A systematic approach to tensor hierarchies

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Abstract

In the present paper, we dwelve into the relationship between tensor hierarchy structures in supergravity theories, and Leibniz algebras. The tensor hierarchies naturally emerge from gauging procedures in supergravity models. Given a set of 1-form fields taking values in a representation V of the symmetry Lie algebra $\mathfrak{g}$ of the model, these techniques provide a tower of p-form fields that induce covariance of the p-form field strengths. The goal of this paper is two-fold: we first show that the choice of an embedding tensor $\Theta : V \to \mathfrak{g}$ induces a Leibniz algebra structure on V. Leibniz algebras are non-skew-symmetric generalizations of Lie algebras. In a Leibniz algebra, the product may admit a non-vanishing symmetric part, so that the usual Jacobi identity may not be satisfied anymore, to the benefits of the Leibniz identity. Then we show how a given Leibniz algebra V gives rise to a family of embedding tensors taking values into sub-Lie algebras of V, showing that the two notions are responding to one another. Finally we propose a systematic construction of the tensor hierarchy algebra that is dependent on the fewest data possible: a Lie algebra $\mathfrak{g}$, a representation V, an embedding tensor Θ and a choice of a lift of the symmetric bracket on V. The construction provides a classification of tensor hierarchies in terms of the lifts of the symmetric bracket.

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1 Introduction

Tensor hierarchies form a class of objects found in supergravity theories, which emerge as compactifications of superstring theories [1, 2, 12]. These models have the particularity of being *ungauged*, i.e. the 1-form fields are not even minimally coupled to any other fields. In the late nineties, a vast amount of new compactifications techniques was discovered, and this let to *gauged* supergravities. One passes from ungauged to gauged models by promoting a suitable sub-Lie algebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of global symmetries to generate the local

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symmetries of the theories. The choice of such a sub-Lie algebra is made through a $\mathfrak{g}$-covariant linear map Θ, called the *embedding tensor*, that relates the space of 1-form fields to the Lie algebra of symmetries $\mathfrak{g}$. The existence of such a map Θ is conditioned by a linear and a quadratic constraint. If they are satisfied, this map uniquely defines the sub-Lie algebra $\mathfrak{h}$.

As in classical gauge theories, the dynamics of the gauge fields A^a is controlled by a corresponding set of field strengths $\mathcal{F}^a$. However, contrary to the classical case, in supergravity models these field strengths are not necessarily covariant. To get rid of this issue, a set of 2-form fields B^I taking value in a representation of $\mathfrak{g}$ are added to the field strength $\mathcal{F}^a$ through a Stuckelberg-like coupling, and their gauge transformation is defined so that the new field strengths happen to be covariant. However, the addition of these new fields B^I necessarily implies to add their corresponding field strengths, i.e. 3-forms $\mathcal{H}^I$. However, they turn not to be covariant either. One then adds 3-form fields to the model so that the field strengths $\mathcal{H}^I$ become covariant. The procedure continues and p-form fields are added to the theory until the dimension of space-time is reached. The set of all these fields form the what is known as a *tensor hierarchy*. If not for the dimension of space-time, in full generality, nothing prevents this tower of fields to be infinite.

Recent developments towards the direction of giving a mathematical framework for this construction has been attempted [5, 6, 8, 9, 11]. In particular, [5] provides a recipe to obtain all p-form fields and their corresponding field strengths, i.e. the content of the hierarchy, given only the family of representations into which the fields take values, together with a differential graded Lie algebra structure on this family of representations. The author in [5] called such an object a *tensor hierarchy algebra*, and we will follow his insights in the present paper. The construction of the tensor hierarchy in [5] is explicited in [9]. We propose in this paper to provide an alternative construction for such a tower of representations, and how it can be equipped with the specific differential graded Lie algebra structure that matches the definition given in [5]. We obtain this result by following rather natural hypothesis, that physicists are often implying in the text without necessarily emphasizing them. The construction crucially relies on the observation that an embedding tensor $\Theta : V \to \mathfrak{g}$ induces a *Leibniz algebra structure* on the representation V, and reciprocally, any Leibniz algebra V gives rise to an embedding tensor taking values in a sub-Lie algebra of V. More precisely we show that tensor hierarchies are actually built from the data of a Leibniz algebra V, the Lie algebra $\mathfrak{g}$, and the embedding tensor $\Theta : V \to \mathfrak{g}$. We call such triple of objects satisfying additional conditions a *Lie-Leibniz triple*.

The original motivation of this paper was actually to provide a systematic construction of the tensor hierarchies from a Leibniz algebras perspective. A Leibniz algebra is a generalization of a Lie algebra, where the product is not necessarily skew-symmetric, and where the Jacobi identity is modified in consequence. The product of a Leibniz algebra V can be split into its symmetric and its skew-symmetric part. The skew-symmetric bracket usually does not satisfy the Jacobi identity either. One can show however that this bracket could be extended to an L_∞*-algebra structure* on some graded vector space that has to be adequately chosen. A unexpected consequence obtained in the course of studying tensor hierarchies is that any tensor hierarchy algebra induces an L_∞-algebra that lifts the skew-symmetric bracket of V. This is actually a simple consequence of [3, 4]. The problem of finding such L_∞*-extensions* of Leibniz algebras will be the object of another paper.

In the present paper, the first part presents the mathematical tools that are used in the second section. Section 2.1 presents the embedding tensor as thought by the physicists, whereas section 2.2 provides the basic notions on Leibniz algebras, and their relationship with the embedding tensor, which leads to the crucial notion of Lie-Leibniz triple. Then we discuss the importance of the symmetric part of the Leibniz product in section 2.3 and we conclude this part by elementary notions on graded geometry, see section 2.4. Then, these mathematical tools are used through the entire second part, through some degree-juggling sessions. Section 3.1 is the core of this paper as it contains Lemma 3.2 that enables to build the *skeleton* of the tensor hierarchy of a Leibniz algebra. Section 3.2 is devoted to the explanation on how to get the tensor hierarchy algebra from any skeleton, and contains the main result of this paper (Theorem 3.6): that any Lie-Leibniz triple $(V, \mathfrak{g}, \Theta)$ induces a family of tensor hierarchies algebra, and we show that these tensor hierarchies are in one-to-one correspondence with the lifts of the symmetric bracket of V.

This paper can be seen as a completion of the results presented in [5,9], where the tensor hierarchy algebra was introduced. We think that the present paper provides an alternative and systematic recipe of building the tensor hierarchy algebra, and as such, it could help physicist to answer questions that are raised in supergravity theories.

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2 Mathematical background

2.1 The embedding tensor

In supergravity theories admitting a Lie algebra of symmetries $\mathfrak{g}$ and a representation V in which 1-form fields take values, a gauging procedure relies on promoting a sub-Lie algebra $\mathfrak{h}$ to the status of gauge algebra. The idea for gauging is to assume that there exists a linear mapping $\Theta : V \to \mathfrak{g}$ such that $\mathfrak{h} \equiv \mathrm{Im}(\Theta)$ is a sub-Lie algebra of $\mathfrak{g}$. Physicists call this map the *embedding tensor*, and the sub-Lie algebra $\mathfrak{h}$ the *gauge algebra*.

The existence of Θ relies on some consistency conditions. For covariance reasons, physicists require the embedding tensor to be an element of a $\mathfrak{g}$-sub-module of $\mathrm{Hom}(V,\mathfrak{g}) \simeq V^*\otimes\mathfrak{g}$. Hence, a careful analysis of the decomposition of $V^*\otimes\mathfrak{g}$ into irreducible representations of $\mathfrak{g}$ needs to be performed. The choice of the correct representation R under which Θ transforms depends on a condition that is called the *linear (or representation) contraint*. This constraint is not well understood today, and needs clarification. In particular it fixes both the representation in which Θ transforms, and the first space W of the tensor hierarchy, in which the two-form fields take values. In the course of this article, we show that the tensor hierarchies associated to the same triple $(V,\mathfrak{g},\Theta)$ are in one-to-one correspondence with the choices of W, hence it is fully adapted to the construction that physicists develop. Since the present paper allows for a decorrelated choice of Θ and W, we will not use the linear constraint, until further investigations.

The condition that $\mathfrak{h} \equiv \mathrm{Im}(\Theta)$ is a sub-Lie algebra of $\mathfrak{g}$ is equivalent to saying that the image of Θ in $\mathfrak{g}$ is closed under the Lie bracket. Given the choice of the sub-module $T_\Theta \subset V^*\otimes\mathfrak{g}$ to which Θ belongs, we denote by $\widehat{\rho} : T_\Theta \to \mathfrak{g}^*\otimes T_\Theta$ the linear mapping that encodes the action of $\mathfrak{g}$ on T_Θ:

$$\begin{aligned}\widehat{\rho}:\ T_\Theta &\longrightarrow \mathfrak{g}^*\otimes T_\Theta\\ \Xi &\longmapsto \widehat{\rho}(\Xi) : a \mapsto \widehat{\rho}_a(\Xi)\end{aligned}$$

In particular, $\widehat{\rho}_a$ is an endomorphism of T_Θ. As an immediate consequence of the definition of the action of $\mathfrak{g}$ on the tensor product $V^*\otimes\mathfrak{g}$, we have:

$$\widehat{\rho}_a(\Theta)(x) = \big[a,\Theta(x)\big] - \Theta\big(\rho_a(x)\big) \tag{2.1}$$

for any $a\in\mathfrak{g}$ and $x\in V$. Physicists ensure that $\mathrm{Im}(\Theta)$ is stable under the Lie bracket by positing that:

$$\widehat{\rho}_a(\Theta) = 0 \qquad \text{for any } a\in\mathrm{Im}(\Theta) \tag{2.2}$$

Indeed, restricting Equation (2.1) to $\mathrm{Im}(\Theta)$, and taking into account Equation (2.2), we obtain that:

$$\Theta\big(\rho_{\Theta(x)}(y)\big) = \big[\Theta(x),\Theta(y)\big] \tag{2.3}$$

for any $x, y \in V$. This implies that $\mathrm{Im}(\Theta)$ is a sub-Lie algebra of $\mathfrak{g}$ that we denote by $\mathfrak{h}$ and that physicists call the *gauge algebra*. The name is justified by the fact that they say that an embedding tensor satisfying Equation (2.2) is gauge invariant. The property that $\mathrm{Im}(\Theta)$ is a sub-Lie algebra of $\mathfrak{g}$, or equivalently that the embedding tensor is gauge invariant, is a condition that physicists require for the consistency of gauging procedures in supergravity theories.

In addition, given that V inherits a $\mathfrak{h}$-module induced by its $\mathfrak{g}$-module structure, Equation (2.3) shows that the embedding tensor Θ is $\mathfrak{h}$-equivariant, with respect to the induced action on V and to the adjoint action of $\mathfrak{h}$ on itself. In supergravity theories, the gauge invariance condition (2.2) is often written under the form of the equivariance condition (2.3) (or in a more compact form in Equation (2.5)), and is called the *quadratic (or closure) constraint*.

The action of $\mathfrak{h}$ on V induces an action of V on V itself by the following formula:

$$x \bullet y \equiv \rho_{\Theta(x)}(y) \tag{2.4}$$

This action may not be symmetric nor skew-symmetric. By the equivariance condition (2.3), we deduce that Θ intertwines the product on V and the Lie bracket on $\mathfrak{h}$:

$$\Theta(x \bullet y) = \big[\Theta(x), \Theta(y)\big] \tag{2.5}$$

This is the most compact form of the quadratic constraint found in supergravity theories. As a side remark, this equation implies that any element of the (symmetric) form $x \bullet y + y \bullet x$ lies in the kernel of the embedding tensor Θ. The ideal (with respect to the product $\bullet$) generated by such elements is called the *ideal of squares*. Hence, the kernel of the embedding tensor contains the ideal of squares, and by definition of the product in Equation (2.4), it is necessarily contained in the *centre of* V: the ideal generated by elements of V whose action on V is trivial. From Equations (2.4) and (2.5), and from the fact that V is a representation of the Lie algebra $\mathfrak{h}$, we deduce the following identity:

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z + y \bullet (x \bullet z) \tag{2.6}$$

In other words, the product $\bullet$ is a derivation of itself. This property is called the *Leibniz identity* and it turns the space $(V, \bullet)$ into what is called a *Leibniz algebra* (cf. Definition 2.1).

To summarize, having a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, the gauging procedure in supergravity theories is basically:

1. to define an embedding tensor $\Theta : V \to \mathfrak{g}$ by the linear constraint,
2. to ensure that its image is stable under Lie bracket (or 'gauge invariant') by the quadratic constraint.

The vector space V can then be equipped with a Leibniz algebra structure, which encodes the gauge transformations of the one-forms in supergravity theories. To make the associated field strengths covariant under the action of V, physicists add a set of two-form fields that take value in a $\mathfrak{g}$-module $W \subset S^2(V)$. This space is chosen such that the symmetric part of the Leibniz bracket $\{.,.\} : S^2(V) \to V$ factors through W. Following further considerations, they arrive at building a whole tower of $\mathfrak{g}$-modules in which higher fields take values. This is what physicists call the *tensor hierarchy* of the model, see details in section 2.4 and [1,5,12].

2.2 Leibniz algebras

In the former section, we have shown that given a Lie algebra $\mathfrak{g}$ and a representation V equipped with an embedding tensor $\Theta : V \to \mathfrak{g}$ defining a sub-Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, we can induce a product in V satisfying the Leibniz identity (2.6). We now turn to the reverse problem: given a vector space V equipped with such a product (what is commonly called a *Leibniz algebra*), can we define a Lie algebra $\mathfrak{g}$ through some embedding tensor such that V is a $\mathfrak{g}$-module ? The answer is positive, and this section is devoted to presenting the notion of Leibniz algebras, and to explaining how it is related to the objects presented in section 2.1. Then, using the relationship between embedding tensors and Leibniz algebras, we will be able to show that Leibniz algebras give rise to a formalism that is at the core of the construction of the tensor hierarchies.

At first, Leibniz algebras have been introduced by Jean Louis Loday in [7] as a non commutative generalization of Lie algebras. In a Lie algebra, the Jacobi identity is equivalent to saying that the adjoint action is a derivation of the bracket. In a Leibniz algebra, we preserve this derivation property but we do not require the bracket to be skew symmetric anymore. More precisely:

Definition 2.1. *A (left)*[1] Leibniz algebra *is a finite dimensional vector space V equipped with a bilinear operation $\bullet$ satisfying the derivation property, or* Leibniz identity*:*

$$x\bullet(y\bullet z)=(x\bullet y)\bullet z+y\bullet(x\bullet z) \tag{2.8}$$

for all $x,y,z\in V$.

Example 1. A Lie algebra $(\mathfrak{g},[\,.\,,.\,])$ is a Leibniz algebra, with product $\bullet=[\,.\,,.\,]$. The Leibniz identity is nothing more than the Jacobi identity on $\mathfrak{g}$. Conversely a Leibniz algebra $(V,\bullet)$ is a Lie algebra when the product does not carry a symmetric part, that is: $x\bullet x=0$ for every $x\in V$. We conclude that the Leibniz identity (2.8) is a possible generalization of the Jacobi identity to non skew-symmetric brackets.

We can split the Leibniz product into its symmetric part $\{\,.\,,.\,\}$ and its skew-symmetric part $[\,.\,,.\,]$:

$$x\bullet y=[x,y]+\{x,y\} \tag{2.9}$$

where

$$[x,y]=\frac{1}{2}\big(x\bullet y-y\bullet x\big) \tag{2.10}$$

$$\{x,y\}=\frac{1}{2}\big(x\bullet y+y\bullet x\big) \tag{2.11}$$

for any $x,y\in V$. As a consequence, the Leibniz product is a derivation of both brackets. An important remark here is that even if the bracket $[\,.\,,.\,]$ is skew-symmetric, it does not satisfy the Jacobi identity, hence it is not a Lie bracket. Rather, using Equation (2.8):

$$\big[x,[y,z]\big]+\big[y,[z,x]\big]+\big[z,[x,y]\big]=\mathrm{Jac}(x,y,z) \tag{2.12}$$

where the Jacobiator is defined by:

$$\mathrm{Jac}(x,y,z)=-\frac{1}{3}\Big(\big\{x,[y,z]\big\}+\big\{y,[z,x]\big\}+\big\{z,[x,y]\big\}\Big) \tag{2.13}$$

for every $x,y,z\in V$.

Remark. Since the Jacobi identity for the skew-symmetric bracket does not close, one is tempted to lean on the notion of L_∞-algebras to extend $[\,.\,,.\,]$. These are algebraic structures that generalize the notion of (differential graded) Lie algebras, by allowing the Jacobi identity to be satisfied only up to homotopy. The construction of such L_∞-extensions of Leibniz algebras from the data contained in a tensor hierarchy algebra will be addressed in another paper.

Let $\mathcal{I}$ be the sub-space of *squares* of the Leibniz algebra, generated by the set of elements of the form $\{x,x\}$. We know that $\mathcal{I}$ contains all symmetric elements of the form $\{x,y\}$, since they can always be written as a sum of squares. By using Equations (2.11) and (2.8), we deduce that $\mathcal{I}$ is an ideal of V for the Leibniz product, and that the action of $\mathcal{I}$ on V is

[1] In a right Leibniz algebra, the product acts from the right, hence the Leibniz identity is:

$$(x\bullet y)\bullet z=x\bullet(y\bullet z)+(x\bullet z)\bullet y \tag{2.7}$$

null. An ideal of V whose action is trivial is said *central*. The set of all elements of V whose action is trivial is called the *centre of* V:

$$\mathcal{Z} = \Big\{ x \in V \;\big|\; x \bullet y = 0 \;\; \text{for all} \;\; y \in V \Big\}$$

By the Leibniz identity (2.8), the sub-space $\mathcal{Z}$ is an ideal of V, and moreover $\mathcal{I} \subset \mathcal{Z}$.

The fact that $\mathcal{I}$ is the ideal of squares implies that the quotient $\mathfrak{h}_{\mathcal{I}} \equiv {}^{V}\!/_{\mathcal{I}}$ is a Lie algebra when equipped with the skew-symmetric bracket defined in Equation (2.10). We can define an action ρ of $\mathfrak{h}_{\mathcal{I}}$ on V by:

$$\rho_a(x) \equiv \widetilde{a} \bullet x \tag{2.14}$$

where $a \in \mathfrak{h}_{\mathcal{I}}$, $x \in V$ and $\widetilde{a}$ is any representant of a in V. First, the action does not depend on the representant since the component of $\widetilde{a}$ which is in $\mathcal{I}$ acts trivially on x. Second, the Leibniz identity, and the fact that the ideal $\mathcal{I}$ is central in V, imply that the action ρ actually defines a representation of $\mathfrak{h}_{\mathcal{I}}$ on V:

$$\rho_{[a,b]}(x) = [\rho_a, \rho_b](x) \tag{2.15}$$

for any $a, b \in \mathfrak{h}_{\mathcal{I}}$ and $x \in V$. Setting $\theta_{\mathcal{I}} : V \to \mathfrak{h}_{\mathcal{I}}$ to be the quotient map, we can rewrite Equation (2.14) as:

$$x \bullet y = \rho_{\theta_{\mathcal{I}}(x)}(y) \tag{2.16}$$

for any $x, y \in V$. Hence see the analogy with Equation (2.4). Since the action (2.14) coincides with the Leibniz product, and since the ideal $\mathcal{I}$ is central, it implies that Equation (2.15) is nothing but the Leibniz identity (2.8), when evaluated in $a = \theta_{\mathcal{I}}(y)$ and $b = \theta_{\mathcal{I}}(z)$.

Now let $\mathcal{K}$ be an ideal of V that contains all the squares. Let $\mathfrak{h}_{\mathcal{K}}$ be the quotient ${}^{V}\!/_{\mathcal{K}}$ and let $\theta_{\mathcal{K}} : V \to \mathfrak{h}_{\mathcal{K}}$ be the corresponding quotient map. A candidate for a skew-symmetric bracket on $\mathfrak{h}_{\mathcal{K}}$ is the projection via $\theta_{\mathcal{K}}$ of the skew symmetric part of the Leibniz product:

$$\big[\theta_{\mathcal{K}}(x), \theta_{\mathcal{K}}(y)\big]_{\mathcal{K}} \equiv \theta_{\mathcal{K}}\big([x,y]\big) \tag{2.17}$$

for any $x, y \in V$. This can be summarized in the following diagram:

$$\begin{tikzcd}
V \otimes V \arrow[r, "{[\,.\,,.\,]}"] \arrow[d, "\theta_{\mathcal{K}} \otimes \theta_{\mathcal{K}}"'] & V \arrow[d, "\theta_{\mathcal{K}}"] \\
\mathfrak{h}_{\mathcal{K}} \otimes \mathfrak{h}_{\mathcal{K}} \arrow[r, "{[\,.\,,.\,]_{\mathcal{K}}}"] & \mathfrak{h}_{\mathcal{K}}
\end{tikzcd}$$

Equation (2.13) shows that the Jacobiator of the skew-symmetric bracket $[\,.\,,.\,]$ takes values in $\mathcal{I} \subset \mathcal{K}$, hence it projects down to zero in $\mathfrak{h}_{\mathcal{K}}$. This turns $\big(\mathfrak{h}_{\mathcal{K}}, [\,.\,,.\,]_{\mathcal{K}}\big)$ into a Lie algebra that we call the *gauge algebra of* V *over* $\mathcal{K}$.

This result is systematic as soon as $\mathcal{K}$ contains the ideal of squares. The set $\mathcal{I}$ is the smallest ideal of V that has this property. The Lie algebra $\mathfrak{h}_{\mathcal{K}}$ is thus defined by the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \xrightarrow{\;\iota\;} V \xrightarrow{\;\theta_{\mathcal{K}}\;} \mathfrak{h}_{\mathcal{K}} \longrightarrow 0$$

This Lie algebra can be seen as a sub-Lie algebra of $\mathfrak{h}_{\mathcal{I}}$ through the injective map $\chi_{\mathcal{K}} : \mathfrak{h}_{\mathcal{K}} \to \mathfrak{h}_{\mathcal{I}}$ defined by:

$$\chi_{\mathcal{K}}(a) = \widehat{a} \tag{2.18}$$

where $\widehat{a} \in V$ is the unique lift of $a \in \mathfrak{h}_{\mathcal{K}}$ that has no component in $\mathcal{K}$, and hence in $\mathcal{I}$. The following diagram shows the relationship between the ideal $\mathcal{K}$ and the ideal of squares $\mathcal{I}$.

$$\begin{tikzcd}
0 \arrow[dr] & & & & \\
 & \mathcal{I} \arrow[d, "\iota"] \arrow[dr] & & & \\
0 \arrow[r] & \mathcal{K} \arrow[r, "\iota"] & V \arrow[r, "\theta_{\mathcal{K}}"] \arrow[dr] & \mathfrak{h}_{\mathcal{K}} \arrow[r] \arrow[d, "\chi_{\mathcal{K}}"] & 0 \\
 & & & \mathfrak{h}_{\mathcal{I}} \arrow[dr] & \\
 & & & & 0
\end{tikzcd}$$

In view of the diagram, the map $\Theta_{\mathcal{K}} = \chi_{\mathcal{K}} \circ \theta_{\mathcal{K}} : V \to \mathfrak{h}_{\mathcal{I}}$ would correspond to the embedding tensor of section 2.1. Then it is inspired by supergravity gauging procedures that we call the quotient map $\theta_{\mathcal{K}}$ the *embedding tensor associated to* $\mathcal{K}$.

Now assume that the ideal $\mathcal{K}$ is central:

$$\mathcal{I} \subset \mathcal{K} \subset \mathcal{Z}$$

Following the same line of arguments as in Equations (2.14) and (2.15), this choice induces an action $\rho^{\mathcal{K}}$ of $\mathfrak{h}_{\mathcal{K}}$ on V that is a representation:

$$\rho^{\mathcal{K}}_{[a,b]}(x) = [\rho^{\mathcal{K}}_a, \rho^{\mathcal{K}}_b](x) \tag{2.19}$$

for any $a, b \in \mathfrak{h}_{\mathcal{K}}$ and $x \in C$. We now show that even if the Lie algebra $\mathfrak{h}_{\mathcal{K}}$ may be smaller than $\mathfrak{h}_{\mathcal{I}}$ (and bigger than $\mathfrak{h}_{\mathcal{Z}}$), the choice of the ideal $\mathcal{K}$ has no influence on the action of $\mathfrak{h}_{\mathcal{K}}$ on V (as soon as it satisfies $\mathcal{I} \subset \mathcal{K} \subset \mathcal{Z}$). Let denote by $\rho^{\mathcal{K}}$ (resp. $\rho^{\mathcal{Z}}$) the action of $\mathfrak{h}_{\mathcal{K}}$ (resp. $\mathfrak{h}_{\mathcal{Z}}$) on V. As a vector space, V is isomorphic to the following decomposition:

$$V \simeq {}^{V}\!/_{\mathcal{Z}} \oplus {}^{\mathcal{Z}}\!/_{\mathcal{K}} \oplus \mathcal{K}$$

Now let $x = x_1 + x_2 + x_3$ be an element of V, whose components in the respective subspaces are: $x_1 \in {}^{V}\!/_{\mathcal{Z}}$, $x_2 \in {}^{\mathcal{Z}}\!/_{\mathcal{K}}$ and $x_3 \in \mathcal{K}$. Then, for any $y \in V$, we have $x \bullet y = (x_1 + x_2) \bullet y = x_1 \bullet y$. This means in particular that:

$$\rho^{\mathcal{K}}_{\theta_{\mathcal{K}}(x)}(y) = \rho^{\mathcal{Z}}_{\theta_{\mathcal{Z}}(x)}(y) \tag{2.20}$$

We have to mention that even though we may have:

$$\big[\theta_{\mathcal{K}}(x), \theta_{\mathcal{K}}(x')\big] \neq \big[\theta_{\mathcal{Z}}(x), \theta_{\mathcal{Z}}(x')\big] \tag{2.21}$$

for any $x, x' \in V$, Equations (2.19) and (2.20) imply that both sides have the same action on V. In other words, the action of a gauge algebra $\mathfrak{h}_{\mathcal{K}}$ on V is independent of the choice of the ideal $\mathcal{K}$ defining it, and is only determined by the center $\mathcal{Z}$ of the Leibniz algebra. This is why the gauge algebra over $\mathcal{Z}$, being the smallest of all gauge algebras over ideals of V, should deserve a name on its own. We will call the gauge algebra of V over the centre $\mathcal{Z}$ the *central gauge algebra of* V. This can be summarized in the following Lemma:

Lemma 2.2. *Let V be a Leibniz algebra and let $\mathcal{K}$ be any central ideal that contains the squares. Then the action of the gauge algebra $\mathfrak{h}_{\mathcal{K}}$ on V is the same as the one of the central gauge algebra $\mathfrak{h}_{\mathcal{Z}}$.*

Remark. In particular, the action of $\mathfrak{h}_{\mathcal{I}}$ on V does not differ from the action of $\mathfrak{h}_{\mathcal{Z}}$, which means that the discussion leading to Equation (2.16) is still valid for $\mathfrak{h}_{\mathcal{Z}}$.

In addition, since $\mathcal{I} \subset \mathcal{K}$, we have the following result:

$$\theta_{\mathcal{K}}(x \bullet y) = \theta_{\mathcal{K}}\big([x, y]\big) \tag{2.22}$$

for any $x, y \in V$. By Equation (2.17) we obtain the following:

$$\theta_{\mathcal{K}}(x \bullet y) = \big[\theta_{\mathcal{K}}(x), \theta_{\mathcal{K}}(y)\big]_{\mathcal{K}} \tag{2.23}$$

Then $\theta_{\mathcal{K}}$ commutes with the respective products on V and $\mathfrak{h}_{\mathcal{K}}$, as in Equation (2.5). When the Lie algebra $\mathfrak{h}_{\mathcal{K}}$ is equipped with the adjoint action, Equation (2.23) states that the quotient map $\theta_\mathcal{K}$ becomes $\mathfrak{h}_\mathcal{K}$-equivariant, and we retrieve Equation (2.3). In other words, the embedding tensor $\theta_\mathcal{K}$ automatically satisfies the quadratic constraint.

Given all these informations, the temptation to compare these results with the data of section 2.1 is strong. Indeed, given a Leibniz algebra V, and a central ideal $\mathcal{K}$ containing the squares, one can see $\mathfrak{h} \equiv \mathfrak{h}_\mathcal{K}$ as a sub-Lie algebra of $\mathfrak{g} \equiv \mathfrak{h}_\mathcal{I}$, that is determined by its embedding tensor $\Theta_\mathcal{K}$. Moreover, we have seen that the action of $\mathfrak{h}_\mathcal{I}$ and $\mathfrak{h}_\mathcal{K}$ on V coincide with the action of the central gauge algebra $\mathfrak{h}_\mathcal{Z}$, and that it gives back the Leibniz product, because they are precisely induced by central ideals. Hence we see that the data of a Leibniz algebra V together with a central ideal $\mathcal{K}$ containing the ideal of squares define the embedding tensor $\Theta_\mathcal{K}$, that satisfies every arguments in section 2.1.

On the other hand, in section 2.1, we have shown that in supergravity theories, the gauging procedure induces a Leibniz algebra structure on some representation V of the Lie algebra $\mathfrak{g}$ of symmetries of the theory. The construction relies on the existence of the embedding tensor $\Theta : V \to \mathfrak{g}$ whose existence depends on a linear and a quadratic constraint. Decomposing the Leibniz product on V into its symmetric and its skew-symmetric part, the quadratic constraint – see Equation (2.5) – induces the following two consistent results: 1) its skew-symmetrization implies that $\mathfrak{h} = \mathrm{Im}(\Theta)$ is a sub-Lie algebra of $\mathfrak{g}$, and 2) its symmetrization implies that $\mathcal{I} \subset \mathrm{Ker}(\Theta)$. Now, from Equation (2.4) we already know that elements in the kernel of Θ are central elements of V, hence:

$$\mathcal{I} \subset \mathrm{Ker}(\Theta) \subset \mathcal{Z}$$

Letting $\mathcal{K} = \mathrm{Ker}(\Theta)$, and $\theta_\mathcal{K} : V \to {}^{V}\!/_{\mathcal{K}}$ be the associated quotient map, we deduce that the map Θ factors through $\mathfrak{h}_\mathcal{K}$:

$$\begin{tikzcd}
V \arrow[r, "\Theta"] \arrow[d, "\theta_\mathcal{K}"'] & \mathfrak{h} \subset \mathfrak{g} \\
\mathfrak{h}_\mathcal{K} \arrow[ur, hook, two heads] &
\end{tikzcd}$$

Hence the gauge algebra $\mathfrak{h}$ defined in section 2.1 is isomorphic to $\mathfrak{h}_\mathcal{K}$, the gauge algebra over $\mathcal{K} = \mathrm{Ker}(\Theta)$. This correspondence justifies the mathematical construction of the tensor hierarchy that is proposed in this paper.

2.3 Factorizing the symmetric bracket

In this section, we discuss the importance of the symmetric bracket of a Leibniz algebra in regard to its relation to the embedding tensor. The construction of a tensor hierarchy is tightly related to the question of lifting the symmetric bracket. The last two sections were devoted to the presentation of the link between the embedding tensor, as seen by a physicist, and the embedding tensor that a mathematician could infer from a Leibniz algebra. It is now time to introduce the following concept that unifies both point of views:

Definition 2.3. *A* Lie-Leibniz triple *is a triple* $(V, \mathfrak{g}, \Theta)$ *where:*

1. $\mathfrak{g}$ *is a Lie algebra;*
2. V *is a* $\mathfrak{g}$*-module that carries a Leibniz algebra structure;*
3. $\Theta : V \to \mathfrak{g}$ *is a linear mapping whose image is a non-trivial sub-Lie algebra* $\mathfrak{h}$ *of* $\mathfrak{g}$*;*
4. Θ *is* $\mathfrak{h}$*-equivariant;*
5. *the Leibniz product on* V *satisfies:*

$$x \bullet y = \rho_{\Theta(x)}(y) \tag{2.24}$$

Example 2. The first example of a Lie-Leibniz triple is the one described in section 2.1. Also in section 2.2, we have seen that to any Leibniz algebra V is associated a family of Lie-Leibniz triples $\big((V, \mathfrak{h}_\mathcal{I}, \Theta_\mathcal{K})\big)_\mathcal{K}$, where $\mathcal{K}$ runs over the central ideals that contain the squares.

Let $(V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple, then the action of $\mathfrak{g}$ is a derivation of the Leibniz product $\bullet$. This derivation property is then automatically satisfied for the symmetric and the skew-symmetric part of the Leibniz product. Also, we canonically extend the action of $\mathfrak{g}$ on V to the symmetric powers of V as a derivation:

$$\rho_a\big(x_1 \odot \ldots \odot x_n\big) = \sum_{i=1}^{n} x_1 \odot \ldots \odot \rho_a(x_i) \odot \ldots \odot x_n \tag{2.25}$$

turning $S^n(V)$ into a $\mathfrak{g}$-module. Here and in part 2, $\odot$ represents the symmetric product.

We observe that the symmetric bracket $\{.,.\}$ is a map from $S^2(V)$ to V, whose image is the ideal of squares $\mathcal{I}$:

$$\{.,.\}(x \odot y) = \{x, y\} \tag{2.26}$$

for all $x, y \in V$. Hence it can be seen as an element of $S^2(V^*) \otimes V$. Let $a \in \mathfrak{g}$, we define the action of a on $\{.,.\}$ by:

$$\rho_a\big(\{.,.\}\big) \equiv \rho_a \circ \{.,.\} - \{.,.\} \circ \rho_a \tag{2.27}$$

After short calculations, one finds that:

$$\rho_a\big(\{.,.\}\big)(x \odot y) = \frac{1}{2}\big(\rho_{\widehat{\rho_a}(\Theta)(x)}(y) + \rho_{\widehat{\rho_a}(\Theta)(y)}(x)\big) \tag{2.28}$$

where $\widehat{\rho}$ denotes the action of $\mathfrak{g}$ on the representation T_Θ to which Θ belongs. In the light of Equation 2.28, physicists say that the symmetric bracket transforms in the same representation as the embedding tensor.

Equivariance of the embedding tensor $\Theta : V \to \mathfrak{g}$ is synonym to the quadratic constraint (2.2). Hence if $a \in \mathfrak{h} \equiv \mathrm{Im}(\Theta)$, then the right hand side of Equation (2.28) vanishes. In other words, the symmetric bracket is naturally $\mathfrak{h}$ equivariant (but not necessarily $\mathfrak{g}$-equivariant). It implies that $\mathrm{Ker}\big(\{.,.\}\big)$ is a $\mathfrak{h}$-sub-module of $S^2(V)$. When $\dim(V) \geq 2$, the kernel of the symmetric bracket cannot be restricted to zero for dimensional reasons. Then the quotient:

$$I = S^2(V) \Big/ \mathrm{Ker}\big(\{.,.\}\big)$$

inherits the quotient representation descending from the action of $\mathfrak{h}$ on $S^2(V)$. The restriction of the symmetric bracket to I is well defined and gives a bijection between I and $\mathcal{I}$.

More generally, if there exists a Lie algebra $\mathfrak{g}$ such that the Leibniz algebra V is a representation of $\mathfrak{g}$, then the symmetric space $L \subset S^2(V)$ can be decomposed into irreducible $\mathfrak{g}$-modules:

$$S^2(V) = L_1 \oplus L_2 \oplus \ldots \oplus L_p$$

for some $p \geq 1$. Now, even though $\mathrm{Ker}\big(\{.,.\}\big)$ may not be a $\mathfrak{g}$-module, if L_i is included in $\mathrm{Ker}\big(\{.,.\}\big)$ for some $1 \leq i \leq p$, then we deduce that the symmetric bracket factorizes through the $\mathfrak{g}$-module $W \equiv L_1 \oplus \ldots \oplus L_{i-1} \oplus L_{i+1} \oplus \ldots \oplus L_p$. In other words, there exists a unique map $\mathrm{d} : W \to \mathcal{I}$ such that:

$$\{.,.\} = \mathrm{d} \circ \varphi_W \tag{2.29}$$

where $\varphi_W : S^2(V) \to W$ be the natural projection map. We call the map d a *W-lift of the symmetric bracket*. By definition it is $\mathfrak{g}$-equivariant, but notice that d may not be $\mathfrak{g}$-equivariant. This discussion leads to the the following result:

Lemma 2.4. *Let $(V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple. There is a one-to-one correspondence between $\mathfrak{g}$-modules $L \subset S^2(V)$ that are included in $\mathrm{Ker}\big(\{.,.\}\big)$, and $\mathfrak{g}$-sub-modules W of the symmetric space $S^2(V)$ through which the symmetric bracket $\{.,.\} : S^2(V) \to V$ factorizes:*

$$\begin{tikzcd} \mathcal{I} & W \arrow[l, "\mathrm{d}"'] \\ S^2(V) \arrow[u, "\{.,.\}"] \arrow[ur] & \end{tikzcd}$$

In the case where we start from a Leibniz algebra V only, the Lie algebra $\mathfrak{g}$ can be taken as $\mathfrak{h}_{\mathcal{I}}$. Recall that by Lemma 2.2, the action of $\mathfrak{h}_{\mathcal{I}}$ on V is the same as the action of the central gauge algebra $\mathfrak{h}_{\mathcal{Z}}$ on V. This result can be extended to all of $S(V)$, the symmetric algebra of V: any $\mathfrak{h}_{\mathcal{I}}$-sub-module of $S^2(V)$ is in fact a $\mathfrak{h}_{\mathcal{Z}}$-sub-module with the same action, and vice-versa. We deduce that the splitting of $S^2(V)$ into its irreducible $\mathfrak{h}_{\mathcal{I}}$-representations coincides with the splitting of $S^2(V)$ into its irreducible $\mathfrak{h}_{\mathcal{Z}}$-representations. Hence $\mathrm{Ker}\big(\{.\,,.\}\big)$ will be a $\mathfrak{h}_{\mathcal{I}}$-module in any case, and the choice of W only depends on the Leibniz algebra structure on V. Among possible candidates for W, the two spaces $I = S^2(V)\Big/\mathrm{Ker}\big(\{.\,,.\}\big)$ and $S^2(V) \simeq S^2V\Big/\{0\}$ are the two extremes:

$$I \subset W \subset S^2(V)$$

In the case that $W = S^2(V)$, the W-lift coincides with the symmetric bracket, then this choice is not very interesting.

In supergravity theories, the choice of W is related to the choice of the representation T_Θ to which the embedding tensor belongs. This is implemented in the theory by the linear constraint. It consists of requiring that the W-lift d belongs to the same representation as the embedding tensor Θ (and thus as the symmetric bracket). This constrains both the choice of the embedding tensor, and the choice of W, because it implies that the tensor products $V^*\otimes\mathfrak{g}$ and $W^*\otimes V$ share a common irreducible representation of $\mathfrak{g}$ in their decomposition into a direct sum of $\mathfrak{g}$-modules. The choice of Θ and W is thus made simultaneously, via this linear constraint.

Usually, this analysis is carefully performed on the fields content of the theory. There is only a finite number of cases to study, because the space-time dimension already constrains the choice for $\mathfrak{g}$ and V. Usually, physicists provide the tensor hierarchies for $D = 3,\ldots,7$ space-time dimensions, but they can go up to dimension 11. Since the linear constraint is not well understood yet, in the present paper, we made possible that we do not depend on it. In other words, we are mostly free of the choice of gauge algebra, as well as of the choice of the lift of the symmetric bracket. We are free to choose both, independently, that is why we do not really bother on the linear constraint. However, as soon as we have fixed the representation W, the procedure to build the tensor hierarchy is very rigid, and follows precisely the lines of what happens in supergravity theories.

Before going further, let us introduce a new notion that will make easier the action of reading through the text, and understanding the results:

Definition 2.5. *An* embedding complex *is a 4-tuple* $(V,\mathfrak{g},\Theta,W)$ *such that:*

1. $(V,\mathfrak{g},\Theta)$ *is a Lie-Leibniz triple;*
2. W *is a* $\mathfrak{g}$*-sub-module of* $S^2(V)$ *that lifts the symmetric bracket on* V*.*

Remark. When V is a Leibniz algebra and that $\mathfrak{g}\equiv\mathfrak{h}_{\mathcal{I}}$, we have shown in section 2.2 that, given any central ideal $\mathcal{K}$ that contains the squares, we can define a Lie algebra $\mathfrak{h}_{\mathcal{K}}$ that injects into $\mathfrak{g}$. We have also seen that $\mathrm{Ker}\big(\{.\,,.\}\big)$ is a $\mathfrak{h}_{\mathcal{I}}$-module. Then we deduce that the 4-tuple $\big(V,\mathfrak{h}_{\mathcal{I}},\Theta_{\mathcal{K}},\mathrm{Ker}\big(\{.\,,.\}\big)\big)$ defines an embedding complex.

2.4 Graded geometry

The construction of the tensor hierarchies will involve many notions from graded algebra. We define a *graded vector space* E as a family of vector spaces $E = (E_i)_{i\in\mathbb{Z}}$. An element x is said *homogeneous of degree* i if $x \in E_i$. The degree of an homogeneous element x is noted $|x|$. A *commutative graded algebra* is a graded vector space $A = (A_i)_{i\in\mathbb{Z}}$ equipped with a product $\odot : A\otimes A\to A$ such that

$$x\odot y = (-1)^{|x||y|}y\odot x$$

for every homogeneous elements $x, y \in A$. We have chosen the symbol $\odot$ for the product to emphasize the analogy with the symmetric product in section 2.3. If the product is associative, successive products of multiple elements make sense whatever the order we perform the products. In that case, given n homogeneous elements $x_1, \ldots, x_n \in A$, and a permutation σ of $\{1, \ldots, n\}$, we define the *Koszul sign of the permutation* (with respect to these elements) as the sign $\epsilon^{\sigma}_{x_1,\ldots,x_n} = \pm 1$ satisfying:

$$x_1 \odot \ldots \odot x_n = \epsilon^{\sigma}_{x_1,\ldots,x_n} x_{\sigma(1)} \odot \ldots \odot x_{\sigma(2)} \tag{2.30}$$

Given two graded vector spaces E and F, a *linear map between E and F* is a family $\phi = (\phi_i)_{i\in\mathbb{Z}}$ of linear applications $\phi_i : E_i \to F_i$. For any two commutative graded algebras A and B, a *homomorphism from A to B* is a collection $\phi = (\phi_i)_{i\in\mathbb{Z}}$ of degree 0 multilinear maps $\phi_i : A_i \to B_i$ that commutes with the respective products of A and B:

$$\phi(x \odot_A y) = \phi(x) \odot_B \phi(y)$$

for any $x, y \in A$. Given a graded vector space $E = (E_i)_{i\in\mathbb{Z}}$, a *linear function on E* is an element of the commutative graded algebra $S(E^*) = \bigoplus_{n\geq 0} S^n(E^*)$, where E^* is the graded vector space defined by the family of dual spaces $E^* = \big((E_i)^*\big)_{i\in\mathbb{Z}}$. A function f is said to be *homogeneous of degree k* if $f \in S(E^*)_k$. A *morphism from E to F* is a (degree 0) commutative graded algebra homomorphism $\Phi : S(F^*) \to S(E^*)$. It induces a degree 0 linear map $\phi^* : F^* \to E^*$ whose dual map is a linear application of graded vector spaces $\phi = E \to F$.

It is now time to define the central mathematical objects attached to a tensor hierarchy:

Definition 2.6. *A* graded Lie algebra *is a graded vector space $L = (L_i)_i \in \mathbb{Z}$ equipped with a graded skew-symmetric bracket $[.\,,.] : L_i \otimes L_j \to L_{i+j}$ that satisfies the* graded Jacobi identity*:*

$$\big[x, [y, z]\big] = \big[[x, y], z\big] + (-1)^{|x||y|}\big[y, [x, z]\big] \tag{2.31}$$

for any $x, y, z \in L$.

A differential graded Lie algebra *is a graded Lie algebra $(L, [.\,,.])$ that admits a differential $\partial = (\partial_i : L_{i-1} \to L_i)_{i\in\mathbb{Z}}$ which is a derivation of the bracket:*

$$\partial\big([x, y]\big) = \big[\partial(x), y\big] + (-1)^{|x|}\big[x, \partial(y)\big] \tag{2.32}$$

for any $x, y \in L$.

There is another formulation of (differential) graded Lie algebras using the notion of *differential graded manifolds*. First, a *graded manifold* $\mathcal{M} = (E, M)$ is a sheaf $\mathcal{C}^\infty_{\mathcal{M}}$ of graded algebras over a smooth manifold M that is called the *base*, such that for every open set $U \subset M$, $\mathcal{C}^\infty_{\mathcal{M}}(U) \simeq \mathcal{C}^\infty(U) \otimes S(E^*)$, where E is a graded vector space called the *fiber*. A *morphism between the graded manifolds $\mathcal{M}$ and $\mathcal{N}$* is a family $\Phi = (\phi_U)_{U\subset M}$ of graded algebra morphisms $\phi_U : \mathcal{C}^\infty_{\mathcal{N}}(U) \to \mathcal{C}^\infty_{\mathcal{M}}(U)$. We define *vector fields on $\mathcal{M}$* as sections to the (graded) vector space of derivations of $\mathcal{C}^\infty_{\mathcal{M}}$. If the base manifold M is reduced to a point, we say that the graded manifold $\mathcal{M}$ is *pointed*, i.e. it is reduced to the graded vector space E. In that case a vector field X can be identified with an element of $S(E^*) \otimes E$. A vector field X on $\mathcal{M}$ is said to be *of arity n* if for any function $f \in S^k(E^*)$, we have $X(f) \in S^{k+n}(E^*)$. Obviously we can decompose a graded vector field by its components of various arities, but they should not be confused with the degree of the vector field.

Definition 2.7. *A* differential graded manifold *is a graded manifold $\mathcal{M}$ equipped with a degree $+1$ vector field Q satisfying $[Q, Q] = 0$.*

Given a graded vector space E, the *suspension of E* is the graded vector space $sE = (sE)_{j\in\mathbb{Z}}$ defined as:

$$(sE)_j = E_{j-1}$$

In other words, the suspension of a graded vector space is the same vector space, but with all degrees shifted by $+1$. Consequently, the degrees of dual elements are shifted by -1 and every graded symmetric object becomes graded skew-symmetric. Hence a function $f \in S^n(E^*)$ of degree k is transformed into a function $sf \in \wedge^n\big((sE)^*\big)$ of degree $k + n$. In particular, given a linear application $F : S^2(E) \to E$ of degree k, it suspension $sF : \wedge^2(sE) \to sE$ has degree

$k-1$ (precise formulas are given in [3]). The suspension isomorphism admits a reverse which is called the *desuspension* and which is noted s^{-1}. The desuspension satisfies the following identity:

$$(s^{-1}E)_j = E_{j+1}$$

We can now give the equivalence that is of interest for us:

Theorem 2.8. *Let $E = (E_i)_{i\in\mathbb{Z}}$ be a graded vector space. Then differential graded Lie algebra structures on a E are in one-to-one correspondence with differential graded manifold structures of arity at most one on the pointed graded manifold $s^{-1}E$.*

Proof. The formulas to pass from one structure to another are taken from [13] and [3]. First, we set ι_u to be the degree $-|u|$ vector field on E satisfying:

$$\iota_u(\alpha) = \langle \alpha, u\rangle \tag{2.33}$$

for any $\alpha \in E^*_{|u|}$. Then, given $x, y \in E$, the relationship between $[x, y]$ in E and the corresponding homological vector field Q is given by:

$$\iota_{s^{-1}[x,y]} = (-1)^{|x|}\big[[Q, \iota_{s^{-1}(x)}], \iota_{s^{-1}(y)}\big] \tag{2.34}$$

where on the right hand side, the use the brackets of (graded) vector fields on $s^{-1}E$. On the other hand, the differential ∂ satisfies:

$$\iota_{s^{-1}(\partial(x))} = -[Q, \iota_{s^{-1}(x)}]\big|_0 \tag{2.35}$$

where the sub-script $|_0$ means that the vector field is constant and its value is the one taken at the origin. Formulas (2.34) and (2.35) provides a one-to-one correspondence between the differential graded Lie algebra structure on E and the differential graded manifold structure on $s^{-1}E$. The Jacobi and Leibniz identities are indeed incapsulated into the homological condition $[Q, Q] = 0$. □

Example 3. Let $\big(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}, \partial, [\,.\,,.]\big)$ be a differential graded Lie algebra. Given a basis $(e_i)_{1\leq i\leq n}$ of $\mathfrak{g}_0$ and $(f_a)_{1\leq a\leq m}$ of $\mathfrak{g}_{-1}$, there exist tensors C^k_{ij}, C^b_{ia} and d^i_a such that:

$$\partial(f_a) = d^i_a\, e_i\,, \qquad [e_i, e_j] = C^k_{ij}\, e_k \qquad \text{and} \qquad [e_i, f_a] = C^b_{ia}\, f_b \tag{2.36}$$

Setting $(\widetilde{e}_i)_{1\leq i\leq n}$ be the basis for $s^{-1}\mathfrak{g}_0$ and $(\widetilde{f}_a)_{1\leq a\leq m}$ be the basis for $s^{-1}\mathfrak{g}_{-1}$, the corresponding homological vector field on $s^{-1}\mathfrak{g}$ is:

$$Q = -d^i_a\, \widetilde{f}^{*a} \otimes \iota_{\widetilde{e}_i} - \frac{1}{2}C^k_{ij}\, \widetilde{e}^{*i}\widetilde{e}^{*j} \otimes \iota_{\widetilde{e}_k} - C^b_{ia}\, \widetilde{e}^{*i}\widetilde{f}^{*a} \otimes \iota_{\widetilde{f}_b} \tag{2.37}$$

where the star denotes the dual basis.

In [5, 9], a *tensor hierarchy algebra* is defined as a very specific differential graded Lie algebra $T = (T_i)_{i\geq -1}$ such that each T_i is a T_0-module, and such that T_{-1} is the representation to which the embedding tensor Θ belongs. Hence the embedding tensor can be seen as an element of T_{-1}. For degree reasons, it is self-commuting: $[\Theta, \Theta] = 0$, so that one requires that the differential ∂ on T_Θ is defined as $\partial \equiv [\Theta, .]$. The representation T_{-2} is determined through the linear constraint. We propose in the present paper to explain an alternative construction to the one given in [9]. The main point is that it is a systematic procedure that relies on a few data. We believe that the definition given in [5] is the correct definition of a tensor hierarchy algebra, but we chose the reverse convention on the grading:

Definition 2.9. *Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple. Then a* tensor hierarchy algebra *associated to $\mathcal{V}$ is a differential graded Lie algebra structure $\big(\mathfrak{T}, \partial, [\,.\,,.]\big)$ that consists of a sequence of $\mathfrak{g}$-modules $\mathfrak{T} = (T_{-i})_{i\geq -1}$ that satisfies:*

1. *$T_1 = s(T_\Theta)$ is the representation to which Θ belongs;*
2. *$T_0 = \mathfrak{g}$ and $T_{-1} = s^{-1}V$;*

3. $T_{-2} \subset S^2(V)$ *is such that* $\big(V, \mathfrak{g}, \Theta, s^2(T_{-2})\big)$ *is an embedding complex;*
4. $T_{-i} \subset s^{-i}\big(V^{\otimes i}\big)$ *for every* $i \geq 3$*;*

and where the differential and the 2-bracket are such that:

1. *the 2-bracket on* T_0 *is the Lie bracket on* $\mathfrak{g}$*;*
2. *the 2-bracket* $[\,.\,,.\,] : T_{-1} \otimes T_{-1} \to T_{-2}$ *satisfies, for all* $x, y \in T_{-1}$*:*
$$[x, y] = 2\, s^{-2} \circ \varphi\big(s(x), s(y)\big) \tag{2.38}$$
where $\varphi : S^2(V) \to s^2 T_{-2}$ *is the canonical projection;*
3. *the 2-bracket* $[\,.\,,.\,] : T_0 \otimes T_{-i} \to T_{-i}$ *corresponds to the action* ρ_{-i} *of* $\mathfrak{g}$ *on* T_{-i}*:*
$$\forall\ a \in \mathfrak{g}, x \in T_{-i} \qquad [a, x] = \rho_{-i,a}(x) = -[x, a] \tag{2.39}$$
4. *the differential* $\partial = \big(\partial_{-i} : T_{-i-1} \to T_{-i}\big)_{i\geq 0}$ *satisfies at highest levels:*
$$\partial_0 = -\Theta \circ s, \qquad \partial_{-1} = -s^{-1} \circ \mathrm{d} \circ s^2 \qquad \textit{and} \qquad \partial_{+1} = -\widehat{\rho}(\Theta) \tag{2.40}$$
where d *is the* T_{-2}*-lift of* $\{\,.\,,.\,\}$*;*
5. *the differential* ∂ *is* $\mathrm{Im}(\Theta)$*-equivariant and the 2-bracket* $[\,.\,,.\,]$ *is* $\mathfrak{g}$*-equivariant.*

Moreover Θ *defines an element of* T_1*, and the bracket of* $\Theta \in T_1$ *with elements of* $\mathfrak{T}$ *is:*
$$[\Theta, .\,] \equiv \partial \tag{2.41}$$

Remark. For degree reasons, the bracket between two elements of T_1 is zero. Hence we obtain that $[\Theta, \Theta] = 0$, which is expected from the homological property of the differential.

3 Building the tensor hierarchy

This section is devoted to the construction of a tensor hierarchy algebra associated to Lie-Leibniz triple $\mathcal{V} = (V, \mathfrak{g}, \Theta)$. This section is focused on constructing a chain complex:
$$\ldots \longrightarrow T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} T_{-1} \xrightarrow{\partial_0} T_0 \xrightarrow{\partial_1} T_1 \longrightarrow 0$$
that can be equipped with a tensor hierarchy algebra structure associated to $\mathcal{V}$. In particular, following Definition 2.9, we expect that $T_{-1} = s^{-1}(V)$. Our goal is to show that, once some choices have been made, the process of constructing this structure is unique and straightforward. We will proceed in two steps: first, from an embedding complex, construct a chain complex:
$$\ldots \longleftarrow U_3 \xleftarrow[\delta_3]{} U_2 \xleftarrow[\delta_2]{} U_1 \xleftarrow[\delta_1]{} U_0 \longleftarrow 0$$
that has some adequate properties. This is worked out in section 3.1. Second, we use the shifted dual of this chain complex $(U_i)_{i\geq 0}$ toinduce the graded vector space that will be equipped with the tensor hierarchy algebra structure that we are looking for:
$$T_{-(i+1)} = (sU_i)^* \qquad \text{for any } i \geq 0$$
Adding T_0 and T_{-1} is not problematic because they are prescribed in Definition 2.9. However the problem of finding a differential graded Lie algebra structure on $\mathfrak{T} = (T_{-i})_{i\geq -1}$ is more subtle. First, we can show that the maps used for the construction of the chain complex $(U_i)_{i\geq 0}$ induce a differential graded Lie algebra structure on the graded vector space $T' = (T_{-i})_{i\geq 2}$. To extend this structure to all of $\mathfrak{T}$, a cautious analysis has to be performed. This last discussion takes place in section 3.2, where the final theorem and main result of the paper is found.

3.1 The skeleton of the tensor hierarchy

The aim of this section is to define the 'skeleton' of the tensor hierarchy algebra, that is: the (possibly infinite) tower of space which is underlying the tensor hierarchy algebra. The construction of this tower of spaces is made by induction. Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple and let $W \subset S^2(V)$ be a $\mathfrak{g}$-module through which the symmetric bracket factorizes (see Lemma 2.4). In other words, the 4-tuple $(V, \mathfrak{g}, \Theta, W)$ is an embedding complex. Then we set $U_0 = V^*$ and $U_1 = s(W^*)$. In this set-up, the shifted dual of the W-lift of the symmetric bracket d becomes a degree $+1$ map that we call δ_1:

$$\delta_1 = -s \circ \mathrm{d}^* : U_0 \to U_1 \tag{3.1}$$

We see that we have obtained from scratch the two first spaces and the first differential of a chain complex:

$$0 \longrightarrow U_0 \xrightarrow{\delta_1} U_1 \xrightarrow{\delta_2} U_2 \xrightarrow{\delta_3} U_3 \longrightarrow \ldots$$

The construction of the hierarchy relies precisely on the choice of $U_1 \simeq W$. Once this space is fixed, the procedure is unique and straightforward. It is now time to define the backbone of the construction of the tensor hierarchies:

Definition 3.1. *Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple. An i-*skeleton* associated to $\mathcal{V}$ (for $i \in \mathbb{N}^* \cup \{\infty\}$) is a 4-tuple (U, δ, π, μ) consisting of:*

1. *a family $U = (U_k)_{0 \leq k < i+1}$ of $\mathfrak{g}$-modules, with respective action $\rho_k : \mathfrak{g} \to \mathrm{End}(U_k)$,*
2. *a degree $+1$ differential $\delta = \big(\delta_k : U_{k-1} \to U_k\big)_{1 \leq k < i+1}$, which is $\mathrm{Im}(\Theta)$-equivariant,*
3. *a family $\pi = (\pi_k)_{0 \leq k < i}$ of $\mathfrak{g}$-equivariant degree -1 linear maps $\pi_k : U_{k+1} \to S^2(U)_k$,*
4. *a family $\mu = (\mu_k)_{0 \leq k < i}$ of degree 0 linear maps $\mu_k : U_k \to S^2(U)_k$,*

that are extended to all of $S(U)$ as derivations, and such that they satisfy the following conditions:

5. *at lowest level, $U_0 = V^*$, equipped with the corresponding contragredient action of $\mathfrak{g}$ on V^*;*
6. *the map π_0 is injective, and for every $1 \leq k < i$, the following sequence is exact:*

$$0 \longrightarrow U_{k+1} \xrightarrow{\pi} S^2(U)_k \xrightarrow{\pi} S^3(U)_{k-1}$$

7. *at lowest order, the map μ_0 satisfies:*

$$\big\langle \mu_0(x), u \otimes v \big\rangle = -\big\langle x, \{u, v\} \big\rangle \tag{3.2}$$

for any $x \in U_0$, $u, v \in V$, whereas for any $k \geq 1$, the map μ_k factors through $\mathfrak{g}^* \otimes U_k$:

$$\begin{tikzcd} & U_0 \otimes U_k \\ U_k \arrow[ur, "\mu_k"] \arrow[r, "\widetilde{\rho_k}"'] & \mathfrak{g}^* \otimes U_k \arrow[u, "\Theta^* \otimes \mathrm{id}"'] \end{tikzcd}$$

where $\widetilde{\rho_k}$ is the canonical map induced by the representation ρ_k of $\mathfrak{g}$ on U_k:

$$\begin{array}{rccc} \widetilde{\rho_k} : & U_k & \longrightarrow & \mathfrak{g}^* \otimes U_k \\ & x & \longmapsto & \rho_k(x) : a \to \rho_{k,a}(x) \end{array} \tag{3.3}$$

8. *the map $\mu : U \to S^2(U)$ is a null-homotopic chain map between U and $S^2(U)$:*

$$\begin{tikzcd}
U_0 \arrow[r, "\delta"] \arrow[d, "\mu"] & U_1 \arrow[r, "\delta"] \arrow[d, "\mu"] \arrow[dl, "\pi"'] & U_2 \arrow[r, "\delta"] \arrow[d, "\mu"] \arrow[dl, "\pi"'] & U_3 \arrow[r, dotted] \arrow[d, dotted] \arrow[dl, "\pi"'] & \cdots \arrow[dl, dotted] \\
S^2(U)|_0 \arrow[r, "\delta"] & S^2(U)|_1 \arrow[r, "\delta"] & S^2(U)|_2 \arrow[r, dotted] & \cdots &
\end{tikzcd}$$

The j-truncation *(for* $1 \leq j < i$*) of the* i*-skeleton* (U, δ, π, μ) *is the* j*-skeleton of* $\mathcal{V}$ *defined by the quadruple* $\left(\bigoplus_{0\leq k\leq j} U_{-k}, \delta|_{W^*}, \pi|_{W^*}, \mu|_{W^*}\right)$.

Remark. 1. By injectivity of π_0 (see item 6.), we deduce that $\mathrm{Ker}(\mu_0) = \mathrm{Ker}(\delta_1)$, and by item 8., we deduce that $\mathrm{Ker}(\mu_0) = \mathcal{I}^\circ$, the annihilator of the ideal of squares. On the other hand, the embedding tensor $\Theta : V \to \mathfrak{g}$ admits a dual map: $\Theta^* : \mathfrak{g}^* \to V^*$ taking values in $\big(\mathrm{Ker}(\Theta)\big)^\circ$. Since the ideal of squares is contained in the kernel of Θ, we have the reverse inclusion on the annihilators: $\big(\mathrm{Ker}(\Theta)\big)^\circ \subset \mathcal{I}^\circ$. It implies the following important result:

$$\delta_1 \circ \Theta^* = 0 \tag{3.4}$$

From this, we deduce that the chain complex (U, δ) admits an augmentation:

$$0 \longrightarrow \mathfrak{g}^* \xrightarrow{\Theta^*} U_0 \xrightarrow{\delta_1} U_1 \xrightarrow{\delta_2} U_2 \xrightarrow{\delta_3} \ldots$$

2. The condition $\delta^2 = 0$ is not necessary (see [12]). A careful analysis shows that item 8. guarantees that the homological condition $\delta^2 = 0$ is induced by Equation (3.4). This last equation is in fact a consequence of the symmetric part of Equation (2.5) (the 'quadratic constraint').

3. The equality $\mu_0 = \pi_0 \circ \delta_1$ means that the symmetric bracket factors through U_1^*:

$$\begin{tikzcd}
U_1^* \arrow[d, "\delta_1^*"'] & \\
V & S^2(V) \arrow[l, "-\{.,.\}"'] \arrow[ul, "\pi_0^*"']
\end{tikzcd}$$

and defines a $\mathfrak{g}$-sub-module $W \subset S^2(V)$ such that $(V, \mathfrak{g}, \Theta, W)$ is an embedding complex.

Example 4. A natural example of a 1-skeleton of a Leibniz algebra V is obtained by dualizing the maps defined in Lemma 2.4. We set $U_0 = V^*$ and $U_1 = s(W^*)$ as in the beginning of the section, and $\delta_1 = -s \circ \mathrm{d}^*$ and $\pi_0 = (\varphi_W)^* \circ s^{-1}$, where $\varphi_W : S^2(V) \to W$ is the projection map presented in Lemma 2.4. This is the converse process of the observation that is given in item 3. of the precedent remark.

Example 5. An example of a 2-skeleton arises in the six-dimensional $(1,0)$ superconformal model in six dimensions presented in [6,8,10]. It involves a Leibniz algebra V, a gauge algebra $\mathfrak{g} = {}^{V}\!/_{\mathcal{I}}$, obtained by quotienting V by the ideal of squares $\mathcal{I}$. Denoting the quotient map by $\Theta : V \to \mathfrak{g}$, this implies that $\mathrm{Im}\big(\{.,.\}\big) = \mathrm{Ker}(\Theta)$. The model involves a set of p-forms (for $p = 1, 2, 3, \ldots$) taking values in some $\mathfrak{g}$-modules. The particularity of this model is that the 3-form fields C_t are dual to the 1-forms A^a. The indices are taken from the beginning and from the end of the alphabet to mark the duality. The theory is governed by a set of constants h^a_I, g^{It}, $f^c_{ab} \equiv f^c_{[ab]}$, $d^I_{ab} \equiv d^I_{(ab)}$, b_{Ita}, subject to the following relations:

$$2\big(d^J_{c(a}d^I_{b)s} - d^I_{cs}d^J_{ab}\big)h^s_J = 2f_{c(a}{}^s d^I_{b)s} - b_{Jsc}d^J_{ab}g^{Is} \tag{3.5}$$
$$\big(d^J_{rs}b_{Iut} + d^J_{rt}b_{Isu} + 2d^K_{ru}b_{Kst}\delta^J_I\big)h^u_J = f_{rs}{}^u b_{Iut} + f_{rt}{}^u b_{Isu} + g^{Ju}b_{Iur}b_{Jst} \tag{3.6}$$
$$f_{[ab}{}^r f_{c]r}{}^s - \frac{1}{3}h^s_I d^I_{r[a}f_{bc]}{}^r = 0 \tag{3.7}$$
$$h^a_I g^{It} = 0 \tag{3.8}$$
$$f_{rb}{}^a h^r_I - d^J_{rb}h^a_J h^r_I = 0 \tag{3.9}$$
$$g^{Js}h^r_K b_{Isr} - 2h^s_I h^r_K d^J_{rs} = 0 \tag{3.10}$$
$$-f_{rt}{}^s g^{It} + d^J_{rt}h^s_J g^{It} - g^{It}g^{Js}b_{Jtr} = 0 \tag{3.11}$$
$$b_{Jt(a}d^J_{bc)} = 0 \tag{3.12}$$

The 1-forms A^a take values in V with generators being noted X_a. The Leibniz product is defined, for any $X_a, X_b \in V$ by:

$$X_a \bullet X_b = -X_{ab}{}^c X_c \tag{3.13}$$

where $X_{ab}{}^c = -f^c_{ab} + d^I_{ab}h^c_I$ are the *structure constants* of the Leibniz algebra. The (skew)-symmetric brackets are thus defined by:

$$[X_a, X_b] = f_{ab}{}^c X_c \qquad \text{and} \qquad \{X_a, X_b\} = -d^I_{ab}h^c_I X_c \tag{3.14}$$

The 2-forms B^I take values in a sub-representation of $S^2(V_0)$ that we note W, that is assigned a -1 degree. A set of generators for W is noted $\{X_I\}$, and the action of $\mathfrak{g}$ on W is written as:

$$\rho_{\Theta(X_a)}(X_I) = -X_{aI}{}^J X_J \tag{3.15}$$

where $X_{aI}{}^J = 2h^c_I d^J_{ac} - g^{Js}b_{Isa}$, and where $X_a \in V$. Incidentally, in the $(1,0)$ superconformal model, the 3-forms C_t are dual to the 1-forms, taking values in $X \equiv V^*$ that is considered as a space of degree -2. By duality, the action of $\mathfrak{g}$ on a generator X^s of X is defined by:

$$\rho_{\Theta(X_a)}(X^s) = X_{at}{}^s X^t \tag{3.16}$$

where $X_{at}{}^s = -f_{at}{}^s + d^I_{at}h^s_I$. The 2-skeleton is hence defined by the following data (the signs and the symbols can directly be read on the Bianchi identities of the field strengths in [10]):

$$\begin{tikzcd}
V^* \arrow[r, "h^a_I"] & W^* \arrow[r, "g^{It}"] \arrow[dl, "d^I_{ab}"'] & X^* \arrow[dl, "-b_{Ita}"'] \\
V^* \odot V^* & V^* \otimes W^* &
\end{tikzcd}$$

where W and X are understood to be spaces of degree -1 and -2 respectively. This diagram says that, if X^a, X^I and X_t are the respective dual elements to X_a, X_I and X^t:

$$\begin{aligned}
\delta_1(X^a) &= h^a_I\, X^I &\qquad \text{and} \qquad& \delta_2(X^I) = g^{It}\, X_t,\\
\pi_0(X^I) &= d^I_{ab}\, X^a \odot X^b &\qquad \text{and} \qquad& \pi_1(X_t) = -b_{Ita}\, X^I \otimes X^a,\\
\mu_0(X^a) &= d^I_{bc}h^a_I\, X^b \odot X^c &\qquad \text{and} \qquad& \mu_1(X^I) = X_{aJ}{}^I\, X^a \otimes X^J
\end{aligned}$$

The corresponding embedding complex is $(V, \mathfrak{g}, \Theta, W)$, that is to say, we have $U_0 = V^*$, $U_1 = W^*$ and $U_2 = X^*$, and the Lie algebra of the embedding complex and the gauge algebra (the image of Θ) coincide in this model.

Under such circumstances, the map μ_1 indeed corresponds to the contragedient representation of $\mathfrak{g}$ on W. Equations (3.5) and (3.6) are those consisting of the equivariance of π_0

and π_1. Equation (3.7) corresponds to the Jacobi identity for the skew-symmetric bracket $[.,.]$, as seen in Equations (2.12) and (2.13). Equation (3.8) can be equivalently be seen as the equivariance of δ_1 or as the condition $\delta_2 \circ \delta_1 = 0$. Equations (3.9) and (3.10) are implied by $\delta_1 \circ \Theta^* = 0$, that is: $\mathcal{I} \subset \mathrm{Ker}(\Theta)$ (it is in fact an equality in this model). Equation (3.11) symbolizes the equivariance of δ_2, and Equation (3.12) is the condition $\pi^2|_{X^*} = 0$. The condition that μ is a null-homotopic map at levels 0 and 1 is satisfied by the specific choice of the maps.

Let us now turn to the problem of extending a given i-skeleton to a $i+1$-skeleton. The following Lemma gives a precise answer:

Lemma 3.2. *Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple, and let $i \geq 1$. Then any i-skeleton $\mathcal{U} = (U, \delta, \pi, \mu)$ associated to $\mathcal{V}$ induces a unique $i+1$-skeleton of $\mathcal{V}$ whose i-truncation is $\mathcal{U}$.*

Proof. The pitch is as follows: the space of degree $i+1$ will be uniquely defined to satisfy exactness of the map π (see item 6. in Definition 3.1). The proof will then mostly consist in showing that every other item is satisfied.

We define the vector space U_{i+1} as:

$$U_{i+1} = s\Big(\mathrm{Ker}\big(\pi|_{S^2(U)_i}\big)\Big)$$

We build the degree -1 injective map π_i by using the inclusion map: $\pi_i = \iota \circ s^{-1} : U_{i+1} \to S^2(U)_i$. In particular we have the following exact sequence:

$$0 \longrightarrow U_{i+1} \xrightarrow{\pi_i} S^2(U)_i \xrightarrow{\pi} S^3(U)_{i-1}$$

Since π is $\mathfrak{g}$-equivariant, $\mathrm{Ker}\big(\pi|_{S^2(U)_i}\big)$ is a $\mathfrak{g}$-sub-module of $S^2(U)_i$. Since π_i is injective, the $\mathfrak{g}$-module structure can be transported to U_{i+1}, turning it into a representation of $\mathfrak{g}$:

$$\rho_{i+1,a}(x) = s \circ \rho_{i,a}\big(\pi_i(x)\big) \tag{3.17}$$

By construction, the map π_i is $\mathfrak{g}$-equivariant. Hence items 1. and 6. of Definition 3.1 are satisfied. It is now time to show that there exists a map μ_i and a map δ_{i+1} that combine with π_i to satisfy all other items (in particular item 8.).

Since U_i admits a $\mathfrak{g}$-action $\rho_i : \mathfrak{g} \to \mathrm{End}(U_i)$, this representation defines a map $\widetilde{\rho_i} : U_i \to \mathfrak{g}^* \otimes U_i$. As in item 7., this map can be lifted to a map $\mu_i : U_i \to U_0 \otimes U_i$ by composition with Θ^*:

$$\begin{tikzcd}
 & U_0 \otimes U_i \\
U_i \arrow[ur, "\mu_i"] \arrow[r, "\widetilde{\rho_i}"] & \mathfrak{g}^* \otimes U_i \arrow[u, "\Theta^* \otimes \mathrm{id}"']
\end{tikzcd}$$

The map Θ^* being the dual map of Θ, $\mathrm{Im}(\Theta^*) = \big(\mathrm{Ker}(\Theta)\big)^\circ$. Setting $\mathfrak{h} \equiv \mathrm{Im}(\Theta) \subset \mathfrak{g}$, we have the following result:

$$\mathrm{Im}(\mu_i) = \big(\mathrm{Ker}(\Theta)\big)^\circ \otimes U_i \simeq \mathfrak{h}^* \otimes U_i$$

This result is actually true for every $k \in \{0, \dots, i\}$. Now let $a \in \mathfrak{h}$ and $x \in U_{i-1}$, then, identifying $\mathfrak{h}^* \otimes U_i$ with $\mathrm{Hom}(\mathfrak{h}, U_i)$, we have:

$$\rho_{i,a}\big(\delta_i(x)\big) = \mu_i\big(\delta_i(x)\big)(a) \tag{3.18}$$

Since $\mathrm{Ker}(\delta_1) = \mathcal{I}^\circ$ and that $\mathfrak{h}^* \subset \mathcal{I}^\circ$, we obtain:

$$\delta_i\big(\rho_{i-1,a}(x)\big) = \delta_i\big(\mu_{i-1}(x)(a)\big) = \big((\delta_1 \otimes \mathrm{id} + \mathrm{id} \otimes \delta_i) \circ \mu_i(x)\big)(a) \tag{3.19}$$

The differential δ is not $\mathfrak{g}$-equivariant, but it is $\mathfrak{h}$-equivariant. Thus we can conclude that:

$$\delta \circ \mu_{i-1} = \mu_i \circ \delta_i \tag{3.20}$$

as shown in the following commutative diagram:

$$
\begin{tikzcd}
U_{i-1} \arrow[r, "\delta_i"] \arrow[d, "\mu_{i-1}"] & U_i \arrow[d, "\mu_i"] \\
S^2(U)_{i-1} \arrow[r, "\delta"] & S^2(U)_i
\end{tikzcd}
$$

The map π_i is $\mathfrak{g}$-equivariant, then following the same arguments, we obtain:

$$\mu \circ \pi_{i-1} = \pi \circ \mu_i \tag{3.21}$$

as illustrated in the following commutative diagram:

$$
\begin{tikzcd}
 & U_i \arrow[dl, hook', "\pi_{i-1}"'] \arrow[d, "\mu_i"] \\
S^2(U)_{i-1} \arrow[d, "\mu"'] & S^2(U)_i \arrow[dl, "\pi"'] \\
S^3(U)_{i-1} &
\end{tikzcd}
$$

Let us show that μ_i is naturally $\mathfrak{h}$-equivariant. Let $a, b \in \mathfrak{h}$ and let $x \in U_i$. Then on the one side:

$$\rho_{i,b}\big(\rho_{i,a}(x)\big) = \mu_i\big(\rho_{i,a}(x)\big)(b) \tag{3.22}$$

and on the other side:

$$\begin{aligned}\rho_{i,b}\big(\rho_{i,a}(x)\big) &= -\rho_{i,[a,b]}(x) + \rho_{i,a}\big(\rho_{i,b}(x)\big) \\ &= \big((\rho_a^* \otimes \mathrm{id} + \mathrm{id} \otimes \rho_{i,a}) \circ \mu_i(x)\big)(b)\end{aligned} \tag{3.23}$$

where ρ^* is the coadjoint action of $\mathfrak{h}$ on $\mathfrak{h}^*$. By still denoting by ρ_i the action of $\mathfrak{g}$ on $S(U)$ (extended naturally by derivation), we finally obtain:

$$\mu_i\big(\rho_{i,a}(x)\big) = \rho_{i,a}\big(\mu_i(x)\big) \tag{3.24}$$

which proves that μ_i is indeed $\mathfrak{h}$-equivariant. Notice that we cannot prove that it is $\mathfrak{g}$-equivariant because the fact that $\mathrm{Im}(\mu_i) \simeq \mathfrak{h}^* \otimes U_i$ was at the core of the proof.

Now that μ_i has been defined, we will show that a map $\delta_{i+1} : U_i \to U_{i+1}$ that satisfies item 2. and 8. of Definition 3.1 exists, is well-defined and unique. First, let us define a degree 0 map h_i by:

$$\begin{array}{rccc} h_i : & U_i & \longrightarrow & S^2(U)_i \\ & x & \longmapsto & \mu_i(x) - \delta \circ \pi_{i-1}(x) \end{array}$$

and we extend it to all of $S(U)$ by derivation. Then, using the identity $\mu = \pi \circ \delta + \delta \circ \pi$ at level $i-1$ (see item 8.), we have the following inclusions:

$$\mathrm{Im}(h_i) \subset \mathrm{Ker}\big(\pi|_{S^2(U)_i}\big) \qquad \text{and} \qquad \mathrm{Im}\big(\delta_i\big) \subset \mathrm{Ker}(h_i).$$

The first inclusion is induced by Equation (3.21) and by the fact that $\pi^2 = 0$ (see item 6.), whereas the second inclusion is induced by Equation (3.20) and by the fact that $\delta^2 = 0$.

Now, let us show that h_i factors through U_{i+1}, i.e. that there exists a unique map $\delta_{i+1} : U_i \to U_{i+1}$ such that the following triangle is commutative:

$$\begin{tikzcd}
U_i \arrow[r, "\delta_{i+1}"] \arrow[d, "h_i"'] & U_{i+1} \arrow[dl, hook, "\pi_i"'] \\
S^2(U)_i &
\end{tikzcd}$$

We first define it. Let $x \in U_i$. Since $\mathrm{Im}(h_i) \subset \mathrm{Ker}(\pi|_{S^2(U)_i})$ and since $\mathrm{Ker}(\pi|_{S^2(U)_i}) = \mathrm{Im}(\pi_i)$, then $h_i(x) \in \mathrm{Im}(\pi_i)$. Thus there exists a unique $u \in U_{i+1}$ such that $\pi_i(u) = h_i(x)$. This choice is unique by injectivity of π_i, then we set:

$$\delta_{i+1}(x) = u \tag{3.25}$$

This automatically implies that $\mathrm{Ker}(h_i) \subset \mathrm{Ker}(\delta_{i+1})$. By the inclusion $\mathrm{Im}(\delta_i) \subset \mathrm{Ker}(h_i)$, we deduce that:

$$\mathrm{Im}\big(\delta_i\big) \subset \mathrm{Ker}(\delta_{i+1})$$

This allows to extend the chain complex (U, δ) one step further.

The $\mathfrak{h}$-equivariance of δ_{i+1} is guaranteed by the fact that μ_i and π_i are both $\mathfrak{h}$-equivariant. Indeed, let $a \in \mathfrak{h}$, let $x \in U_i$, and let $u \in U_{i+1}$ be the (unique) image of x through δ_{i+1} (as in Equation (3.25)). By definition, there exists a unique $v \in U_{i+1}$ such that $\delta_{i+1}\big(\rho_{i,a}(x)\big) = v$. Let us show that $v = \rho_{i+1,a}(u)$, so that we will have:

$$\rho_{i+1,a}\big(\delta_{i+1}(x)\big) = \delta_{i+1}\big(\rho_{i,a}(x)\big) \tag{3.26}$$

We know that $h_i\big(\rho_{i,a}(x)\big) = \pi_i(v)$. But μ_i, π_i and the differential δ are $\mathfrak{h}$-equivariant, hence h_i is $\mathfrak{h}$-equivariant as well, then we have:

$$\pi_i(v) = \rho_{i,a}\big(h_i(x)\big) = \rho_{i,a}\big(\pi_i(u)\big) = \pi_i\big(\rho_{i+1,a}(u)\big) \tag{3.27}$$

Since the map π_i is injective, we deduce that $v = \rho_{i+1,a}(u)$, proving the $\mathfrak{h}$-equivariance of δ_{i+1}.

By construction, the quadruple $\big((U_k)_{0\leq k\leq i+1}, (\delta_k)_{1\leq k\leq i+1}, (\pi_k)_{0\leq k\leq i}, (\mu_k)_{0\leq k\leq i}\big)$ thus satisfies every axioms of Definition 3.1, hence it defines an $i+1$-skeleton of $\mathcal{V}$, and its i-truncation restricts to (U, δ, π, μ). □

Example 6. We have seen in Example 5 a 2-skeleton that appears in the $(1,0)$ superconformal model. In [10], the hierarchy of differential forms A^a, B^I, C_t can be extended by adding a set of 4-forms D_α that take values in a $\mathfrak{g}$-sub-module Y of $(V \otimes X) \oplus (W \odot W)$. The space Y is defined such that Y^* is the kernel of the map $\pi|_{S^2(U)|_2}$, and it is considered as a space of degree -3. The set of new objects satisfy a set of consistency equations:

$$g^{Kt}k^\alpha_t = 0 \tag{3.28}$$
$$4d^J_{ab}c_{\alpha IJ} - b_{Ita}c^t_{\alpha b} - b_{Itb}c^t_{\alpha a} = 0 \tag{3.29}$$
$$k^\alpha_t c_{\alpha IJ} - h^a_{[I}b_{J]ta} = 0 \tag{3.30}$$
$$k^\alpha_t c^s_{\alpha a} - f_{ta}{}^s + b_{Jta}g^{Js} - d^J_{ta}h^s_J = 0 \tag{3.31}$$

These define a 3-skeleton language as follows (the Bianchi identities are given in [10]):

$$\begin{tikzcd}
V^* \arrow[r, "h^a_I"] & W^* \arrow[r, "g^{It}"] \arrow[dl, hook, "d^I_{ab}"'] & X^* \arrow[r, "k^\alpha_t"] \arrow[dl, hook, "-b_{Ita}"'] & Y^* \arrow[r, dotted] \arrow[dl, hook, "-c^s_{\alpha a} + c_{\alpha IJ}"'] & \cdots \arrow[dl, hook, dotted] \\
V^* \odot V^* & V^* \otimes W^* & \begin{matrix} W^* \odot W^* \\ \oplus\, V^* \otimes X^* \end{matrix} & \cdots &
\end{tikzcd}$$

The new maps are:

$$\pi_2(X_\alpha) = -c^t_{\alpha a}\, X_t \otimes X^a + c_{\alpha IJ}\, X^I \odot X^J,$$
$$\delta_2(X_t) = k^\alpha_t\, X_\alpha \qquad \text{and} \qquad \mu_2(X_t) = -X_{at}{}^s\, X^a \otimes X_s$$

The presence of a minus sign in the definition of μ_2 was expected because the index labelling the three forms is at the bottom. Equation (3.28) corresponds to the homological condition $\delta_{-2} \circ \delta_{-1} = 0$, and Equation (3.29) corresponds to the condition $\pi^2|_{U^*_{-3}} = 0$. Equation (3.31) can be written as $f_{at}{}^s - d^J_{at} h^s_J = -k^\alpha_t c^s_{\alpha a} - b_{Jta} g^{Js}$. The left hand side is the structure constant $-X^s_{at}$ of the contragredient action of $\mathfrak{g}$ on X^*, whereas the right hand side corresponds to $\pi_2 \circ \delta_{-2} + \delta \circ \pi_1$. Together with Equation (3.30) we obtain the null-homotopic condition satisfied by μ at level 2: $\mu_2 = \pi_2 \circ \delta_{-2} + \delta \circ \pi_1$. Recall that W is considered as a space of degree -1 hence the symmetric product $W^* \odot W^*$ is actually skew-symmetric on I, J indices. Finally, we notice that Equation (3.11) is obtained by contracting Equation (3.31) with g^{It}.

We observe that the structure of an i-skeleton $\mathcal{U}$ is so rigid, that it uniquely defines the $i+1$-skeleton whose i-truncation is $\mathcal{U}$. Then, given a Lie-Leibniz triple $\mathcal{V} = (V, \mathfrak{g}, \Theta)$, one can start from the 1-skeleton obtained from the data contained in an embedding complex $(V, \mathfrak{g}, \Theta, W)$ to build an i-skeleton associated to $\mathcal{V}$ by applying inductively Lemma 3.2. This i-skeleton is uniquely defined from the choice of W. We can apply the same Lemma again and again, to extend the skeleton to higher degrees. Reproducing the process up to infinity, we are led to the following result:

Proposition 3.3. *Given a Lie-Leibniz triple $\mathcal{V} = (V, \mathfrak{g}, \Theta)$, there is a one-to-one correspondence between ∞-skeletons associated to $\mathcal{V}$ and embedding complexes $(V, \mathfrak{g}, \Theta, W)$.*

We have seen in section 2.2, that any Leibniz algebra admits a set of gauge algebras $\mathfrak{h}_\mathcal{K}$, constructed from a central ideal containing the ideal of squares $\mathcal{I}$. We have seen in Lemma 2.2 that the action of a particular gauge algebra on V is independent of the choice of the ideal $\mathcal{K}$. And we have deduced that the lifts of $\{.,.\}$ are independent of the choice of the gauge algebras. Then, by Proposition 3.3, we know that any choice of an embedding complex $(V, \mathfrak{h}_\mathcal{I}, \Theta_\mathcal{K}, W)$ induces a unique ∞-skeleton. But since each space of the infinite tower is a sub-representation of the symmetric algebra, we deduce by induction that none of them depends on the choice of the gauge algebra $\mathfrak{h}_\mathcal{K}$, and that the ∞-skeleton associated to an embedding complex $(V, \mathfrak{h}_\mathcal{I}, \Theta_\mathcal{K}, W)$ relies only on the choice of W (once V is fixed).

As a final remark, given an embedding complex $(V, \mathfrak{g}, \Theta, W)$, we obtain by Proposition 3.3 an infinite tower of vector spaces U_i of degree i that satisfy all the axioms of Definition 3.1. Item 8. of the definition, together with the fact that the maps δ, π, μ can be extended to all of $S(U)$ as derivations, implies the following result:

Proposition 3.4. *The degree 0 map $\mu : S^\bullet(U) \to S^{\bullet+1}(U)$ is a null-homotopy between the chain complexes $\left(S^k(U), \delta\right)_{1 \leq k}$:*

$$U \xrightarrow{\mu} S^2(U) \xrightarrow{\mu} S^3(U) \xrightarrow{\mu} \ldots$$

3.2 Unveiling the tensor hierarchy algebra

We have shown in the last section that any Lie-Leibniz triple induces an ∞-skeleton. This structure will be at the core of the construction of tensor hierarchies. This section is devoted to showing how to build a tensor hierarchy algebra from the data of an ∞-skeleton. We first show a short Lemma to give rise to a graded Lie bracket, and then we give the formal construction of the tensor hierarchy algebra, and we explain why the axioms of Definition 2.9 are satisfied by construction.

Let us fix an embedding complex $\mathcal{V} = (V, \mathfrak{g}, \Theta, W)$, and let $\mathcal{U} = (U, \delta, \pi, \mu)$ be the unique ∞-skeleton associated to this 4-tuple by Proposition 3.3. Let us turn to the construction of the tensor hierarchy algebra associated to the ∞-skeleton $\mathcal{U}$. Let T be the dual space of the suspension of the graded vector space U:

$$T \equiv s^{-1}(U^*)$$

In other words, $T = (T_{-i})_{i\geq 1}$, with $T_{-1} = s^{-1}V$, $T_{-2} = s^{-1}(U_1^*) = s^{-2}W$, $T_{-3} = s^{-1}(U_2^*)$, and more generally:

$$T_{-i} = s^{-1}(U_{i-1}^*)$$

for any $i \geq 1$. Each vector space T_{-i} is a $\mathfrak{g}$-module, since U_{i-1} is a $\mathfrak{g}$-module. We then have the following result:

Lemma 3.5. *Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple and let $\mathcal{U} = (U, \delta, \pi, \mu)$ be an ∞-skeleton associated to $\mathcal{V}$. Then $T \equiv s^{-1}(U^*)$ inherits a graded Lie algebra structure.*

Proof. Consider the space $s^{-1}T = s^{-2}(U^*)$ which is the graded vector space U^* whose elements have their degree shifted by -2. Since the only modification is that the degree of every element has been shifted by an even number, the map $\pi : U \to S^2(U)$ can also be seen as a linear application $Q_\pi : (s^{-1}T)^* \to S^2\big((s^{-1}T)^*\big)$, and extended to all of $S\big((s^{-1}T)^*\big)$ by derivation. This algebra being the algebra of functions on $s^{-1}T$, this turns Q_π into a vector field on $s^{-1}T$. For degree reasons, the degree of Q_π is no longer -1 but it is $+1$. Moreover it is of arity 1 because π is a map from U to $S^2(U)$. And finally, the identity $(\pi)^2 = 0$ implies that Q_π is a homological vector field on the pointed graded manifold with fiber $s^{-1}T$. In other words, $(s^{-1}T, Q_\pi)$ is a pointed differential graded manifold. Then by Theorem 2.8, we can use the correspondence between a homological vector field od degree $+1$ and of arity 1 and a graded Lie algebra structure.

The graded Lie algebra structure on $T \equiv s^{-1}(U^*)$ is induced by the linear application $Q_\pi : (s^{-1}T)^* \to S^2\big((s^{-1}T)^*\big)$. For any $u \in s^{-1}T$, we define ι_u as the inner derivation of $S\big((s^{-1}T)^*\big)$ which satisfies:

$$\iota_u(\alpha) = \langle \alpha, u\rangle \tag{3.32}$$

for any $\alpha \in (s^{-1}T)^*$. We have a natural identification $u \leftrightarrow \iota_u$, and thus by Theorem 2.8, the graded Lie bracket $[\,.\,,.]'$ on $T = s^{-1}(U^*)$ is given by:

$$\forall\, x, y \in T \qquad\qquad \iota_{s^{-1}[x,y]'} = (-1)^{|x|}\big[[Q_\pi, \iota_{s^{-1}(x)}], \iota_{s^{-1}(y)}\big] \tag{3.33}$$

where on the right side, the bracket is the (graded) bracket of vector fields on the pointed graded manifold with fiber $s^{-1}T$. The sign $(-1)^{|x|}$ in front of the term on the right hand side is necessary to enforce the graded skew symmetry of the bracket. Indeed, due to this sign, for any $x, y \in T$ we have:

$$[x, y]' = -(-1)^{|x||y|}[y, x]' \tag{3.34}$$

This Lie bracket is of degree 0 and the Jacobi identity is satisfied because it is equivalent to the fact that π squares to zero. Thus $T = s^{-1}(U^*)$ can be equipped with a graded Lie algebra structure. □

Now we would like to use T to define a tensor hierarchy algebra that would be associated to the Lie-Leibniz pair $(\mathfrak{g}, V, \Theta)$. For this, we first need to find a differential graded Lie algebra structure on T. Natural candidates to define the differential and the graded Lie bracket would be the maps δ and π of the ∞-skeleton $\mathcal{U}$. This is actually the object of the main theorem of this paper:

Theorem 3.6. *Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple and let $\mathcal{U} = (U, \delta, \pi, \mu)$ be an ∞-skeleton associated to $\mathcal{V}$. Then the graded vector space $\mathfrak{T}_\mathcal{U} \equiv s(T_\Theta) \oplus \mathfrak{g} \oplus s^{-1}(U^*)$ inherits a tensor hierarchy algebra structure.*

Proof. By Lemma 3.5, we know that $T \equiv s^{-1}(U^*)$ can be equipped with a graded Lie algebra bracket $[\,.\,,.]'$, that descends from the map π. This bracket will be be involved in the tensor hierarchy algebra structure, but we have to extend it to all of $\mathfrak{T}_\mathcal{U}$. Before turning to this issue, let us define the map δ' on $s^2(U)$ as the map that acts as δ on U:

$$\forall u \in U \qquad\qquad \delta'(u) = s^2 \circ \delta \circ s^{-2}(u) \tag{3.35}$$

Then, let $\partial' = (\partial_{-k})_{1\leq k}$ be the family of degree $+1$ maps $\partial_{-k} : T_{-k-1} \to T_{-k}$ defined by:

$$\forall\; x \in T_{-k-1} \qquad\qquad \iota_{s^{-1}(\partial_{-k}(x))} = -[\delta'_k, \iota_{s^{-1}(x)}] \tag{3.36}$$

We thus obtain a chain complex:

$$\ldots \longrightarrow T_{-3} \xrightarrow{\partial_{-2}} T_{-2} \xrightarrow{\partial_{-1}} T_{-1} \longrightarrow 0$$

Now we add a vector space of degree 0 and a vector space of degree +1 to the graded Lie algebra $T = s^{-1}(U^*)$. We set $T_0 = \mathfrak{g}$ on the one side, and we require that T_1 is the $\mathfrak{g}$-module T_Θ to which the embedding tensor Θ belongs. In that sense, Θ is an element of T_1, and it is seen as having degree +1. We now have to define the two remaining maps $\partial_0 : T_{-1} \to T_0$ and $\partial_1 : T_0 \to T_1$ that extend the differential ∂'.

First, we define a degree +1 map $\partial_0 : T_{-1} \to T_0$ by composition:

$$\partial_0 = -\Theta \circ s \tag{3.37}$$

It has degree one, and its image coincides with the gauge algebra $\mathfrak{h} \equiv \mathrm{Im}(\Theta)$. We immediately have that $\mathrm{Im}(\partial_{-1}) \subset \mathrm{Ker}(\partial_0)$ by duality of the inclusion $\mathrm{Im}(\Theta^*) \subset \mathrm{Ker}(\delta_1)$. Hence the complex (T, ∂') can be extended through $\mathfrak{g}$:

$$\ldots \longrightarrow T_{-3} \xrightarrow{\partial_{-2}} s^{-2}W \xrightarrow{\partial_{-1}} s^{-1}V \xrightarrow{\partial_0} \mathfrak{g} \longrightarrow 0$$

We now define the map $\partial_1 : T_0 \to T_1$ by using the fact that Θ transforms under the action of $\mathfrak{g}$ via the representation $\widehat{\rho}$. We set, for any $a \in \mathfrak{g}$:

$$\partial_1(a) = -\widehat{\rho}_a(\Theta) \tag{3.38}$$

In particular, if $a \in \mathfrak{h}$, the quadratic constraint imposes that the right hand side vanishes. This implies that $\partial_1 \circ \partial_0 = 0$, and that the above chain complex can be extended one step further:

$$\ldots \longrightarrow T_{-3} \xrightarrow{\partial_{-2}} s^{-2}W \xrightarrow{\partial_{-1}} s^{-1}V \xrightarrow{\partial_0} \mathfrak{g} \xrightarrow{\partial_1} s(T_\Theta) \longrightarrow 0$$

In the following we will change the notations and we set $\partial \equiv (\partial_{-k})_{-1 \leq k}$. Let us summarize what we have so far:

1. a (possibly infinite) tower of $\mathfrak{g}$-modules $\mathfrak{T}_\mathcal{U} = (T_{-i})_{i \geq_1}$;
2. a differential $\partial = (\partial_{-i} : T_{-i-1} \to T_{-i})_{\geq -1}$;
3. a graded Lie algebra bracket $[.\,,.]'$ on $T = (T_{-i})_{i \geq 1}$ (by Lemma 3.5).

Hence, we need to extend the bracket $[.\,,.]'$ to $\mathfrak{g} \otimes \mathfrak{T}_\mathcal{U}$ and $T_1 \otimes \mathfrak{T}_\mathcal{U}$ to obtain a graded Lie bracket $[.\,,.]$ on all of $\mathfrak{T}_\mathcal{U}$, such that the given differential ∂ and the extended bracket $[.\,,.]$ are compatible, as explained in Definition 2.6. We will see that, given the relationship between Θ and the differential ∂, this compatibility condition (2.32) will in fact be expressed by the Jacobi identities involving the embedding tensor.

The Lie algebra $\mathfrak{g}$ comes equipped with its own Lie bracket. We can define the graded Lie bracket between an element $a \in \mathfrak{g}$ and an element of $x \in T_{-i}$ (for $i \geq -1$) by:

$$[a, x] \equiv \rho_{-i+1,a}(x) \tag{3.39}$$

where $\rho_{-i+1} = \rho^*_{i-1}$ is the contragredient action of $\mathfrak{g}$ on $U^*_{i-1} \simeq T_{-i}$. One could even extend this definition to T_0 because this is consistent with the fact that the action of $\mathfrak{g}$ on itself is the adjoint action. For any $a, b \in \mathfrak{g}$ and $x \in T$, the Jacobiator $\mathrm{Jac}(a, b, x)$ turns to be zero because in that case the Jacobi identity corresponds to the condition that the graded vector space $\mathfrak{T}_\mathcal{U}$ is a family of Lie algebra representations. On the other hand, for $a \in \mathfrak{g}$ and $x, y \in T$, the Jacobiator $\mathrm{Jac}(a, x, y)$ vanishes as well, but this time because the map π is $\mathfrak{g}$-equivariant, and that the bracket inherits this property descends through Equation (3.33).

The only remaining Jacobi identities to check are those involving Θ. First, we notice that there are no space of degree higher than +1 in the new family $\mathfrak{T}_\mathcal{U}$. Then for degree reasons, the bracket on T_1 must vanish. And since Θ can be seen as an element of T_1, in particular it implies that $[\Theta, \Theta] = 0$. The procedure to extend the graded Lie bracket to T_1 then goes as follows: first define the bracket between Θ and any other element of lower degree by the following equality:

$$[\Theta, .\,] \equiv \partial \tag{3.40}$$

The fact that $[\Theta,\Theta]=0$ is consistent with the homological property of the differential ∂. As an interesting consequence, we observe that given any $x\in V$, the Jacobi identity:

$$\big[x,[\Theta,\Theta]\big]+\big[\Theta,[\Theta,x]\big]+\big[\Theta,[x,\Theta]\big]=0 \tag{3.41}$$

is nothing but the quadratic constraint $\widehat{\rho}_{\Theta(x)}(\Theta)=0$. Moreover, we see that the definition of $[\Theta,.]$ in Equation (3.40) is consistent with formula (3.39), because one passes from one to another by using Equation (3.38).

We now turn to define the bracket between elements of T and some other elements of T_1. Let S be the subspace of T_1 that is spanned by elements of the form $\widehat{\rho}_a(\Theta)$ for every $a\in\mathfrak{g}$. In other words it is the image of ∂_1. For every $a\in\mathfrak{g}$, we define the bracket between an element $x\in T_{-i}$ (for $i\geq 1$) and elements of S as:

$$\big[\widehat{\rho}_a(\Theta),x\big]\equiv\rho_{-i+2,a}\big(\partial_{-i+1}(x)\big)-\partial_{-i+1}\big(\rho_{-i+1,a}(x)\big) \tag{3.42}$$

and with an element $b\in\mathfrak{g}$ by:

$$\big[\widehat{\rho}_a(\Theta),b\big]\equiv-\widehat{\rho}_a\circ\widehat{\rho}_b(\Theta)+\widehat{\rho}_{[a,b]}(\Theta)=-\widehat{\rho}_b\circ\widehat{\rho}_a(\Theta) \tag{3.43}$$

which is coherent with Equation (3.39). By Equations (3.38), (3.39) (3.40), these definitions are nothing but the corresponding Jacobi identity for a,x,Θ and a,b,Θ, respectively. Also, the right-hand side of Equation (3.42) can be thought of as $\rho_a(\partial)(x)$, hence it is consistent with the fact that the map ∂_{-i+1} transforms in the same representation as Θ (it is the linear contraint that implies this). In particular, if $a\in\mathfrak{h}$, the left hand side vanish by the quadratic constraint, and we recover the result that ∂_{-i+1} is $\mathfrak{h}$-equivariant. For elements of T_1 which are not in S, the bracket with x is zero, whereas the bracket with b as already defined in Equation (3.39), which ends the proof.

Equations (3.41), (3.42) and (3.43) are three sets of Jacobi identities involving Θ. The only ones left are the Jacobi identities involving two elements $x,y\in T$, and Θ. They are the last Jacobi identities to check in order to confirm that $\mathfrak{T}_\mathcal{U}$ is a tensor hierarchy algebra. We will show that this set of Jacobi identities, are equivalent to the compatibility condition between the differential ∂ and the graded Lie bracket $[\,.\,,.]$ on T. Hence we only have to show that for any $x,y\in T=(T_{-i})_{i\geq1}$, the compatibility condition (2.32) is equivalent to the vanishing of the Jacobiator $\mathrm{Jac}(\Theta,x,y)$.

Let us set $U'=\bigoplus_{1\leq k<\infty}U_k$, and its shifted dual: $T'=s^{-1}(U'^*)=\bigoplus_{2\leq k<\infty}T_{-k}$. The corestriction of the map $\mu=[\delta,\pi]$ to $S^2(U')$ is identically zero, because μ_k takes values in $U_0\otimes U_k$. Then we are left with the equality $\delta\circ\pi+\pi\circ\delta=0$. This identity translates as the (graded) commutativity of Q_π and δ' on the pointed graded manifold $s^{-1}T'$. Hence, it implies by duality that the differential ∂ and the bracket on T' commute or, in other words:

$$\partial\big([x,y]\big)=[\partial(x),y]+(-1)^{|x|}[x,\partial(y)] \tag{3.44}$$

for any two homogeneous elements $x,y\in T'$. This equation is nothing but the Jacobi identity for $x,y\in T'$ and Θ. Hence there is only one very last Jacobi identity to check: it is the one with $x\in T_{-1},y\in T$ and Θ. It will be proven by using the null-homotopy property of μ. We now study the corestriction $\mu_k|^{U_0\otimes U_k}$ of μ_k to $U_0\otimes U_k$, for $k\geq1$. In that case we can precisely write the null-homotopy condition as:

$$\mu_k|^{U_0\otimes U_k}=\delta_k\circ\pi_{k-1}+\pi_k\circ\delta_{k+1} \tag{3.45}$$

The two terms on the right-hand side give the following terms on $x\in T_{-1}$ and $y\in T_{-k-1}$: $\big[x,\partial(y)\big]-\partial\big([x,y]\big)$, whereas the left-hand side dually gives the contragredient action of $\mathrm{Im}(\Theta)\subset\mathfrak{g}$ on T_{-k-1}, that is: $\rho_{-k,\Theta(x)}(y)=-[\partial(x),y]$. Gathering every elements together, we obtain Equation (3.44) for $x\in T_{-1}$ and $y\in T'$. The case for $x,y\in T_{-1}$ is made in the same way. Hence, the last Jacobi identities that we were looking for are satisfied by construction. In other words, we have shown that $\big(\mathfrak{T}_\mathcal{U},\partial,[\,.\,,.]\big)$ is a tensor hierarchy algebra. $\square$

To conclude, we have shown that, given the ∞-skeleton $\mathcal{U}$ associated to an embedding complex, the graded vector space $\mathfrak{T}_\mathcal{U}=(T_{-i})_{i\geq-1}$ that we constructed through the proof can be equipped with a unique differential graded Lie algebra structure that satisfies the axioms of Definition 2.9. We emphasize that it is precisely the tensor hierarchy algebra that is obtained in [5], with opposite degrees. We have shown in the present section that the construction of the tensor hierarchy algebra $\mathfrak{T}_{\mathcal{U}}$ is systematic and uniquely defined by $\mathcal{U}$. Recalling Proposition 3.3, we can deduce an important result on the classification of tensor hierarchies:

Proposition 3.7. *Let $\mathcal{V} = (V, \mathfrak{g}, \Theta)$ be a Lie-Leibniz triple. Then there is a one-to-one correspondence between tensor hierarchies associated to $\mathcal{V}$ and embedding complexes $(V, \mathfrak{g}, \Theta, W)$.*

Example 7. We now give the first objects that define the tensor hierarchy algebra that corresponds to the 3-skeleton of Example 6. Considering that the $\mathfrak{g}$-modules V, W, X and Y defined in Example 6 have respective degrees $0, -1, -2$ and -3, we define $T_{-1} \equiv s^{-1}V$, $T_{-2} \equiv s^{-1}W$, $T_{-3} \equiv s^{-1}X$ and $T_{-4} \equiv s^{-1}Y$, so that T_{-i} can be considered as a space of degree $-i$, as desired. The homological vector field Q_π corresponding to the map π acts on functions on $(s^{-1}T_{-i})_{1\leq i\leq 4}$ as follows:

$$Q_\pi(e^a) = 0 \tag{3.46}$$
$$Q_\pi(e^I) = d^I_{bc} e^b \odot e^c \tag{3.47}$$
$$Q_\pi(e_t) = -b_{Ita} e^I \odot e^a \tag{3.48}$$
$$Q_\pi(e_\alpha) = c_{\alpha IJ} e^I \odot e^J - c^t_{\alpha a} e_t \odot e^a \tag{3.49}$$

where $e^{*a}, e^{*b}, e^{*c} \in (s^{-1}T_{-1})^*$, $e^{*I}, e^{*J} \in (s^{-1}T_{-2})^*$, $e^*_t \in (s^{-1}T_{-3})^*$ and $e^*_\alpha \in (s^{-1}T_{-4})^*$. This provides the following graded Lie algebra structure on $T \equiv (T_{-i})_{1\leq i\leq 4}$ through Theorem 2.8:

$$[e_a, e_b] = -2\, d^I_{ab}\, e_I \tag{3.50}$$
$$[e_a, e_I] = b_{Ita}\, e^t \tag{3.51}$$
$$[e_I, e_J] = -2\, c_{\alpha IJ}\, e^\alpha \tag{3.52}$$
$$[e_a, e^t] = c^t_{\alpha a}\, e^\alpha \tag{3.53}$$

where $e_a, e_b \in T_{-1}$, $e_I, e_J \in T_{-2}$, $e^t \in T_{-3}$ and $e^\alpha \in T_{-4}$. Since the degree of e_a, e_b is -1, the bracket is symmetric, whereas the bracket of e_I, e_J is skew-symmetric, for they have degree -3. The differential ∂' on T is defined by Equation (3.36):

$$\partial_{-1}(e_I) = -h^a_I\, e_a\,, \qquad \partial_{-2}(e^t) = -g^{It}\, e_I \quad \text{and} \quad \partial_{-3}(e^\alpha) = -k^\alpha_t\, e^t \tag{3.54}$$

These objects form the beginning of the tensor hierarchy algebra corresponding to the $(1,0)$ superconformal model in six dimensions, and we cannot go further because higher fields have not been defined in [10].

To this structure, we add $T_0 \equiv \mathfrak{g} = {}^{V}\!/_{\mathcal{I}}$ and $T_1 \equiv \mathbb{R}[\Theta]$, and their respective differentials and brackets with elements of T using Equations (3.37), (3.38), (3.39) and (3.40). The space T_1 is one-dimensional because by Equation (2.23), the action of $\mathfrak{g}$ on the embedding tensor Θ is trivial. By the above discussion, this turns $\mathfrak{T} = (T_{-i})_{-1\leq i\leq 4}$ into a differential graded Lie algebra. The Jacobi/Leibniz identities that we can compute, with respect to the corresponding objects we have in our possession, give back Equations (3.5)-(3.12), and (3.28)-(3.31), as well as the definitions of the tensors $X_{ab}{}^c$, $X_{aI}{}^J$ and $X_{as}{}^t$, and no more. In other words, the data of the tensor hierarchy is completely contained in this differential graded algebra $\big(\mathfrak{T}, \partial, [\,.\,,.\,]\big)$. This shows how this tensor hierarchy algebra is the correct object to look at when considering the $(1,0)$ superconformal model in six dimensions.

Remark. Given a Leibniz algebra V, a L_∞-algebra lifting the skew-symmetric bracket is called an L_∞-*extension* of the Leibniz algebra V. L_∞-algebras are algebraic objects that were invented thirty years ago, and they were actually found in many fields of mathematical physics, in particular in string theory. In [6, 8] for example, it is shown that there is an

L_∞-algebra hidden in the $(1,0)$ superconformal model in six dimensions [10]. We see that there is an intricate link between tensor hierarchies and L_∞-algebras, and that the tensor hierarchy algebras associated to V can provide a set of L_∞-extensions of V. This point will be explored further in an upcoming paper. This is a simple application of a result by [3, 4] that states that any differential graded Lie algebra structure on $L = (L_i)_{i\in\mathbb{Z}}$ induces an L_∞-algebra structure on $L' = \big(s(L_i)\big)_{i\leq -1}$ (in our notations).

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