$+[R(\beta^2(z)), \lambda[\beta(x), \alpha(y)]] + [\beta^2(z), [R(\beta(x)), R(\alpha(y))]] +$$
$$+\lambda[\beta^2(z), [R(\beta(x)), \alpha(y)]] + \lambda[\beta^2(z), [\beta(x), R(\alpha(y))]] + \lambda[\beta^2(z), \lambda[\beta(x), \alpha(y)]],$$

i.e., in view of (3.4),

$$\begin{aligned}
&\{\beta^2(x), \{\beta(y), \alpha(z)\}\} + \{\beta^2(y), \{\beta(z), \alpha(x)\}\} + \{\beta^2(z), \{\beta(x), \alpha(y)\}\} \\
= \;& [\beta^2(R(x)), [\beta(R(y)), \alpha(z)]] + [\beta^2(R(x)), [\beta(y), \alpha(R(z))]] + [\beta^2(z), [\beta(R(x)), \alpha(R(y))]] \\
&+[\beta^2(R(y)), [\beta(R(z)), \alpha(x)]] + [\beta^2(y), [\beta(R(z)), \alpha(R(x))]] \\
&+[\beta^2(R(y)), [\beta(z), \alpha(R(x))]] + [\beta^2(x), [\beta(R(y)), \alpha(R(z))]] \\
&+[\beta^2(R(z)), [\beta(R(x)), \alpha(y)]] + [\beta^2(R(z)), [\beta(x), \alpha(R(y))]] \\
&+\lambda[\beta^2(x), [\beta(R(y)), \alpha(z)]] + [\beta^2(R(y)), \lambda[\beta(z), \alpha(x)]] + \lambda[\beta^2(z), [\beta(x), \alpha(R(y))]] \\
&+\lambda[\beta^2(x), [\beta(y), \alpha(R(z))]] + \lambda[\beta^2(y), [\beta(R(z)), \alpha(x)]] + [\beta^2(R(z)), \lambda[\beta(x), \alpha(y)]] \\
&+[\beta^2(R(x)), \lambda[\beta(y), \alpha(z)]] + \lambda[\beta^2(y), [\beta(z), \alpha(R(x))]] + \lambda[\beta^2(z), [\beta(R(x)), \alpha(y)]] \\
&+\lambda[\beta^2(x), \lambda[\beta(y), \alpha(z)]] + \lambda[\beta^2(y), \lambda[\beta(z), \alpha(x)]] + \lambda[\beta^2(z), \lambda[\beta(x), \alpha(y)]] \\
= \;& \circlearrowleft_{R(x),R(y),z} [\beta^2(R(x)), [\beta(R(y)), \alpha(z)]] + \circlearrowleft_{R(y),R(z),x} [\beta^2(R(y)), [\beta(R(z)), \alpha(x)]] \\
&+ \circlearrowleft_{R(z),R(x),y} [\beta^2(R(z)), [\beta(R(x)), \alpha(y)]] + \lambda \circlearrowleft_{x,R(y),z} [\beta^2(x), [\beta(R(y)), \alpha(z)]] \\
&+\lambda \circlearrowleft_{y,R(z),x} [\beta^2(y), [\beta(R(z)), \alpha(x)]] + \lambda \circlearrowleft_{z,R(x),y} [\beta^2(z), [\beta(R(x)), \alpha(y)]] \\
&+\lambda^2 \circlearrowleft_{x,y,z} [\beta^2(x), [\beta(y), \alpha(z)]] \\
= \;& 0,
\end{aligned}$$

where the last equality follows by applying 7 times the BiHom-Jacobi condition for $[\cdot,\cdot]$. □

Corollary 3.14 *Let $(L, [\cdot,\cdot])$ be a Lie algebra, $\lambda \in \Bbbk$ a fixed scalar and $R : L \to L$ a Rota-Baxter operator of weight λ. Define a new multiplication on L by*

$$\{x, y\} = [R(x), y] + [x, R(y)] + \lambda[x, y], \quad \forall\, x, y \in L.$$

Then $(L, \{\cdot,\cdot\})$ is also a Lie algebra (and of course we have $R(\{x, y\}) = [R(x), R(y)]$).

Proof. It follows from Proposition 3.13, by taking $\alpha = \beta = id_L$. □

Proposition 3.15 *Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a a left (respectively right) BiHom-Leibniz algebra, $\lambda \in \Bbbk$ a fixed scalar and $R : L \to L$ a Rota-Baxter operator of weight λ such that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$. Define a new multiplication on L by*

$$\{x, y\} = [R(x), y] + [x, R(y)] + \lambda[x, y], \quad \forall\, x, y \in L.$$

Then $(L, \{\cdot,\cdot\}, \alpha, \beta)$ is a left (respectively right) BiHom-Leibniz algebra.

Proof. We only prove the left-handed version, the right-handed one is similar and left to the reader. In view of Lemma 3.12 we only need to show that $\{\cdot,\cdot\}$ satisfies (3.1). Note also that, by Lemma 3.12 again, (3.8) holds. We compute (for all $x, y, z, t \in L$):

$$\begin{aligned}
\{\alpha\beta(x), t\} &= [R(\alpha\beta(x)), t] + [\alpha\beta(x), R(t)] + \lambda[\alpha\beta(x), t],\\
\{\alpha\beta(x), \{y, z\}\} &= [R(\alpha\beta(x)), [R(y), z]] + [R(\alpha\beta(x)), [y, R(z)]] + [R(\alpha\beta(x)), \lambda[y, z]]\\
&\quad +[\alpha\beta(x), R(\{y, z\})] + \lambda[\alpha\beta(x), [R(y), z]]\\
&\quad +\lambda[\alpha\beta(x), [y, R(z)]] + \lambda[\alpha\beta(x), \lambda[y, z]]\\
&\overset{(3.8)}{=} [R(\alpha\beta(x)), [R(y), z]] + [R(\alpha\beta(x)), [y, R(z)]] + [R(\alpha\beta(x)), \lambda[y, z]]\\
&\quad +[\alpha\beta(x), [R(y), R(z)]] + \lambda[\alpha\beta(x), [R(y), z]]\\
&\quad +\lambda[\alpha\beta(x), [y, R(z)]] + \lambda[\alpha\beta(x), \lambda[y, z]],
\end{aligned}$$

which, by using the fact that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$, may be written as

$$\begin{aligned}
\{\alpha\beta(x), \{y, z\}\} &= [\alpha\beta(R(x)), [R(y), z]] + [\alpha\beta(R(x)), [y, R(z)]] + [\alpha\beta(R(x)), \lambda[y, z]]\\
&\quad +[\alpha\beta(x), [R(y), R(z)]] + \lambda[\alpha\beta(x), [R(y), z]]\\
&\quad +\lambda[\alpha\beta(x), [y, R(z)]] + \lambda[\alpha\beta(x), \lambda[y, z]].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\{\beta(x), y\} &= [R(\beta(x)), y] + [\beta(x), R(y)] + \lambda[\beta(x), y],\\
\{t, \beta(z)\} &= [R(t), \beta(z)] + [t, R(\beta(z))] + \lambda[t, \beta(z)],\\
\{\{\beta(x), y\}, \beta(z)\} &= [R(\{\beta(x), y\}), \beta(z)] + [[R(\beta(x)), y], R(\beta(z))]\\
&\quad +[[\beta(x), R(y)], R(\beta(z))] + [\lambda[\beta(x), y], R(\beta(z))]\\
&\quad +\lambda[[R(\beta(x)), y], \beta(z)] + \lambda[[\beta(x), R(y)], \beta(z)] + \lambda[\lambda[\beta(x), y], \beta(z)]\\
&\overset{(3.8)}{=} [[R(\beta(x)), R(y)], \beta(z)] + [[R(\beta(x)), y], R(\beta(z))]\\
&\quad +[[\beta(x), R(y)], R(\beta(z))] + [\lambda[\beta(x), y], R(\beta(z))]\\
&\quad +\lambda[[R(\beta(x)), y], \beta(z)] + \lambda[[\beta(x), R(y)], \beta(z)] + \lambda[\lambda[\beta(x), y], \beta(z)]
\end{aligned}$$

and

$$\begin{aligned}
\{\beta(y), t\} &= [R(\beta(y)), t] + [\beta(y), R(t)] + \lambda[\beta(y), t],\\
\{\alpha(x), z\} &= [R(\alpha(x)), z] + [\alpha(x), R(z)] + \lambda[\alpha(x), z],\\
\{\beta(y), \{\alpha(x), z\}\} &= [R(\beta(y)), [R(\alpha(x)), z]] + [R(\beta(y)), [\alpha(x), R(z)]]\\
&\quad +[R(\beta(y)), \lambda[\alpha(x), z]] + [\beta(y), R(\{\alpha(x), z\})]\\
&\quad +\lambda[\beta(y), [R(\alpha(x)), z]] + \lambda[\beta(y), [\alpha(x), R(z)]] + \lambda[\beta(y), \lambda[\alpha(x), z]]\\
&\overset{(3.8)}{=} [R(\beta(y)), [R(\alpha(x)), z]] + [R(\beta(y)), [\alpha(x), R(z)]]\\
&\quad +[R(\beta(y)), \lambda[\alpha(x), z]] + [\beta(y), [R(\alpha(x)), R(z)]]\\
&\quad +\lambda[\beta(y), [R(\alpha(x)), z]] + \lambda[\beta(y), [\alpha(x), R(z)]] + \lambda[\beta(y), \lambda[\alpha(x), z]],
\end{aligned}$$

so that

$$\begin{aligned}
&\{\{\beta(x), y\}, \beta(z)\} + \{\beta(y), \{\alpha(x), z\}\}\\
&\quad = [[R(\beta(x)), R(y)], \beta(z)] + [[R(\beta(x)), y], R(\beta(z))] + [[\beta(x), R(y)], R(\beta(z))]
\end{aligned}$$

$$
\begin{aligned}
&+[\lambda[\beta(x), y], R(\beta(z))] + \lambda[[R(\beta(x)), y], \beta(z)] + \lambda[[\beta(x), R(y)], \beta(z)] \\
&+\lambda[\lambda[\beta(x), y], \beta(z)] + [R(\beta(y)), [R(\alpha(x)), z]] + [R(\beta(y)), [\alpha(x), R(z)]] \\
&+[R(\beta(y)), \lambda[\alpha(x), z]] + [\beta(y), [R(\alpha(x)), R(z)]] + \lambda[\beta(y), [R(\alpha(x)), z]] \\
&+\lambda[\beta(y), [\alpha(x), R(z)]] + \lambda[\beta(y), \lambda[\alpha(x), z]],
\end{aligned}
$$

i.e., by using again the fact that $R \circ \alpha = \alpha \circ R$ and $R \circ \beta = \beta \circ R$,

$$
\begin{aligned}
&\{\{\beta(x), y\}, \beta(z)\} + \{\beta(y), \{\alpha(x), z\}\} \\
&\quad = [[\beta(R(x)), R(y)], \beta(z)] + [\beta(R(y)), [\alpha(R(x)), z]] + \\
&\quad\quad +[[\beta(R(x)), y], \beta(R(z))] + [\beta(y), [\alpha(R(x)), R(z)]] \\
&\quad\quad +\lambda[[\beta(R(x)), y], \beta(z)] + \lambda[\beta(y), [\alpha(R(x)), z]] \\
&\quad\quad +[[\beta(x), R(y)], \beta(R(z))] + [\beta(R(y)), [\alpha(x), R(z)]] \\
&\quad\quad +\lambda[[\beta(x), R(y)], \beta(z)] + [\beta(R(y)), \lambda[\alpha(x), z]] \\
&\quad\quad +[\lambda[\beta(x), y], \beta(R(z))] + \lambda[\beta(y), [\alpha(x), R(z)]] \\
&\quad\quad +\lambda[\lambda[\beta(x), y], \beta(z)] + \lambda[\beta(y), \lambda[\alpha(x), z]].
\end{aligned}
$$

By using (3.1) 7 times, we obtain $\{\alpha\beta(x), \{y, z\}\} = \{\{\beta(x), y\}, \beta(z)\} + \{\beta(y), \{\alpha(x), z\}\}$. □

Corollary 3.16 *Let $(L, [\cdot,\cdot], \alpha, \beta)$ be a left (respectively right) BiHom-Lie algebra, $\lambda \in \Bbbk$ a fixed scalar and $R : L \to L$ a Rota-Baxter operator of weight λ such that $R\circ\alpha = \alpha\circ R$ and $R\circ\beta = \beta\circ R$. Define a new multiplication on L by*

$$\{x, y\} = [R(x), y] + [x, R(y)] + \lambda[x, y], \quad \forall\, x, y \in L.$$

Then $(L, \{\cdot,\cdot\}, \alpha, \beta)$ is a left (respectively right) BiHom-Lie algebra.

Proof. It follows by Proposition 3.15 in view of ii) in Lemma 3.12. □

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