# On non-abelian extensions of 3-Lie algebras *

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**Abstract**

In this paper, we study non-abelian extensions of 3-Lie algebras through Maurer-Cartan elements. We show that there is a one-to-one correspondence between isomorphism classes of non-abelian extensions of 3-Lie algebras and equivalence classes of Maurer-Cartan elements in a DGLA. The structure of the Leibniz algebra on the space of fundamental objects is also analyzed.

## 1 Introduction

Ternary Lie algebras (3-Lie algebras) or more generally $n$-ary Lie algebras are a natural generalization of Lie algebras. They were introduced and studied first by Filippov in [10]. This type of algebras appeared also in the algebraic formulation of Nambu Mechanics [21], generalizing Hamiltonian mechanics by considering two hamiltonians, see [24] and also [12]. Moreover, 3-Lie algebras appeared in String Theory and M-theory. In [5], Basu and Harvey suggested to replace the Lie algebra appearing in the Nahm equation by a 3-Lie algebra for the lifted Nahm equations. Furthermore, in the context of Bagger-Lambert-Gustavsson model of multiple M2-branes, Bagger-Lambert managed to construct, using a ternary bracket, an $N = 2$ supersymmetric version of the worldvolume theory of the M-theory membrane, see [1] and also [2, 13, 14, 22].

Several algebraic aspects of $n$-Lie algebras were studied in the last years. See [3, 4] for the construction, realization and classifications of 3-Lie algebras and $n$-Lie algebras. Representation theory of $n$-Lie algebras was first introduced by Kasymov in [15] and cohomologies were studied in [8]. The adjoint representation is defined by the ternary bracket in which two elements are fixed. Through fundamental objects one may also represent a 3-Lie algebra and more generally an $n$-Lie algebra by a Leibniz algebra [6]. Following this approach, deformations of 3-Lie algebras and $n$-Lie algebras are studied in [9, 25], see [20] for a review. In [23], the author defined a graded Lie algebra structure on the cochain complex of an $n$-Leibniz algebra and described an $n$-Leibniz structure as a canonical structure. See the review article [7] for more details. In [16], the authors introduced

[0]*Keyword: 3-Lie algebra, Leibniz algebra, non-abelian extension, Maurer-Cartan element*
[0]*MSC*: 17B10, 17B56, 17A42.
*This research is supported by NSFC (11471139) and NSF of Jilin Province (20170101050JC).

the notion of a generalized representation of a 3-Lie algebra, by which abelian extensions of 3-Lie algebras are studied.

Due to its difficulty and less of tools, non-abelian extensions of 3-Lie algebras are not studied. In this paper, motivated by the work in [11, 16], we find a suitable approach which uses Maurer-Cartan elements to study non-abelian extensions of 3-Lie algebras. We also show that the Leibniz algebra on the space of fundamental objects is a non-abelian extension of Leibniz algebras.

The paper is organized as follows. In Section 2, we give a review of non-abelian extensions of Leibniz algebras and cohomologies of 3-Lie algebras. A characterization of non-abelian extensions of a 3-Lie algebra by another 3-Lie algebra is given in Section 3 and several examples provided. In Section 4, we show that there is a one-to-one correspondence between isomorphism classes of non-abelian extensions of 3-Lie algebras and equivalence classes of Maurer-Cartan elements. Finally, we analyze in Section 5 the corresponding Leibniz algebra structure on the space of fundamental objects and show that it is a non-abelian extension of Leibniz algebras.

# 2 Preliminaries

In this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and all the vector spaces are over $\mathbb{K}$.

## 2.1 Non-abelian extensions of Leibniz algebras

A **Leibniz algebra** is a vector space $\mathfrak{k}$ endowed with a linear map $[\cdot,\cdot]_{\mathfrak{k}}:\mathfrak{k}\otimes\mathfrak{k}\to\mathfrak{k}$ satisfying

$$[x,[y,z]_{\mathfrak{k}}]_{\mathfrak{k}}=[[x,y]_{\mathfrak{k}},z]_{\mathfrak{k}}+[y,[x,z]_{\mathfrak{k}}]_{\mathfrak{k}},\quad \forall x,y,z\in\mathfrak{k}. \tag{1}$$

This is in fact a left Leibniz algebra. In this paper, we only consider left Leibniz algebras which we call Leibniz algebras.

Let $(\mathfrak{k},[\cdot,\cdot]_{\mathfrak{k}})$ be a Leibniz algebra. We denote by $\mathrm{Der}^L(\mathfrak{k})$ and $\mathrm{Der}^R(\mathfrak{k})$ the set of left derivations and the set of right derivations of $\mathfrak{g}$ respectively:

$$\begin{aligned}\mathrm{Der}^L(\mathfrak{k})&=\{D\in\mathfrak{gl}(\mathfrak{k})|D[x,y]_{\mathfrak{k}}=[Dx,y]_{\mathfrak{k}}+[x,Dy]_{\mathfrak{k}},\forall x,y\in\mathfrak{k}\},\\ \mathrm{Der}^R(\mathfrak{k})&=\{D\in\mathfrak{gl}(\mathfrak{k})|D[x,y]_{\mathfrak{k}}=[x,Dy]_{\mathfrak{k}}-[y,Dx]_{\mathfrak{k}},\forall x,y\in\mathfrak{k}\}.\end{aligned}$$

Note that the right derivations are called anti-derivations in [18, 19]. It is easy to see that for all $x\in\mathfrak{k}$, $\mathrm{ad}^L_x:\mathfrak{k}\longrightarrow\mathfrak{k}$, which is given by $\mathrm{ad}^L_x(y)=[x,y]_{\mathfrak{k}}$, is a left derivation; $\mathrm{ad}^R_x:\mathfrak{k}\longrightarrow\mathfrak{k}$, which is given by $\mathrm{ad}^R_x(y)=[y,x]_{\mathfrak{k}}$, is a right derivation.

**Definition 2.1.** (1) *Let $\mathfrak{k}$, $\mathfrak{s}$, $\hat{\mathfrak{k}}$ be Leibniz algebras. A non-abelian extension of Leibniz algebras is a short exact sequence of Leibniz algebras:*

$$0\longrightarrow\mathfrak{s}\xrightarrow{i}\hat{\mathfrak{k}}\xrightarrow{p}\mathfrak{k}\longrightarrow 0.$$

*We say that $\hat{\mathfrak{k}}$ is a non-abelian extension of $\mathfrak{k}$ by $\mathfrak{s}$.*

(2) *A linear section of $\hat{\mathfrak{k}}$ is a linear map $\sigma:\mathfrak{k}\to\hat{\mathfrak{k}}$ such that $p\circ\sigma=\mathrm{Id}$.*

Let $\hat{\mathfrak{k}}$ be a non-abelian extension of $\mathfrak{k}$ by $\mathfrak{s}$, and $\sigma:\mathfrak{k}\to\widehat{\mathfrak{k}}$ a linear section. Define $\omega:\mathfrak{k}\otimes\mathfrak{k}\to\mathfrak{s}$, $l:\mathfrak{k}\longrightarrow\mathfrak{gl}(\mathfrak{s})$ and $r:\mathfrak{k}\longrightarrow\mathfrak{gl}(\mathfrak{s})$ respectively by

$$\begin{aligned}
\omega(x,y) &= [\sigma(x),\sigma(y)]_{\hat{\mathfrak{k}}}-\sigma[x,y]_{\mathfrak{k}}, \quad \forall x,y\in\mathfrak{k}, && (2)\\
l_x(\beta) &= [\sigma(x),\beta]_{\hat{\mathfrak{k}}}, \quad \forall x\in\mathfrak{k},\ \beta\in\mathfrak{s}, && (3)\\
r_y(\alpha) &= [\alpha,\sigma(y)]_{\hat{\mathfrak{k}}}, \quad \forall y\in\mathfrak{k},\ \alpha\in\mathfrak{s}. && (4)
\end{aligned}$$

Given a linear section, we have $\hat{\mathfrak{k}}\cong\mathfrak{k}\oplus\mathfrak{s}$ as vector spaces, and the Leibniz algebra structure on $\hat{\mathfrak{k}}$ can be transferred to $\mathfrak{k}\oplus\mathfrak{s}$:

$$[x+\alpha,y+\beta]_{(l,r,\omega)}=[x,y]_{\mathfrak{k}}+\omega(x,y)+l_x\beta+r_y\alpha+[\alpha,\beta]_{\mathfrak{s}}. \tag{5}$$

**Proposition 2.2.** *With the above notations,* $(\mathfrak{k}\oplus\mathfrak{s},[\cdot,\cdot]_{(l,r,\omega)})$ *is a Leibniz algebra if and only if* $l,r,\omega$ *satisfy the following equalities:*

$$\begin{aligned}
l_x[\alpha,\beta]_{\mathfrak{s}} &= [l_x\alpha,\beta]_{\mathfrak{s}}+[\alpha,l_x\beta]_{\mathfrak{s}}, && (6)\\
r_x[\alpha,\beta]_{\mathfrak{s}} &= [\alpha,r_x\beta]_{\mathfrak{s}}-[\beta,r_x\alpha]_{\mathfrak{s}}, && (7)\\
[l_x\alpha+r_x\alpha,\beta]_{\mathfrak{s}} &= 0, && (8)\\
[l_x,l_y]-l_{[x,y]_{\mathfrak{k}}} &= \mathrm{ad}^L_{\omega(x,y)}, && (9)\\
[l_x,r_y]-r_{[x,y]_{\mathfrak{k}}} &= \mathrm{ad}^R_{\omega(x,y)}, && (10)\\
r_y(r_x(\alpha)+l_x(\alpha)) &= 0, && (11)\\
l_x\omega(y,z)-l_y\omega(x,z)-r_z\omega(x,y) &= \omega([x,y]_{\mathfrak{k}},z)-\omega(x,[y,z]_{\mathfrak{k}})+\omega(y,[x,z]_{\mathfrak{k}}). && (12)
\end{aligned}$$

Eq. (6) means that $l_x\in\mathrm{Der}^L(\mathfrak{s})$ and Eq. (7) means that $r_x\in\mathrm{Der}^R(\mathfrak{s})$. See [17] for more details about non-abelian extensions of Leibniz algebras.

## 2.2 3-Lie algebras and their representations

**Definition 2.3.** *A* 3*-Lie algebra is a vector space* $\mathfrak{g}$ *together with a skew-symmetric linear map* $[\cdot,\cdot,\cdot]_{\mathfrak{g}}:\wedge^3\mathfrak{g}\to\mathfrak{g}$ *such that the following* **Fundamental Identity (FI)** *holds:*

$$\begin{aligned}
&F_{x_1,x_2,x_3,x_4,x_5}\\
\triangleq\ & [x_1,x_2,[x_3,x_4,x_5]_{\mathfrak{g}}]_{\mathfrak{g}}-[[x_1,x_2,x_3]_{\mathfrak{g}},x_4,x_5]_{\mathfrak{g}}-[x_3,[x_1,x_2,x_4]_{\mathfrak{g}},x_5]_{\mathfrak{g}}-[x_3,x_4,[x_1,x_2,x_5]_{\mathfrak{g}}]_{\mathfrak{g}}\\
=\ & 0. \qquad (13)
\end{aligned}$$

**Definition 2.4.** *A morphism of* 3*-Lie algebras* $f:(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})\to(\mathfrak{h},[\cdot,\cdot,\cdot]_{\mathfrak{h}})$ *is a linear map* $f:\mathfrak{g}\to\mathfrak{h}$ *such that*

$$f[x,y,z]_{\mathfrak{g}}=[f(x),f(y),f(z)]_{\mathfrak{h}}. \tag{14}$$

Elements in $\wedge^2\mathfrak{g}$ are called **fundamental objects** of the 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$. There is a bilinear operation $[\cdot,\cdot]_{\mathrm{F}}$ on $\wedge^2\mathfrak{g}$, which is given by

$$[\mathfrak{X},\mathfrak{Y}]_{\mathrm{F}}=[x_1,x_2,y_1]_{\mathfrak{g}}\wedge y_2+y_1\wedge[x_1,x_2,y_2]_{\mathfrak{g}}, \quad \forall \mathfrak{X}=x_1\wedge x_2,\ \mathfrak{Y}=y_1\wedge y_2. \tag{15}$$

It is well-known that $(\wedge^2\mathfrak{g},[\cdot,\cdot]_{\mathrm{F}})$ is a Leibniz algebra [6], which plays an important role in the theory of 3-Lie algebras.

**Definition 2.5.** ([15]) *A representation $\rho$ of a 3-Lie algebra $\mathfrak{g}$ on a vector space $V$ is a linear map $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, such that*

$$\begin{aligned}\rho(x_1,x_2)\rho(x_3,x_4) &= \rho([x_1,x_2,x_3]_{\mathfrak{g}},x_4)+\rho(x_3,[x_1,x_2,x_4]_{\mathfrak{g}})+\rho(x_3,x_4)\rho(x_1,x_2),\\ \rho(x_1,[x_2,x_3,x_4]_{\mathfrak{g}}) &= \rho(x_3,x_4)\rho(x_1,x_2)-\rho(x_2,x_4)\rho(x_1,x_3)+\rho(x_2,x_3)\rho(x_1,x_4).\end{aligned}$$

**Lemma 2.6.** *Let $\mathfrak{g}$ be a 3-Lie algebra, $V$ a vector space and $\rho : \wedge^2\mathfrak{g} \to \mathfrak{gl}(V)$ a skew-symmetric linear map. Then $(V;\rho)$ is a representation of $\mathfrak{g}$ if and only if there is a 3-Lie algebra structure (called the semidirect product) on the direct sum of vector spaces $\mathfrak{g}\oplus V$, defined by*

$$[x_1+v_1,x_2+v_2,x_3+v_3]_{\rho}=[x_1,x_2,x_3]_{\mathfrak{g}}+\rho(x_1,x_2)v_3+\rho(x_2,x_3)v_1+\rho(x_3,x_1)v_2, \tag{16}$$

*for $x_i\in\mathfrak{g}, v_i\in V, 1\leq i\leq 3$. We denote this semidirect product 3-Lie algebra by $\mathfrak{g}\ltimes_{\rho}V$.*

A $p$-cochain on $\mathfrak{g}$ with coefficients in a representation $(V;\rho)$ is a linear map

$$\alpha : \wedge^2\mathfrak{g}\otimes \overset{(p-1)}{\cdots} \otimes \wedge^2\mathfrak{g}\wedge\mathfrak{g} \longrightarrow V.$$

Denote the space of $p$-cochains by $C^{p-1}(\mathfrak{g},V)$. The coboundary operator $\delta : C^{p-1}(\mathfrak{g},V) \longrightarrow C^{p}(\mathfrak{g},V)$ is given by

$$\begin{aligned}&(\delta\alpha)(\mathfrak{X}_1,\cdots,\mathfrak{X}_p,z)\\ =&\sum_{1\leq j<k\leq p}(-1)^j\alpha(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_{k-1},[\mathfrak{X}_j,\mathfrak{X}_k]_{\mathrm{F}},\mathfrak{X}_{k+1},\cdots,\mathfrak{X}_p,z)\\ &+\sum_{j=1}^{p}(-1)^j\alpha(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_p,[\mathfrak{X}_j,z]_{\mathfrak{g}})\\ &+\sum_{j=1}^{p}(-1)^{j+1}\rho(\mathfrak{X}_j)\alpha(\mathfrak{X}_1,\cdots,\hat{\mathfrak{X}}_j,\cdots,\mathfrak{X}_p,z)\\ &+(-1)^{p+1}\Big(\rho(y_p,z)\alpha(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p-1},x_p)+\rho(z,x_p)\alpha(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p-1},y_p)\Big).\end{aligned} \tag{17}$$

# 3 Non-abelian extensions of 3-Lie algebras

**Definition 3.1.** *A non-abelian extension of a 3-Lie algebra $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$ by a 3-Lie algebra $(\mathfrak{h},[\cdot,\cdot,\cdot]_{\mathfrak{h}})$ is a short exact sequence of 3-Lie algebra morphisms:*

$$0\longrightarrow\mathfrak{h}\stackrel{\iota}{\longrightarrow}\hat{\mathfrak{g}}\stackrel{p}{\longrightarrow}\mathfrak{g}\longrightarrow 0,$$

*where $(\hat{\mathfrak{g}},[\cdot,\cdot,\cdot]_{\hat{\mathfrak{g}}})$ is a 3-Lie algebra.*

**Definition 3.2.** *Two extensions of $\mathfrak{g}$ by $\mathfrak{h}$, $(\hat{\mathfrak{g}}_1,[\cdot,\cdot,\cdot]_{\hat{\mathfrak{g}}_1})$ and $(\hat{\mathfrak{g}}_2,[\cdot,\cdot,\cdot]_{\hat{\mathfrak{g}}_2})$, are said to be isomorphic if there exists a 3-Lie algebra morphism $\theta : \hat{\mathfrak{g}}_2\to\hat{\mathfrak{g}}_1$ such that we have the following commutative diagram:*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r, "\iota_2"] \arrow[d, equal] & \hat{\mathfrak{g}}_2 \arrow[r, "p_2"] \arrow[d, "\theta"] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathfrak{h} \arrow[r, "\iota_1"] & \hat{\mathfrak{g}}_1 \arrow[r, "p_1"] & \mathfrak{g} \arrow[r] & 0. \end{tikzcd}$$

Given a section $\sigma$ of $\hat{\mathfrak{g}}$, define $\rho : \wedge^2\mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{h})$, $\nu : \mathfrak{g} \longrightarrow \mathrm{Hom}(\wedge^2\mathfrak{h}, \mathfrak{h})$ and $\omega : \wedge^3\mathfrak{g} \longrightarrow \mathfrak{h}$ by

$$\begin{aligned} \rho(x, y)(u) &= [\sigma(x), \sigma(y), u]_{\hat{\mathfrak{g}}}, \\ \nu(x)(u, v) &= [\sigma(x), u, v]_{\hat{\mathfrak{g}}}, \\ \omega(x, y, z) &= [\sigma(x), \sigma(y), \sigma(z)]_{\hat{\mathfrak{g}}} - \sigma[x, y, z]_{\mathfrak{g}}. \end{aligned}$$

Obviously, $\hat{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Transfer the 3-Lie algebra structure on $\hat{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus \mathfrak{h}$, we obtain a 3-Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{(\rho,\nu,\omega)})$, where $[\cdot, \cdot]_{(\rho,\nu,\omega)}$ is given by

$$\begin{aligned} &[x_1 + v_1, x_2 + v_2, x_3 + v_3]_{(\rho,\nu,\omega)} \\ &= [x_1, x_2, x_3]_{\mathfrak{g}} + \omega(x_1, x_2, x_3) + \rho(x_1, x_2)v_3 + \rho(x_2, x_3)v_1 + \rho(x_3, x_1)v_2 \\ &+\nu(x_1)(v_2, v_3) + \nu(x_2)(v_3, v_1) + \nu(x_3)(v_1, v_2) + [v_1, v_2, v_3]_{\mathfrak{h}}. \end{aligned}$$

The following proposition gives the conditions on $\rho, \nu$ and $\omega$ such that $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{(\rho,\nu,\omega)})$ is a 3-Lie algebra.

**Proposition 3.3.** *With the above notations, $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{(\rho,\nu,\omega)})$ is a 3-Lie algebra if and only if $\rho, \nu$ and $\omega$ satisfy, for all $x_1, \cdots, x_5 \in \mathfrak{g}$, $v_1 \cdots, v_5 \in \mathfrak{h}$, the following conditions*

$$\begin{aligned} 0 &= \omega(x_1, x_2, [x_3, x_4, x_5]_{\mathfrak{g}}) + \rho(x_1, x_2)\omega(x_3, x_4, x_5) - \omega([x_1, x_2, x_3]_{\mathfrak{g}}, x_4, x_5) \\ &\quad -\rho(x_4, x_5)\omega(x_1, x_2, x_3) - \omega(x_3, [x_1, x_2, x_4]_{\mathfrak{g}}, x_5) + \rho(x_3, x_5)\omega(x_1, x_2, x_4) \\ &\quad -\omega(x_3, x_4, [x_1, x_2, x_5]_{\mathfrak{g}}) - \rho(x_3, x_4)\omega(x_1, x_2, x_5), \qquad (18) \\ 0 &= \rho(x_1, x_2)(\rho(x_3, x_4)v_5) - \rho([x_1, x_2, x_3]_{\mathfrak{g}}, x_4)v_5 + \nu(x_4)(\omega(x_1, x_2, x_3), v_5) \\ &\quad -\rho(x_3, [x_1, x_2, x_4]_{\mathfrak{g}})v_5 - \nu(x_3)(\omega(x_1, x_2, x_4), v_5) - \rho(x_3, x_4)(\rho(x_1, x_2)v_5), \qquad (19) \\ 0 &= -\rho(x_1, [x_3, x_4, x_5]_{\mathfrak{g}})v_2 + \nu(x_1)(v_2, \omega(x_3, x_4, x_5)) + \rho(x_4, x_5)(\rho(x_1, x_3)v_2) \\ &\quad -\rho(x_3, x_5)(\rho(x_1, x_4)v_2) + \rho(x_3, x_4)(\rho(x_1, x_5)v_2), \qquad (20) \\ 0 &= \rho(x_1, x_2)(\nu(x_3)(v_4, v_5)) - \nu([x_1, x_2, x_3]_{\mathfrak{g}})(v_4, v_5) - [\omega(x_1, x_2, x_3), v_4, v_5]_{\mathfrak{h}} \\ &\quad -\nu(x_3)(\rho(x_1, x_2)v_4, v_5) - \nu(x_3)(v_4, \rho(x_1, x_2)v_5), \qquad (21) \\ 0 &= \nu(x_1)(v_2, \rho(x_3, x_4)v_5) - \nu(x_4)(\rho(x_1, x_3)v_2, v_5) + \nu(x_3)(\rho(x_1, x_4)v_2, v_5) \\ &\quad -\rho(x_3, x_4)(\nu(x_1)(v_2, v_5)), \qquad (22) \\ 0 &= \nu([x_3, x_4, x_5]_{\mathfrak{g}})(v_1, v_2) + [v_1, v_2, \omega(x_3, x_4, x_5)]_{\mathfrak{h}} - \rho(x_4, x_5)(\nu(x_3)(v_1, v_2)) \\ &\quad +\rho(x_3, x_5)(\nu(x_4)(v_1, v_2)) - \rho(x_3, x_4)(\nu(x_5)(v_1, v_2)), \qquad (23) \\ 0 &= \rho(x_1, x_2)[v_3, v_4, v_5]_{\mathfrak{h}} - [\rho(x_1, x_2)v_3, v_4, v_5]_{\mathfrak{h}} - [v_3, \rho(x_1, x_2)v_4, v_5]_{\mathfrak{h}} \\ &\quad -[v_3, v_4, \rho(x_1, x_2)v_5]_{\mathfrak{h}}, \qquad (24) \\ 0 &= \nu(x_1)(v_2, \nu(x_3)(v_4, v_5)) + [\rho(x_1, x_3)v_2, v_4, v_5]_{\mathfrak{h}} - \nu(x_3)(\nu(x_1)(v_2, v_4), v_5) \\ &\quad -\nu(x_3)(v_4, \nu(x_1)(v_2, v_5)), \qquad (25) \\ 0 &= [v_1, v_2, \rho(x_3, x_4)v_5]_{\mathfrak{h}} + \nu(x_4)(\nu(x_3)(v_1, v_2), v_5) - \nu(x_3)(\nu(x_4)(v_1, v_2), v_5) \\ &\quad -\rho(x_3, x_4)[v_1, v_2, v_5]_{\mathfrak{h}}, \qquad (26) \\ 0 &= \nu(x_1)(v_2, [v_3, v_4, v_5]_{\mathfrak{h}}) - [\nu(x_1)(v_2, v_3), v_4, v_5]_{\mathfrak{h}} - [v_3, \nu(x_1)(v_2, v_4), v_5]_{\mathfrak{h}} \\ &\quad -[v_3, v_4, \nu(x_1)(v_2, v_5)]_{\mathfrak{h}}, \qquad (27) \\ 0 &= [v_1, v_2, \nu(x_3)(v_4, v_5)]_{\mathfrak{h}} - [\nu(x_3)(v_1, v_2), v_4, v_5]_{\mathfrak{h}} - \nu(x_3)([v_1, v_2, v_4]_{\mathfrak{h}}, v_5) \\ &\quad -\nu(x_3)(v_4, [v_1, v_2, v_5]_{\mathfrak{h}}). \qquad (28) \end{aligned}$$

**Proof.** Assume that $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)})$ is a 3-Lie algebra. By

$$\begin{aligned}[x_1,x_2,[x_3,x_4,x_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,x_2,x_3]_{(\rho,\nu,\omega)},x_4,x_5]_{(\rho,\nu,\omega)}+[x_3,[x_1,x_2,x_4]_{(\rho,\nu,\omega)},x_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,x_4,[x_1,x_2,x_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (18) holds. By

$$\begin{aligned}[x_1,x_2,[x_3,x_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,x_2,x_3]_{(\rho,\nu,\omega)},x_4,v_5]_{(\rho,\nu,\omega)}+[x_3,[x_1,x_2,x_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,x_4,[x_1,x_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (19) holds. By

$$\begin{aligned}[x_1,v_2,[x_3,x_4,x_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,v_2,x_3]_{(\rho,\nu,\omega)},x_4,x_5]_{(\rho,\nu,\omega)}+[x_3,[x_1,v_2,x_4]_{(\rho,\nu,\omega)},x_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,x_4,[x_1,v_2,x_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (20) holds. By

$$\begin{aligned}[x_1,x_2,[x_3,v_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,x_2,x_3]_{(\rho,\nu,\omega)},v_4,v_5]_{(\rho,\nu,\omega)}+[x_3,[x_1,x_2,v_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,v_4,[x_1,x_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (21) holds. By

$$\begin{aligned}[x_1,v_2,[x_3,x_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,v_2,x_3]_{(\rho,\nu,\omega)},x_4,v_5]_{(\rho,\nu,\omega)}+[x_3,[x_1,v_2,x_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,x_4,[x_1,v_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (22) holds. By

$$\begin{aligned}[v_1,v_2,[x_3,x_4,x_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[v_1,v_2,x_3]_{(\rho,\nu,\omega)},x_4,x_5]_{(\rho,\nu,\omega)}+[x_3,[v_1,v_2,x_4]_{(\rho,\nu,\omega)},x_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,x_4,[v_1,v_2,x_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (23) holds. By

$$\begin{aligned}[x_1,x_2,[v_3,v_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,x_2,v_3]_{(\rho,\nu,\omega)},v_4,v_5]_{(\rho,\nu,\omega)}+[v_3,[x_1,x_2,v_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[v_3,v_4,[x_1,x_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (24) holds. By

$$\begin{aligned}[x_1,v_2,[x_3,v_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,v_2,x_3]_{(\rho,\nu,\omega)},v_4,v_5]_{(\rho,\nu,\omega)}+[x_3,[x_1,v_2,v_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,v_4,[x_1,v_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (25) holds. By

$$\begin{aligned}[v_1,v_2,[x_3,x_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[v_1,v_2,x_3]_{(\rho,\nu,\omega)},x_4,v_5]_{(\rho,\nu,\omega)}+[x_3,[v_1,v_2,x_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[x_3,x_4,[v_1,v_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (26) holds. By

$$\begin{aligned}[x_1,v_2,[v_3,x_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[x_1,v_2,v_3]_{(\rho,\nu,\omega)},v_4,v_5]_{(\rho,\nu,\omega)}+[v_3,[x_1,v_2,v_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad+[v_3,v_4,[x_1,v_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (27) holds. By

$$\begin{aligned}[v_1,v_2,[x_3,v_4,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)} &= [[v_1,v_2,x_3]_{(\rho,\nu,\omega)},v_4,v_5]_{(\rho,\nu,\omega)}+[x_3,[v_1,v_2,v_4]_{(\rho,\nu,\omega)},v_5]_{(\rho,\nu,\omega)}\\ &\quad +[x_3,v_4,[v_1,v_2,v_5]_{(\rho,\nu,\omega)}]_{(\rho,\nu,\omega)},\end{aligned}$$

we deduce that (28) holds.

Conversely, if (18)-(28) hold, it is straightforward to see that $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)})$ is a 3-Lie algebra. ■

**Example 3.4.** *Let $\mathfrak{g}$ be the simple 4-dimensional 3-Lie algebra defined with respect to a basis $\{x_1,x_2,x_3,x_4\}$ by the skew-symmetric brackets*

$$[x_1,x_2,x_3]=x_4,\ [x_1,x_2,x_4]=x_3,\ [x_1,x_3,x_4]=x_2,\ [x_2,x_3,x_4]=x_1,$$

*and let $\mathfrak{h}$ be the 3-dimensional 3-Lie algebra defined with respect to basis $\{v_1,v_2,v_3\}$ by*

$$[v_1,v_2,v_3]=v_1.$$

*Then every non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ is given by $\rho=0$. The following families of $\nu$ and $\omega$ provide non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$*

- $\nu(x_1)(v_1,v_2)=r_1v_1,\ \nu(x_1)(v_1,v_3)=r_2v_1,\ \nu(x_1)(v_2,v_3)=r_3v_1,$
  $\nu(x_2)(v_1,v_2)=\frac{r_1r_4}{r_2}v_1,\ \nu(x_2)(v_1,v_3)=r_4v_1,\ \nu(x_2)(v_2,v_3)=\frac{r_3r_4}{r_2}v_1,$
  $\nu(x_3)(v_1,v_2)=\frac{r_1r_5}{r_2}v_1,\ \nu(x_3)(v_1,v_3)=r_5v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_3r_5}{r_2}v_1,$
  $\nu(x_4)(v_1,v_2)=\frac{r_1r_6}{r_2}v_1,\ \nu(x_4)(v_1,v_3)=r_6v_1,\ \nu(x_4)(v_2,v_3)=\frac{r_3r_6}{r_2}v_1,$
- $\omega(x_1,x_2,x_3)=-\frac{r_3r_6}{r_2}v_1+r_6v_2-\frac{r_1r_6}{r_2}v_3,$
  $\omega(x_1,x_2,x_4)=-\frac{r_3r_5}{r_2}v_1+r_5v_2-\frac{r_1r_5}{r_2}v_3,$
  $\omega(x_1,x_3,x_4)=-\frac{r_3r_4}{r_2}v_1+r_4v_2-\frac{r_1r_4}{r_2}v_3,$
  $\omega(x_2,x_3,x_4)=-r_3v_1+r_2v_2-r_1v_3,$

*where $r_i$ are parameters in $\mathbb{K}$.*

**Example 3.5.** *Now, let $\mathfrak{g}$ be the 3-dimensional 3-Lie algebra defined with respect to a basis $\{x_1,x_2,x_3\}$ by the skew-symmetric bracket $[x_1,x_2,x_3]=x_1$ and let $\mathfrak{h}$ be the same 3-Lie algebra which we consider with respect to basis $\{v_1,v_2,v_3\}$, that is $[v_1,v_2,v_3]=v_1$. Then every non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$ is given by one of the following triples $(\rho,\nu,\omega)$.*

1. - $\rho(x_1,x_2)(v_1)=0,\ \rho(x_1,x_2)(v_2)=0,\ \rho(x_1,x_2)(v_3)=0,$
     $\rho(x_1,x_3)(v_1)=0,\ \rho(x_1,x_3)(v_2)=0,\ \rho(x_1,x_3)(v_3)=0,$
     $\rho(x_2,x_3)(v_1)=(r_3r_6-r_4r_5)v_1,\ \rho(x_2,x_3)(v_2)=r_1v_1,\ \rho(x_2,x_3)(v_3)=r_2v_1,$
   - $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=0,$
     $\nu(x_2)(v_1,v_2)=r_3v_1,\ \nu(x_2)(v_1,v_3)=r_4v_1,\ \nu(x_2)(v_2,v_3)=\frac{r_2r_3-r_1r_4}{r_4r_5-r_3r_6}v_1,$
     $\nu(x_3)(v_1,v_2)=r_5v_1,\ \nu(x_3)(v_1,v_3)=r_6v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_2r_5-r_1r_6}{r_4r_5-r_3r_6}v_1,$
   - $\omega(x_1,x_2,x_3)=0,$
2. - $\rho(x_1,x_2)(v_1)=0,\ \rho(x_1,x_2)(v_2)=0,\ \rho(x_1,x_2)(v_3)=0,$
     $\rho(x_1,x_3)(v_1)=0,\ \rho(x_1,x_3)(v_2)=0,\ \rho(x_1,x_3)(v_3)=0,$
     $\rho(x_2,x_3)(v_1)=-r_3r_4v_1,\ \rho(x_2,x_3)(v_2)=r_1v_1,\ \rho(x_2,x_3)(v_3)=r_2v_1,$

- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=0,$
  $\nu(x_2)(v_1,v_2)=0,\ \nu(x_2)(v_1,v_3)=r_3v_1,\ \nu(x_2)(v_2,v_3)=\frac{r_1}{r_4}v_1,$
  $\nu(x_3)(v_1,v_2)=r_4v_1,\ \nu(x_3)(v_1,v_3)=r_5v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_2r_4-r_1r_5}{r_3r_4}v_1,$
- $\omega(x_1,x_2,x_3)=0,$

*3.*
- $\rho(x_1,x_2)(v_1)=0,\ \rho(x_1,x_2)(v_2)=0,\ \rho(x_1,x_2)(v_3)=0,$
  $\rho(x_1,x_3)(v_1)=0,\ \rho(x_1,x_3)(v_2)=0,\ \rho(x_1,x_3)(v_3)=0,$
  $\rho(x_2,x_3)(v_1)=0,\ \rho(x_2,x_3)(v_2)=r_1v_1,\ \rho(x_2,x_3)(v_3)=\frac{r_3r_1}{r_2}v_1,$
- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=0,$
  $\nu(x_2)(v_1,v_2)=r_2v_1,\ \nu(x_2)(v_1,v_3)=r_3v_1,\ \nu(x_2)(v_2,v_3)=r_4v_1,$
  $\nu(x_3)(v_1,v_2)=r_5v_1,\ \nu(x_3)(v_1,v_3)=\frac{r_3r_5}{r_2}v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_4r_5+r_1}{r_2}v_1,$
- $\omega(x_1,x_2,x_3)=0,$

*4.*
- $\rho=0$
- $\nu(x_1)(v_1,v_2)=r_1v_1,\ \nu(x_1)(v_1,v_3)=r_2v_1,\ \nu(x_1)(v_2,v_3)=r_3v_1,$
  $\nu(x_2)(v_1,v_2)=r_4v_1,\ \nu(x_2)(v_1,v_3)=\frac{r_2r_4}{r_1}v_1,\ \nu(x_2)(v_2,v_3)=\frac{r_3r_4}{r_1}v_1,$
  $\nu(x_3)(v_1,v_2)=r_5v_1,\ \nu(x_3)(v_1,v_3)=\frac{r_2r_5}{r_1}v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_3r_5}{r_1}v_1,$
- $\omega(x_1,x_2,x_3)=-r_3v_1+r_2v_2-r_1v_3,$

*5.*
- $\rho(x_1,x_2)(v_1)=0,\ \rho(x_1,x_2)(v_2)=0,\ \rho(x_1,x_2)(v_3)=0,$
  $\rho(x_1,x_3)(v_1)=0,\ \rho(x_1,x_3)(v_2)=0,\ \rho(x_1,x_3)(v_3)=0,$
  $\rho(x_2,x_3)(v_1)=0,\ \rho(x_2,x_3)(v_2)=r_1v_1,\ \rho(x_2,x_3)(v_3)=\frac{r_3r_1}{r_2}v_1,$
- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=0,$
  $\nu(x_2)(v_1,v_2)=0,\ \nu(x_2)(v_1,v_3)=0,\ \nu(x_2)(v_2,v_3)=-\frac{r_1}{r_2}v_1,$
  $\nu(x_3)(v_1,v_2)=r_2v_1,\ \nu(x_3)(v_1,v_3)=r_3v_1,\ \nu(x_3)(v_2,v_3)=r_4v_1,$
- $\omega(x_1,x_2,x_3)=0,$

*6.*
- $\rho(x_1,x_2)(v_1)=0,\ \rho(x_1,x_2)(v_2)=0,\ \rho(x_1,x_2)(v_3)=0,$
  $\rho(x_1,x_3)(v_1)=0,\ \rho(x_1,x_3)(v_2)=0,\ \rho(x_1,x_3)(v_3)=0,$
  $\rho(x_2,x_3)(v_1)=0,\ \rho(x_2,x_3)(v_2)=0,\ \rho(x_2,x_3)(v_3)=r_1v_1,$
- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=0,$
  $\nu(x_2)(v_1,v_2)=0,\ \nu(x_2)(v_1,v_3)=r_2v_1,\ \nu(x_2)(v_2,v_3)=r_3v_1,$
  $\nu(x_3)(v_1,v_2)=0,\ \nu(x_3)(v_1,v_3)=r_4v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_3r_4+r_1}{r_2}v_1,$
- $\omega(x_1,x_2,x_3)=0,$

*7.*
- $\rho=0$
- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=r_1v_1,\ \nu(x_1)(v_2,v_3)=r_2v_1,$
  $\nu(x_2)(v_1,v_2)=0,\ \nu(x_2)(v_1,v_3)=r_3v_1,\ \nu(x_2)(v_2,v_3)=\frac{r_2r_3}{r_1}v_1,$
  $\nu(x_3)(v_1,v_2)=0,\ \nu(x_3)(v_1,v_3)=r_4v_1,\ \nu(x_3)(v_2,v_3)=\frac{r_2r_4}{r_1}v_1,$
- $\omega(x_1,x_2,x_3)=-r_2v_1+r_1v_2,$

*8.*
- $\rho=0$
- $\nu(x_1)(v_1,v_2)=r_1v_1,\ \nu(x_1)(v_1,v_3)=r_2v_1,\ \nu(x_1)(v_2,v_3)=0,$
  $\nu(x_2)(v_1,v_2)=r_3v_1,\ \nu(x_2)(v_1,v_3)=\frac{r_2r_3}{r_1}v_1,\ \nu(x_2)(v_2,v_3)=0,$
  $\nu(x_3)(v_1,v_2)=r_4v_1,\ \nu(x_3)(v_1,v_3)=\frac{r_2r_4}{r_1}v_1,\ \nu(x_3)(v_2,v_3)=0,$

- $\omega(x_1,x_2,x_3)=r_2v_2-r_1v_3$,

*9.*

- $\rho=0$
- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=r_1v_1,$
  $\nu(x_2)(v_1,v_2)=0,\ \nu(x_2)(v_1,v_3)=0,\ \nu(x_2)(v_2,v_3)=r_2v_1,$
  $\nu(x_3)(v_1,v_2)=0,\ \nu(x_3)(v_1,v_3)=0,\ \nu(x_3)(v_2,v_3)=r_3v_1,$
- $\omega(x_1,x_2,x_3)=-r_1v_1$,

*10.*

- $\rho(x_1,x_2)(v_1)=0,\ \rho(x_1,x_2)(v_2)=0,\ \rho(x_1,x_2)(v_3)=0,$
  $\rho(x_1,x_3)(v_1)=0,\ \rho(x_1,x_3)(v_2)=0,\ \rho(x_1,x_3)(v_3)=0,$
  $\rho(x_2,x_3)(v_1)=0,\ \rho(x_2,x_3)(v_2)=0,\ \rho(x_2,x_3)(v_3)=r_1v_1,$
- $\nu(x_1)(v_1,v_2)=0,\ \nu(x_1)(v_1,v_3)=0,\ \nu(x_1)(v_2,v_3)=0,$
  $\nu(x_2)(v_1,v_2)=0,\ \nu(x_2)(v_1,v_3)=0,\ \nu(x_2)(v_2,v_3)=-\frac{r_1}{r_2}v_1,$
  $\nu(x_3)(v_1,v_2)=0,\ \nu(x_3)(v_1,v_3)=r_2v_1,\ \nu(x_3)(v_2,v_3)=r_3v_1,$
- $\omega(x_1,x_2,x_3)=0$,

*where $r_i$ are parameters in $\mathbb{K}$.*

Any non-abelian extension, by choosing a section, is isomorphic to $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(\rho,\nu,\omega)})$. Therefore, we only consider in the sequel non-abelian extensions of the form $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(\rho,\nu,\omega)})$.

**Proposition 3.6.** *Let $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot,\cdot]_{(\rho^1,\nu^1,\omega^1)})$ and $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot,\cdot]_{(\rho^2,\nu^2,\omega^2)})$ be two non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$. Then the two extensions are isomorphic if and only if there is a linear map $\xi:\mathfrak{g}\to\mathfrak{h}$ such that the following equalities holds:*

$$\begin{aligned}
\nu^2(x_1)(v_2,v_3)-\nu^1(x_1)(v_2,v_3) &= -[\xi(x_1),v_2,v_3]_{\mathfrak{h}}, && (29)\\
\rho^2(x_1,x_2)v_3-\rho^1(x_1,x_2)v_3 &= -\nu^1(x_1)(\xi(x_2),v_3)-\nu^1(x_2)(v_3,\xi(x_1))\\
&\quad +[\xi(x_1),\xi(x_2),v_3]_{\mathfrak{h}}, && (30)\\
\omega^2(x_1,x_2,x_3)-\omega^1(x_1,x_2,x_3) &= -\rho^1(x_1,x_2)\xi(x_3)-\rho^1(x_2,x_3)\xi(x_1)\\
&\quad -\rho^1(x_3,x_1)\xi(x_2)+\nu^1(x_1)(\xi(x_2),\xi(x_3))\\
&\quad +\nu^1(x_2)(\xi(x_3),\xi(x_1))+\nu^1(x_3)(\xi(x_1),\xi(x_2))\\
&\quad -[\xi(x_1),\xi(x_2),\xi(x_3)]_{\mathfrak{h}}+\xi[x_1,x_2,x_3]_{\mathfrak{g}}. && (31)
\end{aligned}$$

**Proof.** Let $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot,\cdot]_{(\rho^1,\nu^1,\omega^1)})$ and $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot,\cdot]_{(\rho^2,\nu^2,\omega^2)})$ be two non-abelian extensions of $\mathfrak{g}$ by $\mathfrak{h}$. Assume that the two extensions are isomorphic. Then there is a 3-Lie algebra morphism $\theta:\mathfrak{g}\oplus\mathfrak{h}\to\mathfrak{g}\oplus\mathfrak{h}$, such that we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota} & \mathfrak{g}\oplus\mathfrak{h}_{(\rho^2,\nu^2,\omega^2)} & \xrightarrow{\mathrm{pr}} & \mathfrak{g} & \longrightarrow & 0\\
 & & \Big\| & & \theta\Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & \mathfrak{h} & \xrightarrow{\iota} & \mathfrak{g}\oplus\mathfrak{h}_{(\rho^1,\nu^1,\omega^1)} & \xrightarrow{\mathrm{pr}} & \mathfrak{g} & \longrightarrow & 0,
\end{array}$$

where $\iota$ is the inclusion and pr is the projection. Since for all $x\in\mathfrak{g}$, $\mathrm{pr}(\theta(x))=x$, we can assume that $\theta(x+u)=x-\xi(x)+u$ for some linear map $\xi:\mathfrak{g}\to\mathfrak{h}$. By

$$\theta[x_1,v_2,v_3]_{(\rho^2,\nu^2,\omega^2)}=[\theta(x_1),\theta(v_2),\theta(v_3)]_{(\rho^1,\nu^1,\omega^1)},$$

we can deduce that (29) holds. By

$$\theta[x_1,x_2,v_3]_{(\rho^2,\nu^2,\omega^2)}=[\theta(x_1),\theta(x_2),\theta(v_3)]_{(\rho^1,\nu^1,\omega^1)},$$

we can deduce that (30) holds. By

$$\theta[x_1,x_2,x_3]_{(\rho^2,\nu^2,\omega^2)}=[\theta(x_1),\theta(x_2),\theta(x_3)]_{(\rho^1,\nu^1,\omega^1)},$$

we can deduce that (31) holds. ■

# 4 Non-abelian extensions in terms of Maurer Cartan elements

In [23], the author constructed a graded Lie algebra structure by which one can describe an $n$-Leibniz algebra structure as a canonical structure. Here, we give the precise formulas for the 3-Lie algebra case.

Set $C^p(\mathfrak{g},\mathfrak{g})=\mathrm{Hom}(\wedge^2\mathfrak{g}\otimes\overset{(p)}{\cdots}\otimes\wedge^2\mathfrak{g}\wedge\mathfrak{g},\mathfrak{g})$ and $C^*(\mathfrak{g},\mathfrak{g})=\oplus_p C^p(\mathfrak{g},\mathfrak{g})$. Let $\alpha\in C^p(\mathfrak{g},\mathfrak{g}),\beta\in C^q(\mathfrak{g},\mathfrak{g})$, $p,q\geq 0$. Let $\mathfrak{X}_i=x_i\wedge y_i\in\wedge^2\mathfrak{g}$ for $i=1,2,\cdots,p+q$ and $x_i,y_i\in\mathfrak{g}$. A permutation $\sigma\in\mathbb{S}_n$ is called an $(i,n-i)$-unshuffle if $\sigma(1)<\cdots<\sigma(i)$ and $\sigma(i+1)<\cdots<\sigma(n)$. If $i=0$ or $n$, we assume $\sigma=\mathrm{Id}$. The set of all $(i,n-i)$-unshuffles will be denoted by $unsh(i,n-i)$.

**Theorem 4.1.** ([23]) *The graded vector space $C^*(\mathfrak{g},\mathfrak{g})$ equipped with the graded commutator bracket*

$$[\alpha,\beta]^{\mathrm{3Lie}}=\alpha\circ\beta-(-1)^{pq}\beta\circ\alpha, \tag{32}$$

*is a graded Lie algebra where $\alpha\circ\beta\in C^{p+q}(\mathfrak{g},\mathfrak{g})$ is defined by*

$$\begin{aligned}&\alpha\circ\beta(\mathfrak{X}_1,\cdots,\mathfrak{X}_{p+q},x)=\sum_{k=0}^{p-1}(-1)^{kq}\Big(\sum_{\sigma\in unsh(k,q)}\\&(-1)^{\sigma}\big(\alpha(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(k)},\beta(\mathfrak{X}_{\sigma(k+1)},\cdots,\mathfrak{X}_{\sigma(k+q)},x_{k+q+1})\wedge y_{k+q+1},\mathfrak{X}_{k+q+2},\cdots,\mathfrak{X}_{p+q},x)\\&+\alpha(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(k)},x_{k+q+1}\wedge\beta(\mathfrak{X}_{\sigma(k+1)},\cdots,\mathfrak{X}_{\sigma(k+q)},y_{k+q+1}),\mathfrak{X}_{k+q+2},\cdots,\mathfrak{X}_{p+q},x)\big)\Big)\\&+\sum_{\sigma\in unsh(p,q)}(-1)^{pq}(-1)^{\sigma}\alpha(\mathfrak{X}_{\sigma(1)},\cdots,\mathfrak{X}_{\sigma(p)},\beta(\mathfrak{X}_{\sigma(p)},\cdots,\mathfrak{X}_{\sigma(p+q-1)},\mathfrak{X}_{\sigma(p+q)},x)).\end{aligned}$$

Furthermore, $(C^*(\mathfrak{g},\mathfrak{g}),[\cdot,\cdot]^{\mathrm{3Lie}},\overline{\delta})$ is a DGLA, where $\overline{\delta}$ is given by $\overline{\delta}P=(-1)^p\delta P$ for all $P\in C^p(\mathfrak{g},\mathfrak{g})$, and $\delta$ is the coboundary operator of $\mathfrak{g}$ with coefficients in the adjoint representation. See [16] for more details.

**Remark 4.2.** *The coboundary operator $\delta$ associated to the adjoint representation of the 3-Lie algebra $\mathfrak{g}$ can be written as $\delta P=(-1)^p[\mu_{\mathfrak{g}},P]^{\mathrm{3Lie}}$, for all $P\in C^p(\mathfrak{g},\mathfrak{g})$, where $\mu_{\mathfrak{g}}\in C^1(\mathfrak{g},\mathfrak{g})$ is the 3-Lie algebra structure on $\mathfrak{g}$, i.e. $\mu_{\mathfrak{g}}(x,y,z)=[x,y,z]_{\mathfrak{g}}$. Thus, we have $\overline{\delta}P=[\mu_{\mathfrak{g}},P]^{\mathrm{3Lie}}$.*

Now, we describe non-abelian extensions using Maurer-Cartan elements. The set $MC(L)$ of **Maurer-Cartan elements** of a DGLA $(L,[\cdot,\cdot],\mathrm{d})$ is defined by

$$MC(L)\triangleq\{P\in L_1\mid \mathrm{d}P+\frac{1}{2}[P,P]=0\}.$$

Moreover, $P_0, P_1 \in MC(L)$ are called **gauge equivalent** if and only if there exists an element $\xi \in L_0$ such that

$$P_1 = e^{\mathrm{ad}_\xi} P_0 - \frac{e^{\mathrm{ad}_\xi} - 1}{\mathrm{ad}_\xi} \mathrm{d}\xi. \tag{33}$$

We can define a path between $P_0$ and $P_1$. Let

$$P(t) = e^{t\mathrm{ad}_\xi} P_0 - \frac{e^{t\mathrm{ad}_\xi} - 1}{\mathrm{ad}_\xi} \mathrm{d}\xi.$$

Then $P(t)$ is a power series of $t$ in $MC(L)$. We have $P(0) = P_0$ and $P(1) = P_1$. The set of the gauge equivalence classes of $MC(L)$ is denoted by $\mathcal{MC}(L)$.

Let $(\mathfrak{g}, [\cdot,\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot,\cdot]_{\mathfrak{h}})$ be two 3-Lie algebras. Let $\mathfrak{g} \oplus \mathfrak{h}$ be the 3-Lie algebra direct sum of $\mathfrak{g}$ and $\mathfrak{h}$, where the bracket is defined by

$$[x+u, y+v, z+w] = \mu_{\mathfrak{g}}(x,y,z) + \mu_{\mathfrak{h}}(u,v,w).$$

Then there is a DGLA $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{g}\oplus\mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$. Define $C^k_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) \subset C^k(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})$ by

$$C^k(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) = C^k_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) \oplus C^k(\mathfrak{h}, \mathfrak{h}).$$

Denote by $C_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) = \oplus_k C^k_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})$, which is a graded vector space.

**Lemma 4.3.** *With the above notations, we have $\overline{\delta}(C^k_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})) \subset C^{k+1}_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})$, and $(C_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$ is a sub-DGLA of $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{g}\oplus\mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$. Furthermore, its degree 0 part $C^0_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) = \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ is abelian.*

**Proof.** By the definition of the bracket $[\cdot,\cdot]^{3\mathrm{Lie}}$ and $\overline{\delta}$, we obtain that $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$ is a sub-DGLA of $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{g}\oplus\mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$. For $\alpha \in C^p_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})$, we can regard it as $\alpha \in C^p(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})$ such that $\alpha|_{C^p(\mathfrak{h},\mathfrak{h})} = 0$. Moreover, for $\alpha \in C^p_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), \beta \in C^q_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h})$, we have $[\alpha,\beta]^{3\mathrm{Lie}}|_{C^{p+q}(\mathfrak{h},\mathfrak{h})} = 0$ and $\overline{\delta}(\alpha)|_{C^{p+1}(\mathfrak{h},\mathfrak{h})} = [\mu_{\mathfrak{g}} + \mu_{\mathfrak{h}}, \alpha]^{3\mathrm{Lie}}|_{C^{p+1}(\mathfrak{h},\mathfrak{h})} = 0$. Thus, we obtain $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$ is a sub-DGLA of $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$. Therefore, $(C_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$ is a sub-DGLA of $(C(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{g}\oplus\mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$. Obviously, $C^0_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) = \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ is abelian. ■

**Proposition 4.4.** *The following two statements are equivalent:*

(a) $(\mathfrak{g}\oplus\mathfrak{h}, [\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)})$ *is a 3-Lie algebra, which is a non-abelian extension of $\mathfrak{g}$ by $\mathfrak{h}$;*

(b) $\rho+\nu+\omega$ *is a Maurer-Cartan element of the DGLA $(C_>(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}), [\cdot,\cdot]^{3\mathrm{Lie}}, \overline{\delta})$.*

**Proof.** By Proposition 3.3, $(\mathfrak{g}\oplus\mathfrak{h}, [\cdot,\cdot,\cdot]_{(\rho,\nu,\omega)})$ is a 3-Lie algebra if and only if Eqs. (18)-(28) hold.

If $c = \rho+\nu+\omega$ is a Maurer-Cartan element, we have

$$(\overline{\delta}c + \frac{1}{2}[c,c]^{3\mathrm{Lie}})(e_1\wedge e_2, e_3\wedge e_4, e_5) = 0, \quad \forall e_i = x_i + v_i \in \mathfrak{g}\oplus\mathfrak{h}.$$

By straightforward computations, we have

$$\begin{aligned}
\overline{\delta}c(e_1\wedge e_2, e_3\wedge e_4, e_5) &= -(\delta c)(e_1\wedge e_2, e_3\wedge e_4, e_5)\\
&= c([e_1,e_2,e_3]\wedge e_4, e_5) + c(e_3\wedge[e_1,e_2,e_4], e_5)\\
&\quad + c(e_3\wedge e_4, [e_1,e_2,e_5]) - c(e_1\wedge e_2, [e_3,e_4,e_5])\\
&\quad - [e_1,e_2,c(e_3\wedge e_4,e_5)] + [e_3,e_4,c(e_1\wedge e_2,e_5)]\\
&\quad + [e_4,e_5,c(e_1\wedge e_2,e_3)] + [e_5,e_3,c(e_1\wedge e_2,e_4)].
\end{aligned}$$

Recall the definition of the bracket $[\cdot,\cdot,\cdot]$ and the $c$, we have

$$\begin{aligned}
[e_1,e_2,e_3] &= [x_1,x_2,x_3]_{\mathfrak{g}}+[v_1,v_2,v_3]_{\mathfrak{h}},\\
c(e_1,e_2,e_3) &= \rho(x_1,x_2)v_3+\rho(x_2,x_3)v_1+\rho(x_3,x_1)v_2\\
&\quad +\nu(x_1)(v_2,v_3)+\nu(x_2)(v_3,v_1)+\nu(x_3)(v_1,v_2)+\omega(x_1,x_2,x_3).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
c([e_1,e_2,e_3]\wedge e_4,e_5) &= \rho([x_1,x_2,x_3]_{\mathfrak{g}},x_4)(v_5)+\rho(x_4,x_5)([v_1,v_2,v_3]_{\mathfrak{h}})+\rho(x_5,[x_1,x_2,x_3]_{\mathfrak{g}})(v_4)\\
&\quad +\nu([x_1,x_2,x_3]_{\mathfrak{g}})(v_4,v_5)+\nu(x_4)(v_5,[v_1,v_2,v_3]_{\mathfrak{h}})+\nu(x_5)([v_1,v_2,v_3]_{\mathfrak{h}},v_4)\\
&\quad +\omega([x_1,x_2,x_3]_{\mathfrak{g}},x_4,x_5),\\
c(e_3\wedge[e_1,e_2,e_4],e_5) &= \rho(x_3,[x_1,x_2,x_4]_{\mathfrak{g}})(v_5)+\rho([x_1,x_2,x_4]_{\mathfrak{g}},x_5)(v_3)+\rho(x_5,x_3)([v_1,v_2,v_4]_{\mathfrak{h}})\\
&\quad +\nu(x_3)([v_1,v_2,v_4]_{\mathfrak{h}},v_5)+\nu([x_1,x_2,x_4]_{\mathfrak{g}})(v_5,v_3)+\nu(x_5)(v_3,[v_1,v_2,v_4]_{\mathfrak{h}})\\
&\quad +\omega(x_3,[x_1,x_2,x_4]_{\mathfrak{g}},x_5),\\
c(e_3\wedge e_4,[e_1,e_2,e_5]) &= \rho(x_3,x_4)([v_1,v_2,v_5]_{\mathfrak{h}})+\rho(x_4,[x_1,x_2,x_5]_{\mathfrak{g}})(v_3)+\rho([x_1,x_2,x_5]_{\mathfrak{g}},x_3)(v_4)\\
&\quad +\nu(x_3)(v_4,[v_1,v_2,v_5]_{\mathfrak{h}})+\nu(x_4)([v_1,v_2,v_5]_{\mathfrak{h}},v_3)+\nu([x_1,x_2,x_5]_{\mathfrak{g}})(v_3,v_4)\\
&\quad +\omega(x_3,x_4,[x_1,x_2,x_5]_{\mathfrak{g}}),\\
c(e_1\wedge e_2,[e_3,e_4,e_5]) &= \rho(x_1,x_2)([v_3,v_4,v_5]_{\mathfrak{h}})+\rho(x_2,[x_3,x_4,x_5]_{\mathfrak{g}})(v_1)+\rho([x_3,x_4,x_5]_{\mathfrak{g}},x_1)(v_2)\\
&\quad +\nu(x_1)(v_2,[v_3,v_4,v_5]_{\mathfrak{h}})+\nu(x_2)([v_3,v_4,v_5]_{\mathfrak{h}},v_1)+\nu([x_3,x_4,x_5]_{\mathfrak{g}})(v_1,v_2)\\
&\quad +\omega(x_1,x_2,[x_3,x_4,x_5]_{\mathfrak{g}}),\\
[e_1,e_2,c(e_3\wedge e_4,e_5)] &= [v_1,v_2,\rho(x_3,x_4)(v_5)]_{\mathfrak{h}}+[v_1,v_2,\rho(x_4,x_5)(v_3)]_{\mathfrak{h}}+[v_1,v_2,\rho(x_5,x_3)(v_4)]_{\mathfrak{h}}\\
&\quad +[v_1,v_2,\nu(x_3)(v_4,v_5)]_{\mathfrak{h}}+[v_1,v_2,\nu(x_4)(v_5,v_3)]_{\mathfrak{h}}+[v_1,v_2,\nu(x_5)(v_3,v_4)]_{\mathfrak{h}}\\
&\quad +[v_1,v_2,\omega(x_3,x_4,x_5)]_{\mathfrak{h}},\\
[e_3,e_4,c(e_1\wedge e_2,e_5)] &= [v_3,v_4,\rho(x_1,x_2)(v_5)]_{\mathfrak{h}}+[v_3,v_4,\rho(x_2,x_5)(v_1)]_{\mathfrak{h}}+[v_3,v_4,\rho(x_5,x_1)(v_2)]_{\mathfrak{h}}\\
&\quad +[v_3,v_4,\nu(x_1)(v_2,v_5)]_{\mathfrak{h}}+[v_3,v_4,\nu(x_2)(v_5,v_1)]_{\mathfrak{h}}+[v_3,v_4,\nu(x_5)(v_1,v_2)]_{\mathfrak{h}}\\
&\quad +[v_3,v_4,\omega(x_1,x_2,x_5)]_{\mathfrak{h}},\\
[e_4,e_5,c(e_1\wedge e_2,e_3)] &= [v_4,v_5,\rho(x_1,x_2)(v_3)]_{\mathfrak{h}}+[v_4,v_5,\rho(x_2,x_3)(v_1)]_{\mathfrak{h}}+[v_4,v_5,\rho(x_3,x_1)(v_2)]_{\mathfrak{h}}\\
&\quad +[v_4,v_5,\nu(x_1)(v_2,v_3)]_{\mathfrak{h}}+[v_4,v_5,\nu(x_2)(v_3,v_1)]_{\mathfrak{h}}+[v_4,v_5,\nu(x_3)(v_1,v_2)]_{\mathfrak{h}}\\
&\quad +[v_4,v_5,\omega(x_1,x_2,x_3)]_{\mathfrak{h}},\\
[e_5,e_3,c(e_1\wedge e_2,e_4)] &= [v_5,v_3,\rho(x_1,x_2)(v_4)]_{\mathfrak{h}}+[v_5,v_3,\rho(x_2,x_4)(v_1)]_{\mathfrak{h}}+[v_5,v_3,\rho(x_4,x_1)(v_2)]_{\mathfrak{h}}\\
&\quad +[v_5,v_3,\nu(x_1)(v_2,v_4)]_{\mathfrak{h}}+[v_5,v_3,\nu(x_2)(v_4,v_1)]_{\mathfrak{h}}+[v_5,v_3,\nu(x_4)(v_1,v_2)]_{\mathfrak{h}}\\
&\quad +[v_5,v_3,\omega(x_1,x_2,x_4)]_{\mathfrak{h}}.
\end{aligned}$$

Furthermore, by the definition of the bracket in Theorem 4.1, we have

$$\begin{aligned}
(\frac{1}{2}[c,c]^{\mathrm{3Lie}})(e_1\wedge e_2,e_3\wedge e_4,e_5) &= (c\circ c)(e_1\wedge e_2,e_3\wedge e_4,e_5)\\
&= c(c(e_1\wedge e_2,e_3)\wedge e_4,e_5)+c(e_3\wedge c(e_1\wedge e_2,e_4),e_5)\\
&\quad -c(e_1\wedge e_2,c(e_3\wedge e_4,e_5))+c(e_3\wedge e_4,c(e_1\wedge e_2,e_5)).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
c(c(e_1\wedge e_2,e_3)\wedge e_4,e_5) &= \rho(x_4,x_5)(\omega(x_1,x_2,x_3))+\rho(x_4,x_5)(\rho(x_1,x_2)v_3)+\rho(x_4,x_5)(\rho(x_2,x_3)v_1)\\
&+\rho(x_4,x_5)(\rho(x_3,x_1)v_2)+\rho(x_4,x_5)(\nu(x_1)(v_2,v_3))+\rho(x_4,x_5)(\nu(x_2)(v_3,v_1))\\
&+\rho(x_4,x_5)(\nu(x_3)(v_1,v_2))+\nu(x_4)(v_5,\omega(x_1,x_2,x_3))+\nu(x_4)(v_5,\rho(x_1,x_2)v_3)\\
&+\nu(x_4)(v_5,\rho(x_2,x_3)v_1)+\nu(x_4)(v_5,\rho(x_3,x_1)v_2)+\nu(x_4)(v_5,\nu(x_1)(v_2,v_3))\\
&+\nu(x_4)(v_5,\nu(x_2)(v_3,v_1))+\nu(x_4)(v_5,\nu(x_3)(v_1,v_2))+\nu(x_5)(\omega(x_1,x_2,x_3),v_4)\\
&+\nu(x_5)(\rho(x_1,x_2)v_3,v_4)+\nu(x_5)(\rho(x_2,x_3)v_1,v_4)+\nu(x_5)(\rho(x_3,x_1)v_2,v_4)\\
&+\nu(x_5)(\nu(x_1)(v_2,v_3),v_4)+\nu(x_5)(\nu(x_2)(v_3,v_1),v_4)+\nu(x_5)(\nu(x_3)(v_1,v_2),v_4),\\
c(e_3\wedge c(e_1\wedge e_2,e_4),e_5) &= \rho(x_5,x_3)(\rho(x_1,x_2)v_4)+\rho(x_5,x_3)(\rho(x_2,x_4)v_1)+\rho(x_5,x_3)(\rho(x_4,x_1)v_2)\\
&+\rho(x_5,x_3)(\nu(x_1)(v_2,v_4))+\rho(x_5,x_3)(\nu(x_2)(v_4,v_1))+\rho(x_5,x_3)(\nu(x_4)(v_1,v_2))\\
&+\rho(x_5,x_3)(\omega(x_1,x_2,x_4))+\nu(x_3)(\rho(x_1,x_2)v_4,v_5)+\nu(x_3)(\rho(x_2,x_4)v_1,v_5)\\
&+\nu(x_3)(\rho(x_4,x_1)v_2,v_5)+\nu(x_3)(\nu(x_1)(v_2,v_4),v_5)+\nu(x_3)(\nu(x_2)(v_4,v_1),v_5)\\
&+\nu(x_3)(\nu(x_4)(v_1,v_2),v_5)+\nu(x_3)(\omega(x_1,x_2,x_4),v_5)+\nu(x_5)(v_3,\rho(x_1,x_2)v_4)\\
&+\nu(x_5)(v_3,\rho(x_2,x_4)v_1)+\nu(x_5)(v_3,\rho(x_4,x_1)v_2)+\nu(x_5)(v_3,\nu(x_1)(v_2,v_4))\\
&+\nu(x_5)(v_3,\nu(x_2)(v_4,v_1))+\nu(x_5)(v_3,\nu(x_4)(v_1,v_2))+\nu(x_5)(v_3,\omega(x_1,x_2,x_4)),\\
c(e_1\wedge e_2,c(e_3\wedge e_4,e_5)) &= \rho(x_1,x_2)(\rho(x_3,x_4)v_5)+\rho(x_1,x_2)(\rho(x_4,x_5)v_3)+\rho(x_1,x_2)(\rho(x_5,x_3)v_4)\\
&+\rho(x_1,x_2)(\nu(x_3)(v_4,v_5))+\rho(x_1,x_2)(\nu(x_4)(v_5,v_3))+\rho(x_1,x_2)(\nu(x_5)(v_3,v_4))\\
&+\rho(x_1,x_2)(\omega(x_3,x_4,x_5))+\nu(x_1)(v_2,\rho(x_3,x_4)v_5)+\nu(x_1)(v_2,\rho(x_4,x_5)v_3)\\
&+\nu(x_1)(v_2,\rho(x_5,x_3)v_4)+\nu(x_1)(v_2,\nu(x_3)(v_4,v_5))+\nu(x_1)(v_2,\nu(x_4)(v_5,v_3))\\
&+\nu(x_1)(v_2,\nu(x_5)(v_3,v_4))+\nu(x_1)(v_2,\omega(x_3,x_4,x_5))+\nu(x_2)(\rho(x_3,x_4)v_5,v_1)\\
&+\nu(x_2)(\rho(x_4,x_5)v_3,v_1)+\nu(x_2)(\rho(x_5,x_3)v_4,v_1)+\nu(x_2)(\nu(x_3)(v_4,v_5),v_1)\\
&+\nu(x_2)(\nu(x_4)(v_5,v_3),v_1)+\nu(x_2)(\nu(x_5)(v_3,v_4),v_1)+\nu(x_2)(\omega(x_3,x_4,x_5),v_1),\\
c(e_3\wedge e_4,c(e_1\wedge e_2,e_5)) &= \rho(x_3,x_4)(\rho(x_1,x_2)v_5)+\rho(x_3,x_4)(\rho(x_2,x_5)v_1)+\rho(x_3,x_4)(\rho(x_5,x_1)v_2)\\
&+\rho(x_3,x_4)(\nu(x_1)(v_2,v_5))+\rho(x_3,x_4)(\nu(x_2)(v_5,v_1))+\rho(x_3,x_4)(\nu(x_5)(v_1,v_2))\\
&+\rho(x_3,x_4)(\omega(x_1,x_2,x_5))+\nu(x_3)(v_4,\rho(x_1,x_2)v_5)+\nu(x_3)(v_4,\rho(x_2,x_5)v_1)\\
&+\nu(x_3)(v_4,\rho(x_5,x_1)v_2)+\nu(x_3)(v_4,\nu(x_1)(v_2,v_5))+\nu(x_3)(v_4,\nu(x_2)(v_5,v_1))\\
&+\nu(x_3)(v_4,\nu(x_5)(v_1,v_2))+\nu(x_3)(v_4,\omega(x_1,x_2,x_5))+\nu(x_4)(\rho(x_1,x_2)v_5,v_3)\\
&+\nu(x_4)(\rho(x_2,x_5)v_1,v_3)+\nu(x_4)(\rho(x_5,x_1)v_2,v_3)+\nu(x_4)(\nu(x_1)(v_2,v_5),v_3)\\
&+\nu(x_4)(\nu(x_2)(v_5,v_1),v_3)+\nu(x_4)(\nu(x_5)(v_1,v_2),v_3)+\nu(x_4)(\omega(x_1,x_2,x_5),v_3).
\end{aligned}$$

Thus, $c=\rho+\nu+\omega$ is a Maurer-Cartan element if and only if (18)-(28) hold. ■

**Corollary 4.5.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two 3-Lie algebras. Then there is a one-to-one correspondence between non-abelian extensions of the 3-Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$ and Maurer-Cartan elements in the DGLA $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\cdot,\cdot]^{3\mathrm{Lie}},\overline{\delta})$.*

**Theorem 4.6.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two 3-Lie algebras. Then the isomorphism classes of non-abelian extensions $\mathfrak{g}$ by $\mathfrak{h}$ one-to-one correspond to the gauge equivalence classes of Maurer-Cartan elements in the DGLA $(C_>(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h}),[\cdot,\cdot]^{3\mathrm{Lie}},\overline{\delta})$.*

**Proof.** Two elements $c=\rho+\nu+\omega$ and $c'=\rho'+\nu'+\omega'$ in $MC(L)$ are equivalent if there exists

$\xi \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{h})$ such that

$$c' = e^{\mathrm{ad}_\xi} c - \frac{e^{\mathrm{ad}_\xi} - 1}{\mathrm{ad}_\xi} \overline{\delta}\xi.$$

More precisely, for all $e_i = x_i + v_i \in \mathfrak{g} \oplus \mathfrak{h}$, we have

$$\begin{aligned} c'(e_1 \wedge e_2, e_3) &= \big((\mathrm{Id} + \mathrm{ad}_\xi + \frac{1}{2!}\mathrm{ad}_\xi^2 + \frac{1}{3!}\mathrm{ad}_\xi^3 + \cdots + \frac{1}{n!}\mathrm{ad}_\xi^n + \cdots)c\big)(e_1 \wedge e_2, e_3) \\ &\quad -\big((\mathrm{Id} + \frac{1}{2!}\mathrm{ad}_\xi + \frac{1}{3!}\mathrm{ad}_\xi^2 + \cdots + \frac{1}{n!}\mathrm{ad}_\xi^{n-1} + \cdots)\overline{\delta}\xi\big)(e_1 \wedge e_2, e_3). \end{aligned}$$

Furthermore, by the bracket in Theorem 4.1, we have

$$\begin{aligned} [\xi, c]^{3\mathrm{Lie}}(e_1 \wedge e_2, e_3) &= (\xi \circ c)(e_1 \wedge e_2, e_3) - (c \circ \xi)(e_1 \wedge e_2, e_3) \\ &= -\big(c(\xi(e_1) \wedge e_2, e_3) + c(e_1 \wedge \xi(e_2), e_3) + c(e_1 \wedge e_2, \xi(e_3))\big) \\ &= -c(\xi(x_1) \wedge e_2, e_3) - c(e_1 \wedge \xi(x_2), e_3) - c(e_1 \wedge e_2, \xi(x_3)) \\ &= -\rho(x_2, x_3)\xi(x_1) - \nu(x_2)(v_3, \xi(x_1)) - \nu(x_3)(\xi(x_1), v_2) \\ &\quad -\rho(x_3, x_1)\xi(x_2) - \nu(x_1)(\xi(x_2), v_3) - \nu(x_3)(v_1, \xi(x_2)) \\ &\quad -\rho(x_1, x_2)\xi(x_3) - \nu(x_1)(v_2, \xi(x_3)) - \nu(x_2)(\xi(x_3), v_1). \end{aligned}$$

Thus, we have

$$\begin{aligned} & [\xi, [\xi, c]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}(e_1 \wedge e_2, e_3) \\ = & -[\xi, c]^{3\mathrm{Lie}}(\xi(x_1) \wedge e_2, e_3) - [\xi, c]^{3\mathrm{Lie}}(e_1 \wedge \xi(x_2), e_3) - [\xi, c]^{3\mathrm{Lie}}(e_1 \wedge e_2, \xi(x_3)) \\ = & \; 2\nu(x_1)(\xi(x_2), \xi(x_3)) + 2\nu(x_2)(\xi(x_3), \xi(x_1)) + 2\nu(x_3)(\xi(x_1), \xi(x_2)). \end{aligned}$$

Moreover, we have

$$[\xi, [\xi, [\xi, c]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}(e_1 \wedge e_2, e_3) = 0.$$

More generally, for $n \geq 3$

$$\mathrm{ad}_\xi^n c = 0.$$

For all $e_i = x_i + v_i \in \mathfrak{g} \oplus \mathfrak{h}$, we have

$$\begin{aligned} \overline{\delta}\xi(e_1 \wedge e_2, e_3) &= [\mu_\mathfrak{g} + \mu_\mathfrak{h}, \xi]^{3\mathrm{Lie}}(e_1 \wedge e_2, e_3) \\ &= ((\mu_\mathfrak{g} + \mu_\mathfrak{h}) \circ \xi)(e_1 \wedge e_2, e_3) - (\xi \circ (\mu_\mathfrak{g} + \mu_\mathfrak{h}))(e_1 \wedge e_2, e_3) \\ &= (\mu_\mathfrak{g} + \mu_\mathfrak{h})(\xi(e_1) \wedge e_2, e_3) + (\mu_\mathfrak{g} + \mu_\mathfrak{h})(e_1 \wedge \xi(e_2), e_3) \\ &\quad +(\mu_\mathfrak{g} + \mu_\mathfrak{h})(e_1 \wedge e_2, \xi(e_3)) - \xi((\mu_\mathfrak{g} + \mu_\mathfrak{h})(e_1 \wedge e_2, e_3)) \\ &= [\xi(x_1), v_2, v_3]_\mathfrak{h} + [v_1, \xi(x_2), v_3]_\mathfrak{h} + [v_1, v_2, \xi(x_3)]_\mathfrak{h} - \xi[x_1, x_2, x_3]_\mathfrak{g}. \end{aligned}$$

Thus, we have

$$\begin{aligned} [\xi, \overline{\delta}\xi]^{3\mathrm{Lie}}(e_1 \wedge e_2, e_3) &= -\overline{\delta}\xi(\xi(x_1) \wedge e_2, e_3) - \overline{\delta}\xi(e_1 \wedge \xi(x_2), e_3) - \overline{\delta}\xi(e_1 \wedge e_2, \xi(x_3)) \\ &= -2[\xi(x_1), \xi(x_2), v_3]_\mathfrak{h} - 2[v_1, \xi(x_2), \xi(x_3)]_\mathfrak{h} - 2[\xi(x_1), v_2, \xi(x_3)]_\mathfrak{h}, \end{aligned}$$

and

$$\begin{aligned} & [\xi, [\xi, \overline{\delta}\xi]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}(e_1 \wedge e_2, e_3) \\ = & -[\xi, \overline{\delta}\xi]^{3\mathrm{Lie}}(\xi(x_1) \wedge e_2, e_3) - [\xi, \overline{\delta}\xi]^{3\mathrm{Lie}}(e_1 \wedge \xi(x_2), e_3) - [\xi, \overline{\delta}\xi]^{3\mathrm{Lie}}(e_1 \wedge e_2, \xi(x_3)) \\ = & \; 6[\xi(x_1), \xi(x_2), \xi(x_3)]_\mathfrak{h}. \end{aligned}$$

Moreover, we have

$$[\xi,[\xi,[\xi,\overline{\delta}\xi]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}(e_1\wedge e_2,e_3)=0.$$

More generally, for $n\geq 3$

$$\mathrm{ad}_\xi^n\overline{\delta}\xi=0.$$

Therefore,we have

$$c'=(c+[\xi,c]^{3\mathrm{Lie}}+\frac{1}{2!}[\xi,[\xi,c]^{3\mathrm{Lie}}]^{3\mathrm{Lie}})-(\overline{\delta}\xi+\frac{1}{2!}[\xi,\overline{\delta}\xi]^{3\mathrm{Lie}}+\frac{1}{3!}[\xi,[\xi,\overline{\delta}\xi]^{3\mathrm{Lie}}]^{3\mathrm{Lie}}).$$

Thus, two elements $c=\rho+\nu+\omega$ and $c'=\rho'+\nu'+\omega'$ in $MC(L)$ are equivalent if and only (29)-(31) hold. ■

# 5 Non-abelian extensions of Leibniz algebras

In this section, we always assume that $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(\rho,\nu,\omega)})$ is a non-abelian extension of the 3-Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$. We aim to analyze the corresponding Leibniz algebra structure on the space of fundamental objects. Note that $\wedge^2(\mathfrak{g}\oplus\mathfrak{h})\cong((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}))\oplus(\wedge^2\mathfrak{g})$ naturally. We use $[\cdot,\cdot]_{\hat{\mathrm{F}}}$ to denote the Leibniz bracket on the space of fundamental objects of the 3-Lie algebra $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(\rho,\nu,\omega)})$.

First we introduce a Leibniz algebra structure on $(\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h})$. Define a linear map $\{\cdot,\cdot\}:((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}))\otimes((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}))\ \to\ (\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h})$ by

$$\begin{aligned}&\{u_1\wedge v_1+x_1\otimes w_1,u_2\wedge v_2+x_2\otimes w_2\}\\&=[u_1,v_1,u_2]_{\mathfrak{h}}\wedge v_2+u_2\wedge[u_1,v_1,v_2]_{\mathfrak{h}}+\nu(x_2)(u_1,v_1)\wedge w_2+x_2\otimes[u_1,v_1,w_2]_{\mathfrak{h}}\\&+\nu(x_1)(w_1,u_2)\wedge v_2+u_2\wedge\nu(x_1)(w_1,v_2)-\rho(x_1,x_2)(w_1)\wedge w_2+x_2\otimes\nu(x_1)(w_1,w_2).\end{aligned}\quad(34)$$

**Proposition 5.1.** *With the above notations, $((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}),\{\cdot,\cdot\})$ is a Leibniz algebra.*

**Proof.** By direct computation, we have

$$\{u_1\wedge v_1+x_1\otimes w_1,u_2\wedge v_2+x_2\otimes w_2\}=[u_1\wedge v_1+x_1\otimes w_1,u_2\wedge v_2+x_2\otimes w_2]_{\hat{\mathrm{F}}}.$$

Thus, $((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}),\{\cdot,\cdot\})$ is a Leibniz subalgebra of the Leibniz algebra $(\wedge^2(\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\hat{\mathrm{F}}})$. ■

We define $\varpi:(\wedge^2\mathfrak{g})\otimes(\wedge^2\mathfrak{g})\ \to\ (\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h})$, $l:(\wedge^2\mathfrak{g})\ \to\ \mathfrak{gl}((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}))$, and $r:(\wedge^2\mathfrak{g})\ \to\ \mathfrak{gl}((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}))$ respectively by

$$\begin{aligned}\varpi(x\wedge y,z\wedge t)&=-t\otimes\omega(x,y,z)+z\otimes\omega(x,y,t),&(35)\\ l(x\wedge y)(u\wedge v+z\wedge w)&=\rho(x,y)(u)\wedge v+u\wedge\rho(x,y)(v)+[x,y,z]_{\mathfrak{g}}\otimes w\\&\quad+\omega(x,y,z)\wedge w+z\otimes\rho(x,y)(w),&(36)\\ r(x\wedge y)(u\wedge v+z\wedge w)&=-y\otimes\nu(x)(u,v)+x\otimes\nu(y)(u,v)\\&\quad y\otimes\rho(z,x)(w)-x\otimes\rho(z,y)(w),&(37)\end{aligned}$$

for all $x,y,z,t\in\mathfrak{g},\ u,v,w\in\mathfrak{h}$.

Now we are ready to give the main result of this section.

**Theorem 5.2.** *Let $(\mathfrak{g},[\cdot,\cdot,\cdot]_{\mathfrak{g}})$ and $(\mathfrak{h},[\cdot,\cdot,\cdot]_{\mathfrak{h}})$ be two 3-Lie algebras and $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]_{(\rho,\nu,\omega)})$ a non-abelian extension of the 3-Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$. Then the Leibniz algebra $(\wedge^2(\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\hat{\mathrm{F}}})$ is a non-abelian extension of the Leibniz algebra $(\wedge^2\mathfrak{g},[\cdot,\cdot]_{\mathrm{F}})$ by the Leibniz algebra $((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}),\{\cdot,\cdot\})$.*

**Proof.** One can show that conditions (6)-(12) in Proposition 2.2 hold directly. Thus, $(\wedge^2(\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\hat{\mathrm{F}}})$ is a non-abelian extension of the Leibniz algebra $(\wedge^2\mathfrak{g},[\cdot,\cdot]_{\mathrm{F}})$ by the Leibniz algebra $((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}),\{\cdot,\cdot\})$. Here we use a different approach to prove this theorem. Using the isomorphism between $\wedge^2(\mathfrak{g}\oplus\mathfrak{h})$ and $((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}))\oplus(\wedge^2\mathfrak{g})$, the Leibniz algebra structure on $\wedge^2(\mathfrak{g}\oplus\mathfrak{h})$ is given by

$$\begin{aligned}
&[u_1\wedge v_1+x_1\otimes w_1+y_1\wedge z_1,u_2\wedge v_2+x_2\otimes w_2+y_2\wedge z_2]_{\hat{\mathrm{F}}}\\
&=[u_1\wedge v_1+x_1\otimes w_1,u_2\wedge v_2+x_2\otimes w_2]_{\hat{\mathrm{F}}}+[y_1\wedge z_1,u_2\wedge v_2+x_2\otimes w_2]_{\hat{\mathrm{F}}}\\
&+[u_1\wedge v_1+x_1\otimes w_1,y_2\wedge z_2]_{\hat{\mathrm{F}}}+[y_1\wedge z_1,y_2\wedge z_2]_{\hat{\mathrm{F}}}\\
&=\{u_1\wedge v_1+x_1\otimes w_1,u_2\wedge v_2+x_2\otimes w_2\}+l(y_1\wedge z_1)(u_2\wedge v_2+x_2\otimes w_2)\\
&+r(y_2\wedge z_2)(u_1\wedge v_1+x_1\otimes w_1)+\varpi(y_1\wedge z_1,y_2\wedge z_2)+[y_1\wedge z_1,y_2\wedge z_2]_{\mathrm{F}}.
\end{aligned}$$

Thus, by (5), we deduce that $(\wedge^2(\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{\hat{\mathrm{F}}})$ is a non-abelian extension of the Leibniz algebra $(\wedge^2\mathfrak{g},[\cdot,\cdot]_{\mathrm{F}})$ by the Leibniz algebra $((\wedge^2\mathfrak{h})\oplus(\mathfrak{g}\otimes\mathfrak{h}),\{\cdot,\cdot\})$. ■

# References


[1] J. Bagger and N. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes gauge theories. *Phys. Rev. D* **77** (2008), 065008.

[2] J. Bagger and N. Lambert, Three-algebras and N=6 Chern-Simons gauge theories. *Phys. Rev. D* **79** (2009), no. 2, 025002, 8 pp.

[3] R. Bai, C. Bai and J. Wang, Realizations of 3-Lie algebras. *J. Math. Phys.* 51 (2010), 063505.

[4] R. Bai, G. Song and Y. Zhang, On classification of $n$-Lie algebras. *Front. Math. China* **6** (2011), no. 4, 581–606.

[5] A. Basu and J. A. Harvey, The M2-M5 brane system and a generalized Nahm's equation. *Nucl. Phys. B 713*, 136 (2005), 136–150.

[6] Y. Daletskii and L. Takhtajan, Leibniz and Lie algebra structures for Nambu algebra. *Lett. Math. Phys.* **39** (1997), 127–141.

[7] J. A. de Azcárraga and J. M. Izquierdo, $n$-ary algebras: a review with applications, *J. Phys. A: Math. Theor.* **43** (2010), 293001.

[8] J. A. de Azcárraga and J. M. Izquierdo, Cohomology of Filippov algebras and an analogue of Whitehead's lemma. *J. Phys. Conf. Ser.* **175**: 012001, (2009).

[9] J. Figueroa-O′Farrill, Deformations of 3-algebras. *J. Math. Phys.* **50** (2009), no. 11, 113514, 27 pp.

[10] V. T. Filippov, $n$-Lie algebras. *Sib. Mat. Zh.* **26** (1985) 126–140.

[11] Y. Frégier, Non-abelian cohomology of extensions of Lie algebras as Deligne groupoid. *J. Algebra* **398** (2014) 243–257.

[12] P. Gautheron, Some remarks concerning Nambu mechanics. *Lett. Math. Phys.* **37** (1996) 103–116.

[13] J. Gomis, D. Rodríguez-Gómez, M. Van Raamsdonk and H. Verlinde, Supersymmetric Yang-Mills theory from Lorentzian three-algebras. *J. High Energy Phys.* no. 8, 094, 18 pp (2008).

[14] P. Ho, R. Hou and Y. Matsuo, Lie 3-algebra and multiple $M_2$-branes. *J. High Energy Phys.* no. 6, 020, 30 pp (2008).

[15] Sh. M. Kasymov, On a theory of $n$-Lie algebras. (Russian) *Algebra i Logika* **26**, no. 3 (1987) 277–297.

[16] J. Liu, A. Makhlouf and Y. Sheng, A new approach to representations of 3-Lie algebras and abelian extensions. *Algebr Represent Theor.* (2017), doi:10.1007/s10468-017-9693-0.

[17] J. Liu, Y. Sheng and Q. Wang, On non-abelian extensions of Leibniz algebras. *Comm. Algebra.* (2018), DOI:10.1080/00927872.2017.1324870.

[18] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Enseign. Math.* (2), 39 (1993), 269-293.

[19] J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.* 296 (1993), 139-158.

[20] A. Makhlouf, On Deformations of $n$-Lie Algebras, Chapter 4 in Non Associative & Non Commutative Algebra and Operator Theory, C.T. Gueye, M.S. Molina (eds.), Springer Proceedings in Mathematics & Statistics **160**, (2016).

[21] Y. Nambu, Generalized Hamiltonian dynamics. *Phys. Rev. D* **7** (1973) 2405–2412.

[22] G. Papadopoulos, M2-branes, 3-Lie algebras and Plucker relations. *J. High Energy Phys.* (2008), no. 5, 054, 9 pp.

[23] M. Rotkiewicz, Cohomology ring of $n$-Lie algebras. *Extracta Math.* **20** no. 3, (2005) 219–232.

[24] L. Takhtajan, On foundation of the generalized Nambu mechanics. *Comm. Math. Phys.* **160** (1994) 295–315.

[25] L. Takhtajan, A higher order analog of Chevalley-Eilenberg complex and deformation theory of $n$-algebras. *St. Petersburg Math. J.* **6** (1995) 429–438.