MINIMAL NONNILPOTENT LEIBNIZ ALGEBRAS

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Abstract. We classify all nonnilpotent, solvable Leibniz algebras with the property that all proper subalgebras are nilpotent. This generalizes the work of [12] and [13] in Lie algebras. We show several examples which illustrate the differences between the Lie and Leibniz results.

1. Introduction

Leibniz algebras were defined by Loday in [11]. They are a generalization of Lie algebras, removing the restriction that the product must be anti-commutative or that the squares of elements must be zero. One immediate consequence of this is that while the Lie algebra generated by a single element is necessarily one-dimensional, the Leibniz algebra generated by a single element (called a cyclic algebra) could be of any dimension.

Recent work in Leibniz algebra often involves studying certain classes of Leibniz algebras, such as cyclic algebras [5], algebras with a certain nilradical [3, 9], or algebras of a certain dimension [7, 8, 10]. Many of these articles involve generalizing results from Lie algebras to Leibniz algebras. Some of these results only hold over the field of complex numbers.

An algebra L is called *minimal nonnilpotent* if L is nonnilpotent, solvable, and all proper subalgebras of L are nilpotent. Minimal nonnilpotent Lie algebras were studied by Stitzinger in [12]. Later Towers classified all such Lie algebras in [13]. It is the goal of this work to generalize these results to Leibniz algebras. Our results hold over any field.

2. Results

A Leibniz algebra L is a vector space equipped with a bilinear product or bracket $ab = [a, b]$ which satisfies the Leibniz identity $a(bc) = (ab)c + b(ac)$ for all $a, b, c \in L$. For convenience we suppress the bracket notation for the product of individual elements of the algebra. Note

2010 *Mathematics Subject Classification.* 17D99.
Key words and phrases. Leibniz, maximal, solvable, Lie.

that we follow the notation in [1, 6] and use "left" Leibniz algebras; some authors [9, 10] instead use "right" Leibniz algebras.

The following result was proven in [6] Theorem 4.16.

Proposition 2.1. *A Leibniz algebra L is nilpotent if and only if every proper subalgebra of L is properly contained in its normalizer.*

Definition 2.2. Let M be a subalgebra of a Leibniz algebra L. Define the *core* of M to be the maximal ideal of L contained in M.

Proposition 2.3. *Let L be a solvable Leibniz algebra and let M be a self-normalizing maximal subalgebra of L. Let N be the core of M. Then*

(1) *L/N contains a unique minimal ideal A/N.*
(2) *L/N is the semidirect sum of A/N and M/N.*
(3) *The Frattini ideal, $\phi(L/N) = 0$.*
(4) *L/N is not nilpotent.*

The proof is identical to the Lie case in [12] and makes use of Proposition 2.1.

Theorem 2.4. *Let L be a nonnilpotent, solvable Leibniz algebra all of whose proper subalgebras are nilpotent. Then $L = A \oplus \operatorname{span}\{x\}$ and $A = nilrad(L) = \operatorname{span}\{a_0, \dots, a_k\} \oplus N$, with N an ideal of L and $x \in L$ is described by the following products:*

$$xa_0 = a_1, \quad xa_1 = a_2, \quad \dots, \quad xa_{k-1} = a_k, \quad xa_k = c_0a_0 + \cdots + c_ka_k$$

where $c_0 \neq 0$. Additionally $N = \langle x \rangle^2 + (\operatorname{span}\{a_0, \dots, a_k\})^2$, $A^3 \leq Leib(L)$, and $p(\lambda) = \lambda^{k+1} - c_k\lambda^k - \cdots - c_1\lambda - c_0$ is irreducible. Finally, either L is cyclic or $Leib(L) \leq N$.

Proof. L contains a self-normalizing maximal subalgebra M, which is a Cartan subalgebra of L. Let N be the core of M. By Proposition 2.3, L/N contains a unique minimal ideal A/N which complements M/N in L/N. So $L/A \cong M/N$ and since M is nilpotent, L/A is nilpotent. Since A/N is nilpotent and minimal, $(A/N)^2 = 0$ so A/N is abelian. Since L/N

is not nilpotent, by Engel's Theorem [1, 4], there exists $x \in L/N$ with $x \notin A/N$ such that left-multiplication by x, denoted ℓ_x, is not nilpotent on L/N. Without loss of generality, we can assume $x \in M$, $x \notin N$. Since M/N is nilpotent and complements A/N in L/N, this implies that ℓ_x restricted to A/N is not nilpotent. Thus the subalgebra B/N of L/N generated by A/N and x is not nilpotent, so by the hypothesis of the theorem $B/N = L/N$.

Since A/N is an ideal, $L = \langle x \rangle + A$. We claim that $x^2 \in N \subseteq A$. Since M is nilpotent, $x^{n+1} = 0$ for some n. Let $N_1 = \operatorname{span}\{x^2, \ldots, x^n\} + N$, so that $N \leq N_1 \lneq M$. Since $N \trianglelefteq L$ and left-multiplication by x^i is zero for $i > 1$, $[N_1, L] \leq N \leq N_1$. Since A/N is a minimal ideal of L/N, by Lemma 1.9 of [1], $[A/N, L/N]$ is 0 or anticommutative. But since $[x^i, A] = 0$ for all $i > 1$, $[A, x^i]$ is contained in N. From this, using the decompositions $L = \langle x \rangle + A$ and $N_1 = \operatorname{span}\{x^2, \ldots, x^n\} + N$ it follows that $[L, N_1] \leq N_1$. Thus N_1 is an ideal of L and by the maximality of N, $N_1 = N$. Therefore $x^2 \in N$ and $L = \operatorname{span}\{x\} \oplus A$. Thus $\dim L = 1 + \dim A$, and $1 = \dim L/A = \dim M/N$. Define F to be the one-dimensional subspace $F = \operatorname{span}\{x\}$. Then we have $L = \langle x \rangle + A$ and $L = F \oplus A$, but unless $x^2 = 0$ the first sum is not direct and F is not a subalgebra.

Let $L = M \oplus L_1$ be the Fitting decomposition of L with respect to left-multiplication by M. Then M/N is a Cartan subalgebra of L/N and $(L_1 + N)/N$ is the Fitting one-component of L/N with respect to left-multiplication by M/N. Since $L/N = A/N + M/N$, L/N is not nilpotent and A/N is a minimal ideal, we have that M/N acts nontrivially and irreducibly on A/N. Since $[M/N, A/N] = A/N$, the Fitting one-component of L/N with respect to left-multiplication by M/N is A/N. Therefore $A/N = (L_1 + N)/N$ and $A = L_1 \oplus N$. In addition, $[N, L_1] \subseteq [M, L_1] = L_1$, so $[N, L_1] \subseteq N \cap L_1 = 0$.

Let T be the subalgebra of L generated by L_1. Since left-multiplication by M acts irreducibly on L_1 and N is nilpotent, $[F, L_1] = L_1$. This implies $[F, T] = T$ and further that $[\langle x \rangle, T] = T$. Thus $\langle x \rangle + T$ is a nonnilpotent subalgebra of L, hence $\langle x \rangle + T = L$. Notice $x^2 \in N \leq A$ and $L_1 \leq A$ imply that $\langle x \rangle^2 + T \leq A$. However $\langle x \rangle^2 + T$ is a codimension 1 subalgebra of L, so $A = \langle x \rangle^2 + T = \langle x \rangle^2 + \langle L_1 \rangle$.

Recalling that A/N is abelian, we know that $A^2 \leq N$, so it follows that $(L_1)^2 + \langle x\rangle^2 \leq N$. However, $(L_1)^2 + \langle x\rangle^2$ and N have the same dimension, so $(L_1)^2 + \langle x\rangle^2 = N$. Hence, $N^2 = [N, N] \leq Leib(L)$. Because $A = L_1 \oplus N$, we know $[N, A] \leq Leib(L)$. By definition of $Leib(L)$, this implies $[A, N] \leq Leib(L)$. Thus, $A^3 = [A, A^2] \leq [A, N] \leq Leib(L)$.

Since $\ell_x|_{L_1}$ is not nilpotent, there exists an $a \in L_1$ such that ℓ_x is not nilpotent on a. Then $\text{span}\{a, xa, x(xa), \ldots, (\ell_x)^k(a)\} \subseteq L_1$, where we choose the largest k such that this set is linearly independent. Since M/N acts irreducibly on A/N, it follows that $F \simeq M/N$ acts irreducibly on $L_1 \simeq A/N$, so $\text{span}\{a, xa, x(xa), \ldots, (\ell_x)^k(a)\} = L_1$. Because L_1 is the Fitting one-component, $(\ell_x)^{k+1}(a) = c_0a + c_1xa + \cdots + c_k(\ell_x)^k(a)$, and $c_0 \neq 0$. Note that the matrix for ℓ_x acting on L_1 is in rational canonical form, and therefore the characteristic polynomial is the minimal polynomial $p(\lambda)$, as given in the theorem.

If $Leib(L/N) = 0$, then $Leib(L) \leq N$. Now suppose that $Leib(L/N) \neq 0$. Then there exists a minimal ideal inside of $Leib(L/N)$, and so $A/N \leq Leib(L/N)$. Since A/N is a codimension 1 subalgebra of L/N, then $A/N = Leib(L/N)$. Thus, $Leib(L/N)$ has codimension 1 in L/N, which implies that L/N is cyclic: $L/N = \langle \bar{z}\rangle$. Since $\langle \bar{z}\rangle$ is nonnilpotent, then $\langle z\rangle$ is nonnilpotent, and $L = \langle z\rangle$ is cyclic. □

Note that the products listed in this theorem are not necessarily the only nonzero products in L. However we know that $x^2 \in \langle x\rangle^2 \leq N$, $nx \in N$ for any $n \in N$, and $a_ix = -xa_i + Leib(L)$, and in the noncyclic case $Leib(L) \leq N$. Also, A/N abelian means that $a_ia_j \in N$. Thus the description in the proof shows all nontrivial products in L/N.

Using the notation from the proof, the theorem can be restated in the following way.

Corollary 2.5. *Let L be a minimal nonnilpotent Leibniz algebra. Let M be a self-normalizing maximal subalgebra of L with core N, and $L = M \oplus L_1$ be the Fitting decomposition of L with respect to M. Then L is the vector space direct sum of N, L_1, and F where F is a one-dimensional subspace of L and $M = N \oplus F$. Furthermore, $A = N \oplus L_1$ is an ideal of L with $A^3 \leq Leib(L)$.*

In Lie algebras [12, 13] prove that $A^3 = 0$. We recover this result for the case where L is a Lie algebra and generalize to $A^3 \leq Leib(L)$ in the non-Lie case. This is due to the fact that, $A^2 = N$ in Lie algebras but in Leibniz algebras we have that $A^2 \leq N$, since $N = (L_1)^2 + \langle x \rangle^2 = A^2 + \langle x \rangle^2$. If $x^2 = 0$, then we would have $A^2 = N$ and $A^3 = 0$.

Note that many nonnilpotent, cyclic Leibniz algebras have all proper subalgebras nilpotent (see Example 3.1). However there also exist nonnilpotent, cyclic Leibniz algebras with nonnilpotent subalgebras. One example is $L = \mathrm{span}\{z, z^2, z^3\}$ with $zz^3 = z^2 + 2iz^3$ over $\mathbb{C}$, which has a nonnilpotent subalgebra $M = \mathrm{span}\{ia - a^2, a^2 + ia^3\}$. Our theorem shows the structure required for minimal nonnilpotent, cyclic Leibniz algebras. For a more exhaustive study of cyclic Leibniz algebras, see [2, 5].

3. EXAMPLES

For the following examples we adopt the convention that when we list products of a Leibniz algebra, those not mentioned are assumed to be zero. Note that whenever L is cyclic, its generator will never be an element of either M or A. In Example 3.1, neither x nor a is a generator, but $z = x + a$ is a generator.

Example 3.1. Let L be the cyclic Leibniz algebra $L = \mathrm{span}\{z, z^2\}$ with $zz^2 = z^2$. This is a minimal nonnilpotent Leibniz algebra. Then $x = z - z^2$, $a = z^2$, $N = 0$, $M = \mathrm{span}\{x\}$, and $A = \mathrm{span}\{a\}$.

In Lie algebras $F = \mathrm{span}\{x\}$ is a subalgebra, however in Leibniz algebras this only guaranteed to be a subspace of L. See Example 3.2. In Lie algebras, either A is a minimal ideal or $A^2 = Z(A)$. Either case would imply $A^3 = 0$, but this is clearly not the case in Example 3.2 when $k \geq 3$.

Example 3.2. Let $L = \mathrm{span}\{x, x^2, \ldots, x^j, a, a^2, \ldots, a^k\}$ for some $j, k \in \mathbb{N}$ with $x^{j+1} = 0$, $a^{k+1} = 0$, $xa = a = -ax$, and $xa^i = ia^i$. This is a minimal nonnilpotent Leibniz algebra. Then $N = Leib(L) = \mathrm{span}\{x^2, \ldots, x^j, a^2, \ldots, a^k\}$, $F = \mathrm{span}\{x\}$, $M = F \oplus N$, and $A =$

$\text{span}\{a\} \oplus N$. In this example $c_0 = 1$ and $p(\lambda) = \lambda - 1$, which is irreducible over any field. Here $A^3 = \text{span}\{a^3, \ldots, a^k\} \neq 0$ for $k \geq 3$.

Over an algebraically closed field every irreducible polynomial has degree one, so the dimension of A/N is one and $A = \text{span}\{a\} \oplus N$. Over the field of real numbers every irreducible polynomial is linear or quadratic, so either $A = \text{span}\{a\} \oplus N$ or $A = \text{span}\{a_0, a_1\} \oplus N$. Over the rational numbers, we can construct a Leibniz algebra of this type with A/N having any dimension:

Example 3.3. Over the field of rational numbers there is an irreducible polynomial of form $p(\lambda) = \lambda^{k+1} - c_k\lambda^k - \cdots - c_1\lambda - c_0$ for any k. Define $L = \text{span}\{x, a_0, a_1, \ldots, a_k\}$ with $xa_i = a_{i+1}$ for $0 \leq i < k$ and $xa_k = c_0a_0 + c_1a_1 + \cdots + c_ka_k$. This is a minimal nonnilpotent Leibniz algebra. Then $N = Leib(L) = \text{span}\{a_1, \ldots, a_k\}$, $M = \text{span}\{x\} \oplus N$, $A = \text{span}\{a_0\} \oplus N$.

References

[1] Barnes, D. Some theorems on Leibniz algebras, *Comm. Algebra.* (7) 39, (2011) 2463-2472.

[2] Batten-Ray, C., A. Combs, N. Gin, A. Hedges, J.T. Hird, L. Zack. Nilpotent Lie and Leibniz algebras, *Comm. Algebra.* (42) 6, (2014) 2404-2410 DOI:10.1080/00927872.2012.717655

[3] Bosko-Dunbar, L., J. Dunbar, J.T. Hird, K. Stagg. Solvable Leibniz algebras with Heisenberg nilradical, *Comm. Algebra.* (43) 6, (2015) 2272-2281.

[4] Bosko, L., A. Hedges, J.T. Hird, N. Schwartz, K. Stagg. Jacobson's refinement of Engel's theorem for Leibniz algebras, *Involve.* (4) 3, (2011) 293-296.

[5] Bugg, K., A. Hedges, M. Lee, B. Morell, D. Scofield, S. McKay Sullivan. Cyclic Leibniz algebras. To appear, arXiv:1402.5821.

[6] Demir, I., K. Misra, E. Stitzinger. On some structures of Leibniz algebras, *Contemp. Math.* 623 (2014), 41-54.

[7] Demir, I., K. Misra, E. Stitzinger. On classification of four-dimensional nilpotent Leibniz algebras. *Comm. Algebra.* (45) 3, (2017) 10121018.

[8] Demir, I. Classification of 5-dimensional complex nilpotent Leibniz algebras. To appear, arXiv:1706.00951.

[9] Karimjanov, I. A.; Khudoyberdiyev, A. Kh.; Omirov, B. A. Solvable Leibniz algebras with triangular nilradicals. *Linear Algebra Appl.* 466 (2015), 530546.

[10] Khudoyberdiyev, A. Kh.; Rakhimov, I. S.; Said Husain, Sh. K. On classification of 5-dimensional solvable Leibniz algebras. *Linear Algebra Appl.* 457 (2014), 428454.

[11] Loday, J. Une version non commutative des algebres de Lie: les algebres de Leibniz, *Enseign. Math.* 39 (1993) 269-293.

[12] Stitzinger, E. Minimal Nonnilpotent Solvable Lie Algebras, *Proc. Amer. Math. Soc.* **28** 1 (1971) 47-49.

[13] Towers, D. Lie Algebras All of Whose Proper Subalgebras are Nilpotent, *Linear Algebra Appl.* **32** (1980) 61-73.

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