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GRADINGS OF NULL-FILIFORM AND NATURALLY GRADED FILIFORM LEIBNIZ ALGEBRAS

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ABSTRACT. In the paper we describe all gradings admitting on null-filiform and naturally graded filiform Leibniz algebras.

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1. INTRODUCTION

Gradings by abelian groups have played a key role in the study of Lie algebras and superalgebras, starting with the root space decomposition of the semisimple Lie algebras over the complex field, which is an essential ingredient in the Killing-Cartan classification of these algebras. Gradings by a cyclic group appear in the connection between Jordan algebras and Lie algebras through the Tits-Kantor-Koecher construction, and in the theory of Kac-Moody Lie algebras. Gradings by the integers or the integers modulo 2 are ubiquitous in Geometry.

In 1989, Patera and Zassenhaus [24] began a systematic study of gradings by abelian groups on Lie algebras. They raised the problem of classifying the fine gradings, up to equivalence, on the simple Lie algebras over the complex numbers. This problem has been settled now thanks to the work of many colleagues. After that, were described gradings of simple alternative and simple Malcev algebras [14], simple Kac Jordan superalgebra [5], countless simple Lie algebras [6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19], filiform Lie algebras [4].

The concept of length of a Lie algebra was introduced by Gómez, Jiménez-Merchán and Reyes in [20], [21]. Where, they distinguished an interesting family: algebras admitting a grading with the greatest possible number of non-zero subspaces. Actually, the gradings with a large number of non-zero subspaces enable us to describe the multiplication on the algebra more exactly.

In the past years, Leibniz algebras have been under active research. The main result on the structure of finite-dimensional Leibniz algebras asserts that a Leibniz algebra decomposes into a semidirect sum of the solvable radical and a semisimple Lie algebra [3]. Therefore, the main problem of the description of finite-dimensional Leibniz algebras consists of the study of solvable Leibniz algebras. Similarly to the case of Lie algebras the study of solvable Leibniz algebras is reduced to nilpotent ones [7].

Since the description of all n-dimensional nilpotent Leibniz algebras is an unsolvable task (even in the case of Lie algebras), we have to study nilpotent Leibniz algebras under certain conditions (conditions on index of nilpotency, various types of grading, characteristic sequence etc.) [1, 22, 23]. The well-known natural grading of nilpotent Lie and Leibniz algebras is very helpful when investigating of the properties of those algebras without restrictions on the grading. Indeed, we can always choose an homogeneous basis and thus the grading allows to obtain more explicit conditions for the structural constants. Moreover, such grading is useful for the investigation of cohomologies for the considered algebras, because it induces the corresponding grading of the group of cohomologies. Thus, it is very crucial to know what kind of grading admits a nilpotent Leibniz algebra. The paper is devoted to the description of all gradings of null-filiform and naturally graded filiform Leibiz algebras.

2. PRELIMINARIES

In this section we give necessary definitions and preliminary results.

Definition 1. *A vector space with bilinear bracket $(L, [-,-])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big]$$

holds.

From the Leibniz identity we conclude that the right annihilator $Ann_r(L) = \{x \in L \mid [y,x] = 0, \text{ for all } y \in L\}$ of the Leibniz algebra L is a two-sided ideal of L. Moreover, for a given element x of a Leibniz algebra L, the right multiplication operators $\mathcal{R}_x \colon L \to L, \mathcal{R}_x(y) = [y,x], y \in L$, are derivations.

For a given Leibniz algebra $(L, [-,-])$ the sequence of two-sided ideals defined recursively as follows:

$$L^1 = L, \; L^{k+1} = [L^k, L], \; k \geq 1,$$

is said to be *the lower central series of* L.

Definition 2. *A Leibniz algebra L is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $L^n = 0$. The minimal number n with such property is said to be the index of nilpotency of the algebra L.*

Evidently, the index of nilpotency of an n-dimensional nilpotent algebra is not greater than $n+1$.

Definition 3. *An n-dimensional Leibniz algebra L is said to be null-filiform if* $\dim L^i = n+1-i, \; 1 \leq i \leq n+1$.

Evidently, null-filiform Leibniz algebras have maximal index of nilpotency.

Theorem 1 ([2]). *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra*

$$NF_n : \quad [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n-1,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra NF_n.

Actually, a nilpotent Leibniz algebra is null-filiform if and only if it is one-generated algebra. Notice that this notion has no sense in Lie algebras case, because they are at least two-generated.

Definition 4. *An n-dimensional Leibniz algebra L is said to be filiform if* $\dim L^i = n - i$*, for* $2 \leq i \leq n$.

Let G be an abelian group. An algebra L is a G-graded algebra if and only if the vector space L has the following decomposition $L = \bigoplus\limits_{g \in G} L_g$ and the multiplication law of L has the following property $[L_g, L_h] \subset L_{g+h}, \forall g, h \in G$. For a G-graded algebra $L = \bigoplus\limits_{g \in G} L_g$ we will use the following notation $L_g := \langle e_{i_1}, \ldots, e_{i_k} \rangle_g$, if the vector space L_g is generated by $e_{i_1}, \ldots, e_{i_k}$.

Now let us define a natural graduation for a filiform Leibniz algebra. Given a filiform Leibniz algebra L, put $L_i = L^i / L^{i+1}, \; 1 \leq i \leq n-1$, and $gr(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $gr(L)$. If $gr(L)$ and L are isomorphic, then we say that an algebra L is naturally graded.

Thanks to [25] it is well known that there are two types of naturally graded filiform Lie algebras. In fact, the second type will appear only in the case when the dimension of the algebra is even.

Theorem 2 ([25]). *Any complex naturally graded filiform Lie algebra is isomorphic to one of the following non isomorphic algebras:*

$$L_n : [e_i, e_1] = -[e_1, e_i] = e_{i+1}, \quad 2 \leq i \leq n-1.$$

$$Q_n(n - even) : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^{i+1} e_n, & 2 \leq i \leq n-1. \end{cases}$$

In the following theorem we recall the classification of the naturally graded filiform non-Lie Leibniz algebras given in [2].

Theorem 3 ([2]). *Any complex n-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non isomorphic algebras:*

$$F_n^1 = \{ \; [e_i, e_1] \; = \; e_{i+1}, \quad 2 \leq i \leq n-1,$$

$$F_n^2 = \{ \; [e_i, e_1] \; = \; e_{i+1}, \quad 1 \leq i \leq n-2.$$

3. GRADINGS

In the description of all abelian grading of ours algebras, we are using ideas from [5]. In the first, we are calculating the group of automorphisms of our algebra. In the second, we are proving that the normalizer of maximal torus is the same torus. In the third, we are constructing all toral gradings on our algebra.

3.1. **Gradings of the null-filiform Leibniz algebra.** Let NF_n be the n-dimensional null-filiform Leibniz algebra given by:

$$NF_n: \ [e_i, e_1] = e_{i+1}, \quad 1 \le i \le n-1.$$

3.1.1. *Automorphism of the family* NF_n. Clearly, $Center(NF_n) = \langle e_n \rangle$. Let f be an arbitrary element of $Aut(NF_n)$, then $f(e_n) = \lambda e_n \neq 0$ and

$$f(e_1) = \sum_{i=1}^{n} \alpha_i e_i, \ f(e_{n-1}) = \sum_{j=1}^{n} \beta_j e_j.$$

From $[e_{n-1}, e_1] = e_n$ we have that

$$[\sum_{j=1}^{n} \beta_j e_j, \sum_{i=1}^{n} \alpha_i e_i] = \lambda e_n \Longrightarrow \sum_{i=1, j=1}^{n} \beta_j \alpha_i [e_j, e_i] = \lambda e_n$$

that is,

$$\beta_1\alpha_1 e_2 + \beta_2\alpha_1 e_3 + \cdots + \beta_{n-2}\alpha_1 e_{n-1} + \beta_{n-1}\alpha_1 e_n = \lambda e_n \Longrightarrow \beta_{n-1}\alpha_1 = \lambda$$
$$\Longrightarrow \beta_{n-1}, \alpha_1 \neq 0, \text{ and } \beta_1 = \beta_2 = \cdots = \beta_{n-2} = 0.$$

Thus, $f(e_{n-1}) = \beta_{n-1} e_{n-1} + \beta_n e_n$.

We denote $f(e_{n-2}) = \sum_{j=1}^{n} \gamma_j e_j$.

Consider the following chain of equalities

$$\beta_{n-1} e_{n-1} + \beta_n e_n = f(e_{n-1}) = f([e_{n-2}, e_1]) = [f(e_{n-2}), f(e_1)] = [\sum_{j=1}^{n} \gamma_j e_j, \sum_{i=1}^{n} \alpha_i e_i].$$

Then we deduce $\gamma_1 = \gamma_2 = \cdots = \gamma_{n-3} = 0, \ \gamma_{n-1} = \frac{\beta_n}{\alpha_1}, \ \gamma_{n-2} = \frac{\lambda}{\alpha_1^2}$ and

$$f(e_{n-2}) = \gamma_{n-2} e_{n-2} + \gamma_{n-1} e_{n-1} + \gamma_n e_n.$$

By induction it is easy to prove the following expression for any $f \in Aut(NF_n)$:

$$f(e_i) = \alpha^i e_i + \sum_{j=i+1}^{n} \frac{\beta_{n+1-j}}{\alpha^{n-j}} e_j, \quad 1 \le i \le n,$$

with $\alpha \neq 0$.

3.1.2. *Maximal Torus.* The maximal torus is formed by:

$$\mathcal{T} = \left\{ \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha^n \end{pmatrix} : \alpha \in \mathbb{K}^* \right\} \cong \mathbb{K}^*.$$

Lemma 1. *Let $\mathcal{N}(\mathcal{T})$ be the normalizer of maximal torus. Then, $\mathcal{N}(\mathcal{T}) = \mathcal{T}$.*

Proof. We have $\mathcal{N}(\mathcal{T}) = \{M \in Aut(A) : \ MTM^{-1} \in \mathcal{T}, \ \forall\, T \in \mathcal{T}\}$.

For

$$M = \begin{pmatrix} \alpha & 0 & \dots & 0 & 0 \\ \frac{\beta_{n-1}}{\alpha^{n-2}} & \alpha^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\beta_2}{\alpha} & \frac{\beta_3}{\alpha} & \dots & \alpha^{n-1} & 0 \\ \beta_1 & \beta_2 & \dots & \beta_{n-1} & \alpha^n \end{pmatrix} \text{ and } T = \begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 0 & \lambda^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda^{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda^n \end{pmatrix}.$$

we have

$$MT = \begin{pmatrix} \alpha\lambda & 0 & \dots & 0 & 0 \\ \frac{\beta_{n-1}\lambda}{\alpha^{n-2}} & \alpha^2\lambda^2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\beta_2\lambda}{\alpha} & \frac{\beta_3\lambda^2}{\alpha} & \dots & \alpha^{n-1}\lambda^{n-1} & 0 \\ \beta_1\lambda & \beta_2\lambda^2 & \dots & \beta_{n-1}\lambda^{n-1} & \alpha^n\lambda^n \end{pmatrix}.$$

We need to prove the existence of $T' \in \mathcal{T}$ such that $MT = T'M$. For

$$T' = \begin{pmatrix} \delta & 0 & \dots & 0 \\ 0 & \delta^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \delta^n \end{pmatrix}$$

we conclude that $\delta = \lambda$ and by choosing $\lambda \neq 1, 0$ we get $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 0$ and $M \in \mathcal{T}$. $\square$

3.1.3. *Toral gradings.* Let A be a null-filiform Leibniz algebra. We distinguish the following cases:

- $\alpha^i \neq 1$ for $i = 1, 2, \dots, n$. In this case, we have the following grading:
$$A = \langle e_1\rangle_1 \oplus \langle e_2\rangle_2 \oplus \cdots \oplus \langle e_n\rangle_n : \ \mathbb{Z}\text{-grading.}$$
- $\alpha = 1$, we get
$$A = \langle e_1, e_2, \dots, e_n\rangle.$$
- $\alpha = -1$:
  - If n even, then we obtain
$$A = \langle e_1, e_3, \dots, e_{n-1}\rangle_1 \oplus \langle e_2, e_4, \dots, e_n\rangle_0 : \ \mathbb{Z}_2\text{-grading.}$$
  - If n odd, then we obtain
$$A = \langle e_1, e_3, \dots, e_n\rangle_1 \oplus \langle e_2, e_4, \dots, e_{n-1}\rangle_0 : \ \mathbb{Z}_2\text{-grading.}$$
- $\alpha = \xi$ with $\xi^3 = 1$ and $\xi \neq \pm 1$.
  - If $n = 3m$, we have a $\mathbb{Z}_3$-grading:
$$A = \langle e_1, e_4, e_7, \dots, e_{n-2}\rangle_1 \oplus \langle e_2, e_5, \dots, e_{n-1}\rangle_2 \oplus \langle e_3, e_6, \dots, e_n\rangle_0.$$
  - If $n = 3m-1$, we have a $\mathbb{Z}_3$-grading:
$$A = \langle e_1, e_4, \dots, e_{n-1}\rangle_1 \oplus \langle e_2, e_5, \dots, e_n\rangle_2 \oplus \langle e_3, e_6, \dots, e_{n-2}\rangle_0.$$
  - If $n = 3m-2$, we have a $\mathbb{Z}_3$-grading:
$$A = \langle e_1, e_4, \dots, e_n\rangle_1 \oplus \langle e_2, e_5, \dots, e_{n-2}\rangle_2 \oplus \langle e_3, e_6, \dots, e_{n-1}\rangle_0.$$
- $\sqrt[i]{\alpha} = 1$ and α is a primitive root of 1.

  Let $n = mi + p$ be with $0 \leq p \leq i$, we have a $\mathbb{Z}_i$-grading with the following homogeneous subspaces:

$$\begin{array}{lcl} A_{\overline{1}} & = & \langle e_1, e_{i+1}, \dots, e_{mi+1}\rangle_1 \\ A_{\overline{2}} & = & \langle e_2, e_{i+2}, \dots, e_{mi+2}\rangle_2 \\ & & \dots \\ A_{\overline{p}} & = & \langle e_p, e_{i+p}, \dots, e_{mi+p}\rangle_p \\ A_{\overline{p+1}} & = & \langle e_{p+1}, e_{i+p+1}, \dots, e_{mi+p+1}\rangle_{p+1} \\ & & \dots \\ A_{\overline{i-1}} & = & \langle e_{i-1}, e_{2i-1}, \dots, e_{mi+i-1}\rangle_{i-1} \\ A_{\overline{0}} & = & \langle e_i, e_{2i}, \dots, e_{mi}\rangle_0. \end{array} \tag{1}$$

Lemma 2. *Let A be a null-filiform Leibniz algebra of dimension n. Then, up to equivalence, all cyclic toral gradings are the following:*

1. *The trivial grading gives by $A = \langle e_1, e_2, \dots, e_n\rangle$;*
2. *The $\mathbb{Z}$-grading gives by $A = \langle e_1\rangle_1 \oplus \langle e_2\rangle_2 \oplus \cdots \oplus \langle e_n\rangle_n$;*
3. *The $\mathbb{Z}_i$-grading with $1 < i < n$ give by $A = A_{\overline{0}} \oplus A_{\overline{1}} \oplus \cdots \oplus A_{\overline{i-1}}$, where graded spaces are given in (1) and $n = mi + p,\ m \in \mathbb{N}$.*

Lemma 3. *If e_1 is an homogeneous element of a group grading (that is, $e_1 \in A_x$ for some $x \in G$), then the grading is one of the list of Lemma 2.*

Proof. Let $e_1 \in A_x$ and $x \in G$ be. Let i be the order of x.

- $i > n$. Let $j \leq n$ be, we have $e_j \in [[[e_1, \underbrace{e_1], e_1], \ldots, e_1]}_{j-1} \in A_{jx}$ and the $\mathbb{Z}$-grading
$$A = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_n \rangle.$$
- $i \leq n$. Then, $e_1 \in A_x,\ e_2 \in A_{2x}, \ldots, e_{i-1} \in A_{(i-1)x}$ with $|\{x, 2x, \ldots, (i-1)x\}| = i-1$. Let $j \geq i$ be, $j = im + p$ with $0 \leq p < i$. Then $e_j \in [[[e_1, \underbrace{e_1], e_1], \ldots, e_1]}_{j-1} \in A_{im+p} = A_{\overline{p}}$. Thus,
$$A = A_{\overline{1}} \oplus A_{\overline{2}} \oplus A_{\overline{3}} \oplus \cdots \oplus A_{\overline{i-1}} \oplus A_{\overline{0}}$$
with
$$\begin{array}{lcl} A_{\overline{1}} & = & \langle e_1, e_{1+i}, \ldots \rangle, \\ A_{\overline{2}} & = & \langle e_2, e_{2+i}, \ldots \rangle, \\ & & \cdots \\ A_{\overline{i-1}} & = & \langle e_{i-1}, e_{2i-1}, \ldots \rangle, \\ A_{\overline{0}} & = & \langle e_i, e_{2i}, \ldots \rangle. \end{array}$$

□

We conclude the next theorem.

Theorem 4. *Any group grading of a null-filiform Leibniz algebra is equivalent to one of the list of Lemma 2.*

3.2. **Grading of F_n^2.** Let F_n^2 be the n-dimensional filiform Leibniz algebra given by:
$$F_n^2 : \ [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n-2.$$

Theorem 5. *Let F_n^2 be a naturally graded filiform Leibniz algebra of dimension n. Any group grading of $F_n^2$is equivalent to one of the following list:*

1. *The trivial grading gives by $A = \langle e_1, e_2, \ldots, e_n \rangle$;*
2. *The $\mathbb{Z}$-grading gives by $A = \langle e_1 \rangle_1 \oplus \langle e_2 \rangle_2 \oplus \cdots \oplus \langle e_n \rangle_n$;*
3. *The $\mathbb{Z}$-grading gives by $A = \langle e_1 \rangle_1 \oplus \langle e_2 \rangle_2 \oplus \cdots \oplus \langle e_k, e_n \rangle_k \oplus \cdots \oplus \langle e_{n-1} \rangle_{n-1}$;*
4. *The $\mathbb{Z}_i$-grading with $1 < i < n$ give by $A = (A_{\overline{0}})_0 \oplus (A_{\overline{1}})_1 \oplus \cdots \oplus (A_{\overline{k}} \oplus \langle e_n \rangle)_k \oplus \cdots \oplus (A_{\overline{i-1}})_{i-1}$ in the equations (1) and $n-1 = mi + p,\ m \in \mathbb{N}$.*
5. *The $\mathbb{Z} \times \mathbb{Z}_i$-grading with $1 < i < n-1$ give by $A = \langle e_n \rangle_{(1,0)} \oplus (A_{\overline{0}})_{(0,0)} \oplus (A_{\overline{1}})_{(0,1)} \oplus \cdots \oplus (A_{\overline{i-1}})_{(0,i-1)}$ in the equations (1) and $n-1 = mi + p,\ m \in \mathbb{N}$.*

Proof. Note that if an algebra A has a grading $G : A = \bigoplus\limits_{g \in G} A_g$ and B is the one dimensional algebra with zero multiplication, then the algebra $A \oplus B$ has the following gradings:

1. $G : A \oplus B = \bigoplus g \in G(A \oplus B)_g$, where there is one element $h \in G$, such that $(A \oplus B)_h = A_h \oplus B$ and for others elements $h^* \in G : (A \oplus B)_{h^*} = (A)_{h^*}$;
2. $\mathbb{Z} \times G : A \oplus B = B \oplus \bigoplus\limits_{g \in G} A_g$.

It is easy to see that the n-dimensional algebra F_n^2 is the direct sum of the $(n-1)$-dimensional algebra NF_{n-1} and the one-dimensional abelian algebra.

□

3.3. **Gradings of the algebra F_n^1.** Let F_n^1 be the n-dimensional filiform Leibniz algebra given by:
$$F_n^1 : [e_i, e_1] = e_{i+1}, 2 \leq i \leq n-1.$$

3.3.1. *Automorphism of the family F_n^1.* Let $f \in Aut(F_n^1)$ then

$$f(e_1)=\sum_{k=1}^{n} a_k e_k,\ f(e_2)=\sum_{k=1}^{n} b_k e_k,$$
$$f(e_i)=[f(e_{i-1}),f(e_1)]=a_1^{i-2}\sum_{k=i}^{n} b_{k-i+2}e_k,\ 3\le i\le n.$$

with $a_1b_2(a_1b_2-a_2b_1)\neq 0$.

From the product $[f(e_1),f(e_1)]=0$ we derive $a_1\neq 0$, $a_k=0$ with $2\le i\le n-1$.

The equality $[f(e_2),f(e_2)]=0$ implies $b_2\neq 0$, $b_1=0$.

Thus, we have

$$f(e_1)=a_1e_1+a_ne_n,\ f(e_2)=\sum_{k=2}^{n} b_k e_k,$$
$$f(e_i)=[f(e_{i-1}),f(e_1)]=a_1^{i-2}\sum_{k=i}^{n} b_{k-i+2}e_k,\ 3\le i\le n.$$

with $a_1b_2\neq 0$.

3.3.2. *Maximal Torus.* The maximal torus is formed by:

$$\mathcal{T}=\left\{\begin{pmatrix} a & 0 & 0 & \dots & 0\\ 0 & b & 0 & \dots & 0\\ 0 & 0 & ab & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & a^{n-2}b\end{pmatrix} : a,b\in\mathbb{K}^*\right\}\cong \mathbb{K}^*\times\mathbb{K}^*.$$

By some similar calculations (see, Lemma 1), one can prove the following result:

Lemma 4. *Let $\mathcal{N}(\mathcal{T})$ be the normalizer of maximal torus. Then, $\mathcal{N}(\mathcal{T})=\mathcal{T}$.*

3.3.3. *Toral gradings.* Let A be a naturally graded filiform Leibniz algebra. We note $d_1=a$, $d_2=b$ and $d_i=a^{i-2}b$ with $3\le i\le n$ the diagonal of the matrix of maximal torus. We distinguish the following cases:

(1) $a=b$.

(1.1) If $a=b=1$. In this case, we have the trivial grading:

$$A=\langle e_1,e_2,\dots,e_n\rangle.$$

(1.2) If $a=b$ and $a^i=1, 1<i<n$. Let $n=im+p$ be with $0\le p<i$. Then, we get the $\mathbb{Z}_i$-grading

$$\begin{array}{rcllllll}
A & = & \langle e_1, & e_2, & e_{2+i}, & e_{2+2i}, & \dots, & e_{2+(m-1)i}, \quad e_{2+mi}\rangle_1\\
 & & \oplus & \langle e_3, & e_{3+i}, & e_{3+2i}, & \dots, & e_{3+(m-1)i}, \quad e_{3+mi}\rangle_2\\
 & & & \dots & & & & \\
 & & \oplus & \langle e_p, & e_{p+i}, & e_{p+2i}, & \dots, & e_{p+(m-1)i}, \quad e_{p+mi}\rangle_{p-1}\\
 & & \oplus & \langle e_{p+1}, & e_{p+1+i}, & e_{p+1+2i}, & \dots, & e_{p+1+(m-1)i}\rangle_p\\
 & & & \dots & & & & \\
 & & \oplus & \langle e_{i+1}, & e_{2i+1}, & e_{3i+1}, & \dots, & e_{i+1+(m-1)i}\rangle_i.
\end{array}$$

(1.3) If $a=b$ and $a^i\neq 1, 0<i<n$. We have the $\mathbb{Z}$-grading

$$A=\langle e_1,e_2\rangle_1\oplus\langle e_3\rangle_2\oplus\langle e_4\rangle_3\oplus\dots\oplus\langle e_{n-1}\rangle_{n-2}\oplus\langle e_n\rangle_{n-1}.$$

(2) $a\neq b$. We have $d_1=a$, $d_2=b$, $d_i=a^{i-2}b$ with $3\le i\le n$.

(2.1) If $a=1$. We obtain a $\mathbb{Z}_2$-grading

$$A=\langle e_1\rangle_0\oplus\langle e_2,e_3,\dots,e_n\rangle_1.$$

(2.2) If $a=-1$ and $b=1$.
We get the $\mathbb{Z}_2$-grading

$$A=\langle e_2,e_4,e_6\dots\rangle_0\oplus\langle e_1,e_3,e_5,e_7\dots\rangle_1.$$

(2.3) If $a=-1$.
We get the $\mathbb{Z}\times\mathbb{Z}_2$-grading

$$A=\langle e_1\rangle_{(0,1)}\oplus\langle e_2,e_4,e_6,\dots\rangle_{(1,0)}\oplus\langle e_3,e_5,e_7,\dots\rangle_{(1,1)}.$$

(2.4) If $a\neq\{1,-1\}$.

(2.4.1) If $b=1$, then $d_1=a$, $d_2=1$, $d_i=a^{i-2}$ with $3\le i\le n$. We can distinguish two cases:

(A) If there exists i with $3\le i\le n-2$ such that $a^i=1$. Let $n=mi+p$, $0\le p\le i-1$. We have the following $\mathbb{Z}_i$-grading

$$\begin{array}{rclllllll} A & = & & \langle e_2, & e_{2+i}, & e_{2+2i}, & \dots, & e_{2+(m-1)i}, & e_{2+mi}\rangle_0 \\ & \oplus & \langle e_1, & e_3, & e_{3+i}, & e_{3+2i}, & \dots, & e_{3+(m-1)i}, & e_{3+mi}\rangle_1 \\ & & \dots & & & & & & \\ & \oplus & & \langle e_p, & e_{p+i}, & e_{p+2i}, & \dots, & e_{p+(m-1)i}, & e_{p+mi}\rangle_{p-2} \\ & \oplus & & \langle e_{p+1}, & e_{p+1+i}, & e_{p+1+2i}, & \dots, & e_{p+1+(m-1)i}\rangle_{p-1} & \\ & & \dots & & & & & & \\ & \oplus & & \langle e_{i+1}, & e_{2i+1}, & e_{3i+1}, & \dots, & e_{i+1+(m-1)i}\rangle_{i-1}. & \end{array}$$

(B) If $a^i\neq1$ for any i, $3\le i\le n-2$. We have the following $\mathbb{Z}$-grading:

$$A=\langle e_2\rangle_0\oplus\langle e_1,e_3\rangle_1\oplus\langle e_4\rangle_2\oplus\dots\oplus\langle e_{n-1}\rangle_{n-3}\oplus\langle e_n\rangle_{n-2}.$$

(2.4.2) If $b\neq1$.

(A) $d_i\neq d_j$ for all i,j with $1\le i,j\le n$. We have the following $\mathbb{Z}$-grading:

$$A=\langle e_1\rangle_1\oplus\langle e_2\rangle_k\oplus\langle e_3\rangle_{k+1}\oplus\langle e_4\rangle_{k+2}\oplus\dots\oplus\langle e_{n-1}\rangle_{k+n-3}\oplus\langle e_n\rangle_{k+n-2}.$$

(B) there exist k,l with $k\neq l$ such that $d_k=d_l$ with $3\le k<l\le n$. Thus, $a^i=1$. Let $n=mi+p$ be. We have the following $\mathbb{Z}\times\mathbb{Z}_i$-grading:

$$\begin{array}{rclllllll} A & = & \langle e_1\rangle_{(0,1)} & & & & & & \\ & & \oplus\langle e_2, & e_{2+i}, & e_{2+2i}, & \dots, & e_{2+(m-1)i}, & e_{2+mi}\rangle_{(1,0)} \\ & & \oplus\langle e_3, & e_{3+i}, & e_{3+2i}, & \dots, & e_{3+(m-1)i}, & e_{3+mi}\rangle_{(1,1)} \\ & & & \dots & & & & \\ & & \oplus\langle e_p, & e_{p+i}, & e_{p+2i}, & \dots, & e_{p+(m-1)i}, & e_{p+mi}\rangle_{(1,p-2)} \\ & & \oplus\langle e_{p+1}, & e_{p+1+i}, & e_{p+1+2i}, & \dots, & e_{p+1+(m-1)i}\rangle_{(1,p-1)} & \\ & & & \dots & & & & \\ & & \oplus\langle e_{i+1}, & e_{2i+1}, & e_{3i+1}, & \dots, & e_{i+1+(m-1)i}\rangle_{(1,i-1)}. & \end{array}$$

(C) there exists i, $3\le i\le n$ such that $d_1=d_i$ and $d_1\neq d_j, i\neq j$. Thus, $b=\frac{1}{a^{i-3}}$. We put $a^{i-3}\neq1$ because in other case $b=1$. We have the following $\mathbb{Z}$-grading:

$$A=\langle e_2\rangle_{3-i}\oplus\langle e_3\rangle_{2-i}\oplus\dots\oplus\langle e_1,e_i\rangle_1\oplus_1\oplus\dots\oplus\langle e_n\rangle_{n-i+1}.$$

(D) there exist k,l with $k>l$ such that $d_1=d_k=d_l$ and $k-l=i$. Thus $b=a^{3-l}$ and $a^i=1$. Let $n=mi+p$ with $0\le p<k-l$ be. We have the following $\mathbb{Z}_i$-grading:

$$\begin{array}{rcllllll} A & = & & \langle e_{k-1}, & e_{k+i-1}, & e_{k+2i-1}, & \dots\rangle_0 \\ & \oplus & \langle e_1, & e_k, & e_{k+i}, & e_{k+2i}, & \dots\rangle_1 \\ & & & \dots & & & \\ & \oplus & & \langle e_{k+i-2}, & e_{k+2i-2}, & e_{k+3i-2}, & \dots\rangle_{i-1}. \end{array}$$

Lemma 5. *Let F_n^1 be the naturally graded filiform Leibniz algebra of dimension n. Then, up to equivalence, all cyclic toral gradings are the following:*

(1) *The trivial grading gives by $A=\langle e_1,e_2,\dots,e_n\rangle$;*

(2) *The $\mathbb{Z}_2$-grading gives by*

$$A=\langle e_1\rangle_0\oplus\langle e_2,e_3,\dots,e_n\rangle_1;$$

(3) *The $\mathbb{Z}$-grading gives by*

$$A=\langle e_1\rangle_1\oplus\langle e_2\rangle_k\oplus\langle e_3\rangle_{k+1}\oplus\langle e_4\rangle_{k+2}\oplus\dots\oplus\langle e_{n-1}\rangle_{k+n-3}\oplus\langle e_n\rangle_{k+n-2}.$$

(4) *The $\mathbb{Z}_i$-grading gives by*

$$\begin{array}{rcllllll} A & = & & \langle e_{k-1}, & e_{k+i-1}, & e_{k+2i-1}, & \dots\rangle_0 \\ & \oplus & \langle e_1, & e_k, & e_{k+i}, & e_{k+2i}, & \dots\rangle_1 \\ & & & \dots & & & \\ & \oplus & & \langle e_{k+i-2}, & e_{k+2i-2}, & e_{k+3i-2}, & \dots\rangle_{i-1}. \end{array}$$

(5) *The $\mathbb{Z}\times\mathbb{Z}_i$-grading gives by*

$$\begin{array}{rclllllll} A & = & \langle e_1\rangle_{(0,1)} & & & & & \\ & & \oplus\langle e_2, & e_{2+i}, & e_{2+2i}, & \dots, & e_{2+(m-1)i}, & e_{2+mi}\rangle_{(1,0)} \\ & & \oplus\langle e_3, & e_{3+i}, & e_{3+2i}, & \dots, & e_{3+(m-1)i}, & e_{3+mi}\rangle_{(1,1)} \\ & & & \dots & & & & \\ & & \oplus\langle e_p, & e_{p+i}, & e_{p+2i}, & \dots, & e_{p+(m-1)i}, & e_{p+mi}\rangle_{(1,p-2)} \\ & & \oplus\langle e_{p+1}, & e_{p+1+i}, & e_{p+1+2i}, & \dots, & e_{p+1+(m-1)i}\rangle_{(1,p-1)} & \\ & & & \dots & & & & \\ & & \oplus\langle e_{i+1}, & e_{2i+1}, & e_{3i+1}, & \dots, & e_{i+1+(m-1)i}\rangle_{(1,i-1)}. & \end{array}$$

Lemma 6. *If e_1, e_2 are homogeneous elements of a group grading (that is, $e_1 \in A_x, e_2 \in A_y$ for some $x, y \in G$ for the algebra A), then the grading is one of the list of Lemma 5.*

Proof. Let $e_1 \in A_x, e_2 \in A_y$ and $x, y \in G$ be. Let i be the order of x.

$x = y$.

- $i > n-1$. Let $j \leq n$ be, we have $e_j \in [[[e_2, \underbrace{e_1], e_1], \dots, e_1]}_{j-2} \in A_{(j-1)x}$ and the $\mathbb{Z}$-grading from Lemma 5(3) for $k=1$.
- $i \leq n$. Then, $e_1, e_2 \in A_x,\ e_3 \in A_{2x}, \dots, e_i \in A_{(i-1)x}$ with $|\{x, 2x, \dots, (i-1)x\}| = i-1$. Let $j \geq i$ be, $j = im+p$ with $0 \leq p < i$. Then we have $\mathbb{Z}_i$-grading from Lemma 5(4) for $k=2$.

$x \neq y$.

- $i > n-2$. Let $j \leq n$ be, we have $e_j \in [[[e_2, \underbrace{e_1], e_1], \dots, e_1]}_{j-2} \in A_{y+(j-2)x}$ and if $x \neq 0$, it is the $\mathbb{Z}$-grading from Lemma 5(3) for $k \neq 1$; on the other side for $x = 0$, we have the $\mathbb{Z}_2$-grading from Lemma 5(2).
- $i \leq n$. Then, $e_1 \in A_x, e_2 \in A_y,\ e_3 \in A_{y+x}, \dots, e_{i+1} \in A_{y+(i-1)x}$ with $|\{x, 2x, \dots, (i-1)x\}| = i-1$. Let $j \geq i$ be, $j = im+p$ with $0 \leq p < i$. Then we have $\mathbb{Z}_i$-grading from Lemma 5(4) for $k \neq 2$ or $\mathbb{Z}\times\mathbb{Z}_i$-grading from Lemma 5(5).

□

We conclude the next theorem.

Theorem 6. *Any group grading of a naturally graded filiform Leibniz algebra is equivalent to one of the list of Lemma 5.*

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