FORUM

Questions are in reverse chronological order

- Question December 2

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 On page 12, author mentions
that m is a dense ring of linear transformations on R over C' (Jacobson,
Lecture in Abstract Algebra, vol. II, p274). Is there any way that I can
refer to this resource? (This is being mention on page 12 right before
theorem 1) I am also wondering that is the dense ting being mention here
related to Jacobson density theorem?

  In addition, when I search the word
'priori', it does show me any useful information. In the book, author
indicates "As an algebra over C0 (or any subfield F of C0), R is simple
since any ideal of R as an algebra is a priori an ideal of R as a ring".
The word 'priori' is italic in the book. Often, the author will italic
some terminologies. But for this one, should I understand it as explained
in the dictionary or does it have any mathematical meanings?
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- Answer to Question December 2

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I am sending you two files.  One is an explanation of some of the concepts on
page 12 of Schaferbook.  The other is a file from my 2012 transfer seminar
which explains some terms like "module, field, etc"  It is 97 pages long
so you only need to look at pp. 67-72 and 77-89.

Here is the transfer seminar file. See the next page for more of my answer.

http://www.math.uci.edu/~brusso/slid022812full3sht.pdf
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Answer to Question December 2

p. 12 of Schaferbook

R simple ring ~~over $\mathbb{K}$~~ $\mathbb{Z}$ (the only ideals are (0) & R)

associative multiplication algebra $M = M(R) =$ algebra generated by $\{L_x, R_y : x, y \in R\}$

$\subseteq \text{End}\, R$

<u>M is irreducible</u> This means if $S \subset R$ subspace & $L_x(S) \subset S$, $R_y(S) \subset S$ then $S = (0)$ or R.

let $S \subset R$ be a subspace such that $L_x(S) \subset S$, $R_y(S) \subset S$ $\Rightarrow$ S ideal $\Rightarrow S = (0)$ or R (since R is simple)

$\therefore$ M is irreducible.

~~M is an R-module~~

~~if $T = \sum t_i$ each T_i = a product of L_x's and R_y's~~

Let $C' = \{ T \in \text{End}(R) : TL_x = L_xT, \; TR_y = R_yT \}$

$\therefore$

$$\left(\begin{array}{l} T(xy) = T(L_x y) = L_x(Ty) = x(Ty) \\ T(xy) = T(R_y x) = R_y(Tx) = (Tx)y \end{array} \right) \quad (18)$$

C' is commutative (see p. 12 of Schafer book)

$Tx = 0 \Rightarrow T(xy) = 0 \quad \forall y \in R$

$T(yx) = y(Tx) = 0 \quad \forall y \in R$

$\Rightarrow$

$\ker T$ is an ideal so $\left\{ \begin{array}{l} \ker T = (0) \\ \text{or } \ker T = R \end{array} \right\}$ i.e. T is invertible OR $T = 0$

every $\neq 0$ element has an inverse

so C' is a field.

Denote $T \in C'$ by α, define $\alpha x = T(x)$ (scalar) $\quad x \in R$

R is a vector space over C' since

$$\alpha(xy) = T(xy) = x\,T(y) = x(\alpha y)$$
$$\alpha(xy) = T(xy) = (Tx)y = (\alpha x)y$$

R is a simple algebra <u>a priori</u>

Don't worry about Theorem IV and Jacobson density theorem and the rest of the chapter. We would have to talk in person to discuss that. Maybe next quarter "a priori" is a latin term, meaning that something is obvious as a special case.

For example on page 12 since an algebra is a special case of a ring (ignore the scalars), if you ignore the field, the algebra is a ring and an ideal in an algebra is ~~the "a priori"~~ subspace ~~closed~~ of the vector space structure of the algebra, so an ideal in an algebra is "a priori" a ideal in the ring structure of the algebra because it only needs to satisfy some of the axioms, namely ab & ba ∈ I (ideal) if a ∈ I & b ∈ A ~~and not worry about~~

Your problem was googling "priori" instead of "a priori" (For some definitions see the file slid022812full3sht.pdf)

field, module, ...

- Question December 1

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Page 22, definition of "quadratic algebra". There's not much
information about it that I can find on Google.

          Page 23. Involution and later its connection with quadratic
algebra, namely (27). The book says "clearly 27 implies 25", but to me it's
really not clear..
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- Answer to Question December 1—see the next page

Answer to Question December 1,

An algebra A with unit 1 over a field F is called a quadratic algebra

if for each $x \in A$ $\exists$ scalars $t(x), n(x) \in F$

such that $x^2 - t(x)x + n(x)1 = 0$.

Note 1 if $x \notin F1$ then $t(x)$ and $n(x)$ are uniquely determined

[proof: if $x^2 - \alpha x + \beta 1 = 0$ and $x^2 - \alpha' x + \beta' = 0$

then $(\alpha - \alpha')x + (\beta' - \beta) \cdot 1 = 0$ but

x and 1 are linearly independent since $x \notin F1$

so $\alpha - \alpha' = 0$ and $\beta' - \beta = 0$]

Note 2 if $x \in F1$, say $x = \alpha 1$ we define

$t(x) = 2\alpha$ and $n(x) = \alpha^2$

This makes t a linear functional and n a quadratic form

[For the proof it must be shown that $t(x + \lambda y) = t(x) + \lambda t(y)$

for all $x, y \in A$ and $\lambda \in F$ AND

n is quadratic form — ignore this for now]

(quadratic form will be defined later)

3 cases (1) $x, y \in F1$

(2) $x \in F1$ $y \notin F1$

seems to be harder → (3) $x, y \notin F1$

<u>Regarding (27) ⇒ (25)</u>, note that

(27) : $x+\bar{x} \in F1$ $\qquad x\bar{x} = \bar{x}x \in F1$

allows us to define $t(x)$ by

$$x+\bar{x} = t(x)1$$

and $n(x)$ by $\quad x\bar{x} = \bar{x}x = n(x)1$

Then $x^2 - t(x)x + n(x)1$

$$= x^2 - (t(x)1)x + n(x)1$$

$$= x^2 - (x+\bar{x})x + x\bar{x}$$

$$= x^2 - \overline{x^2} \bar{x}x + x\bar{x} = 0 \text{ as required.}$$

- Question November 30

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There's something on page 20, Chapter 3, Schaefer that
really confuses me.

Under Theorem 4's explanation, what exactly is B*
and its relationship to B? Why if x^j = 0 then T^(2j-1) = 0?
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- Answer to Question November 30—see the next page

Answer to Question of November 30

For any subset B of an algebra A

B^* denotes the algebra generated by $\{R_x, L_x : x \in B\}$.

B^* is an <u>associative</u> subalgebra of End (A),

and consists of all linear combinations

of products $S_1 S_2 \ldots S_n$ $\quad n \geq 1$

where $S_i \in \{R_x, L_x : x \in A\}$

Regarding Theorem 4 on page 20, if

B is the subalgebra of A generated by a

single ~~element~~ element x and $T \in B^*$,

then $T = \sum_{i=1}^{n} T_i$ where $T_i \in \{R_x^{j_1}, L_x^{j_2}, R_x^{j_3} L_x^{j_4}\}$

$1 \leq j_i \leq$

If ~~x~~ $x^j = 0$ then each $j_i < j$ but we don't need this fact.

$$T^m = \left(\sum_{i_1=1}^{n} T_{i_1}\right)\left(\sum_{i_2=1}^{n} T_{i_2}\right) \cdots \left(\sum_{i_m=1}^{n} T_{i_m}\right) \quad (m \text{ terms})$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} (T_{i_1} T_{i_2} \cdots T_{i_m})$$

this $= R_x^p L_x^q$ where $p+q \geq m$

Note $L_x R_x = R_x L_x$ (6'')
& $R_{x^j} = R_x^j$, $L_{x^j} = L_x^j$ (17)

If $p < j$ and $q < j$ and we set $m = 2j-1$ then

$2j-1 = m \leq p+q < 2j$ a contradiction. So $T^{2j-1} = 0$ ⌧

- Question November 29

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This question refers to page 31 of Schafer's book ``Nonassociative algebras,''
and involves symmetric bilinear forms.

I'm not sure if the radical of A (and J) is a subalgebra of
A (and J), and does the radical works the same as the "null space" of
a vector space?
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- Answer to Question of November 29—see the next page

Answer to Question of November 29

Let (x,y) be a symmetric bilinear form on an algebra A. This means that

$$(\cdot,\cdot): A \times A \to \Phi \quad (\Phi = \mathbb{R} \text{ or } \mathbb{C})$$

is bilinear: $(x_1+x_2, y) = (x_1,y) + (x_2,y)$

$(x, y_1+y_2) = (x,y_1) + (x,y_2)$.

and symmetric: $(x,y) = (y,x)$.

$(\cdot,\cdot)$ is called a "<u>trace form</u>" or an "<u>associative form</u>", or an "<u>invariant form</u>" if

$$(xy, z) = (x, yz) \quad \forall x,y,z \in A.$$

If J is any subset of A (for example, an ideal) then $J^{\perp} = \{x \in A : (x,y) = 0 \ \forall y \in J\}$

If J is an ideal, so is $J^{\perp}$.

(see p. 31 for the proof)

In particular, $A^{\perp}$ is an ideal, which is called "radical" in the book. Ignore this.

With regard to your question about "null space" (or "kernel") if $(\cdot,\cdot)$ is a symmetric bilinear form and $x \in A$ is fixed, then $T_x : y \to (x,y)$ is a linear transformation $T_x : A \to \Phi$ $(\Phi = \mathbb{R} \text{ or } \mathbb{C})$. Then

$J^{\perp} = \{x \in A : J \subset \ker T_x\}$ where $\ker T_x$ = nullspace of T_x

- Question November 20

  I have 4 question from reading the Schaferbook. The
  following is my summary of these 4 questions.
  1. On page 7. For number (4) equalities. The third one. I initially thought
  it would be (b1, c1)(b2, c2) = (b1b2, c1c2). But the book indicates (b1, c1
)(b2, c2) = (b1c1, b2c2). I am wondering how does this come?

  2. On page 9. This is a paragraph from page 9: For finite-dimensional A,
  the scalar extension AK (K an arbitrary extension of F) may be defined in a
  non-invariant way (without recourse to tensor products) by use of a basis
  as above. Let u1, . . . , un be any basis for A over F; multiplication in A
  is given by the multiplication table (10). Let AK be an n-dimensional
  algebra over K with the same
  multiplication table (this is valid since the ijk, being in F, are in K). *What
  remains to be verified is that a different choice of basis for A over F
  would yield an algebra isomorphic (over K) to this one.*

  The underlined sentence is my question. I don't know how to verify it.
  Every time I saw 'one can verify that...', I would like to do the same
  thing as we would do in Meyberg book (like the exercises. I think by doing
  this kind of proof is good to test if I really understand it. In addition,
  I would like to verify some of the identities or properties, like the third
  question I have below.

  3. Deleted

  4. Go back to page 9. I think I just had a bad time understanding page 9.
  This is another paragraph from page 9: For the classes of algebras
  mentioned in the Introduction (Jordan algebras of characteristic 6= 2, and
  Lie and alternative algebras of arbitrary characteristic), *one may verify
  that algebras remain in the same class under scalar extensiona property
  which is not shared by classes of algebras defined by more general
  identities *(as, for example, in V)

  The underlined sentence is my question for this paragraph.

- Answer Question November 20

  1. DO NOT BELIEVE EVERYTHING YOU READ! THIS IS A MISPRINT IN THE BOOK. YOU ARE CORRECT.

  2. I ALREADY TOLD YOU THAT I TRIED TO PROVE THIS WITHOUT SUCCESS. I WILL WORK ON IT AND TRY TO EXPLAIN TO EVERYONE SOMETIME. THERE IS ALSO ANOTHER QUESTION HERE: WHY IS THE BASIS FOR A OVER F, STILL A BASIS OF A

OVER K. THE LINEAR INDEPENCENCE IS NOT CLEAR. ACTUALLY BOTH OF THESE QUESTIONS CAN BE ANSWERED USING TENSOR PRODUCTS. I WILL EXPLAIN ALL THIS AFTER I REVIEW IT.

3. DELETED

4. I HAD THE SAME QUESTION FROM ANOTHER STUDENT. SEE THE ANSWER TO QUESTION OF NOVEMBER 9 IN "FORUM"

- Question November 15

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I have a question regarding to the project. On page 30 of Schafer's book
(Non associative algebras) there  is a word called "enveloping algebra".
I searched on the Internet but it has multiple definitions, which are universal,
 non-associative, and two more definitions. Can you clarify this part?
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- ANSWER to Question November 15

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Look at page 11 in the middle of the page.  Here you will find
the definition of the enveloping algebra, and the
corresponding notation.
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- Question November 10

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1. On page 2 of Schaferbook, 4th line.
    what does it mean  that the bilinear multiplication in an algebra is
completely determined by n^3 multiplication constants...
2. on page 3, line 12.
    In the example (A: associative algebra of n by n matrices...). why the
dimension of L is (1/2)n(n-1).
3. on page 6.
    I do not really understand the customary isomorphism theorem. I
can't easily see why this would hold.
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- ANSWERS to Question November 10

1. On page 2 of Schaferbook, 4th line. what does it mean that the bilinear multiplication in an algebra is completely determined by n^3 multiplication constants...

ANSWER: An algebra A is in particular a vector space. Let $u_1, \dots, u_n$ be a basis for this vector space. The product $u_i u_j$ of basis elements is another vector so it has coordinates with respect to the given basis, call them $\gamma_{ij1}, \gamma_{ij2}, \dots, \gamma_{ijn}$ so that

$$u_i u_j = \sum_k \gamma_{ijk} u_k.$$

This gives you the n^3 numbers $\{\gamma_{ijk} : \ i,j,k \in \{1,2,\dots n\}\}$. To find the coordinates of the product of two arbitrary elements a, b of A, you need to know the coordinates of a and b with respect to the given basis, and the numbers γ_{ijk}, that is, if $a = \sum_i \alpha_i u_i$ and $b = \sum_j \beta_j u_j$, then

$$ab = \sum_i \sum_j \alpha_i \beta_j u_i u_j = \sum_i \sum_j \alpha_i \beta_j \left(\sum_k \gamma_{ijk} u_k \right) = \sum_k \left(\sum_i \sum_j \alpha_i \beta_j \gamma_{ijk} \right) u_k.$$

So the coordinates of the product ab are $\sum_i \sum_j \alpha_i \beta_j \gamma_{ijk}, \ k = 1,2,\dots n$. This is the meaning of the statement.

2. on page 3, line 12. In the example (A: associative algebra of n by n matrices...). why the dimension of L is (1/2)n(n-1).

ANSWER: First of all, recall that the algebra M_n of all n by n matrices has dimension n^2. This is because there are n^2 entries in each matrix, and the set of matrices with all entries 0 except one which has the value 1, forms a basis (linearly independent and generating). In symbols, if $X = [x_{ij}]$ is a matrix, then

$$X = \sum_{i,j} x_{ij} E_{ij}$$

where E_{ij} is the matrix with a 1 in the (i,j)-entry and zeros everywhere else. (Write this out for $n = 2$ and 3)

Now let us consider the set A_n of all n by n skew-symmetric matrices, that is, $X^T = -X$. Such a matrix has zeros along the diagonal, so that takes away n dimensions from n^2. Each entry above the diagonal is the negative of its reflection below the diagonal. This means that of the $n^2 - n$ possible dimensions remaining, each E_{ij} above the diagonal ($i < j$) is paired with $-E_{ji}$. So the dimension is $(n^2 - n)/2 = \frac{1}{2} n(n-1)$. In symbols, a basis for A_n is $\{E_{ij} - E_{ji} : i < j\}$ (Again write this out for $n = 2$ and 3)

3. on page 6. I do not really understand the customary isomorphism theorem. I can't easily see why this would hold.

ANSWER: First of all, the statements (1) and (2) on page 6 of Schafer are stated in (ii) of Theorem 1 on page 3 of Meyberg. See the proof in my notes on the website:

Second meeting October 7, 2016. Meyberg Chapter 1 Exercises 1-3-SOLUTIONS (click here)

As for the customary isomorphism theorems, the statement (i) on page 6 of Shafer was not mentioned by me in class. (You can safely ignore statement (i) on page 6 of Shafer, but after you understand the answers to the other parts, you can easily prove it. I recommend you do that!) The statement (ii) on page 6 of Shafer is the same as (iii) of Theorem 1 on page 3 of Meyberg, and is also proved in the above mentioned notes.

- Question November 9

One definition that I am still confused with is "scalar extension". I searched on the Internet and it says "Extension of scalars changes R-modules into S-modules."

So the claim "Any scalar extension Jk of a Jordan algebra is a Jordan algebra" confused me. Does it mean there is an element from Jordan algebra and we want to extend it to field F?

- ANSWER to Question November 9

On page 29 note that (3) is equivalent to (1) (reason: we derived (3) from (1) and to go from (3) to (1) you just set w=z=x in (3))

(3) is linear in each of its variables x,y,z,w

Now look at page 9, the definition of scalar extension. If u_1,,u_n is a basis for A over F, and K contains F, then u_1,,u_n is a basis for A_K over K (the proof of this is not trivial, we need to work on it later; lets assume this for now)

Since J is a Jordan algebra over F, (3) holds for the basis u_1,,u_n. But u_1,,u_n is a basis over K, so each x, y, z, w is a linear combination with coefficients in K. In (3) these coefficients all factor out of each term and so (3) holds for x,y,z,w in J_K.

- Question November 6

How do you do? I have met a lot of problems with the project. I'm talking about the book (Schafer, Nonassociative algebras) that we use for the project.

1) Why we do linearization of (xy)x^2=x(yx^2)?
And why we assume char(F) not equals 2 while linearizing it?

2) I can't understand "Replacing x in (x,y,x^2)=0 by x+?z, ??F, and ?=0 since F contains at least 3 elements."

3) If ?=0, why can we get 2(x,y,zx)+(z,y,x^2)=0?

- ANSWER to Question November 6

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1) Why we do linearization of (xy)x^2=x(yx^2)?
And why we assume char(F) not equals 2 while linearizing it?
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ANSWER: Linearization, also called polarization, gives you additional information. You want identities to be linear in each variable. For example, look at the solution to Exercise 7 from the transfer seminar. You need char(F) not equal to 2 because you will get 2 as a coefficent after linearizing, and in fields of characteristic 2, $2 = 0$, which destroys the new information (On page 29 notice at some point that you divide by 2)

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2) I can't understand "Replacing x in (x,y,x^2)=0 by x+?z, ??F, and
?=0 since F contains at least 3 elements."

3) If ?=0, why can we get 2(x,y,zx)+(z,y,x^2)=0?
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ANSWERS: I am sending you a scan of my notes for page 29. There is an error sign on the third page of the scan. Here is one problem: Schaefer writes RS for the operator R followed by S. We write this as SR (R first , S second) so when he writes [R,S]=RS-SR it is the negative of what we write as [R,S]

Start over

$$(xy)x^2 = x(yx^2)$$

$$(x,y,x^2) = (xy)x^2 - x(yx^2) = 0$$

$$(x+\lambda z,\, y,\, (x+\lambda z)^2) = 0$$

$$((x+\lambda z)y)(x+\lambda z)^2 - (x+\lambda z)(y(x+\lambda z)^2) = 0$$

$$(xy+\lambda zy)(x^2+2\lambda xz+\lambda^2 z^2) - (x+\lambda z)(yx^2+2\lambda y(xz)+\lambda^2 y z^2) = 0$$

coeff of λ "

$$(zy)x^2 + 2(xy)(xz) - 2x(y(xz)) - z(yx^2) = 0$$

$$(z,y,x^2) + 2(x,y,xz) = 0$$

$$(z,y,(x+\lambda w)^2) + 2(x+\lambda w,\, y,\, (x+\lambda w)z) = 0$$

$$(zy)(x^2+2\lambda xw+\lambda^2 w^2) - z(yx^2+2\lambda y(xw)+\lambda^2 yw^2)$$

$$+2((x+\lambda w)y)((x+\lambda w)z) - 2(x+\lambda w)(y((x+\lambda w)z)) = 0$$

coeff of λ

$$2(zy)(xw) - 2z(y(xw)) + 2(wy)(xz) + 2(xy)(wz)$$

$$-2x(y(wz)) - 2w(y(xz)) = 0$$

$$(zy,xw) + (x,y,wz) + (w,y,xz) = 0$$

$$(zy)(xw) - z(y(xw)) + (xy)(wz) - x(y(wz)) + (wy)(xz) - w(y(xz)) = 0$$

act on y $\quad [R_x, R_{wz}]^{(y)} + [R_w, R_{zx}]^{(y)} + [R_z, R_{xw}]^{(y)} = 0$

$$R_x R_{wz} - R_{wz} R_x + R_w R_{zx} - R_{zx} R_w + R_z R_{xw} - R_{xw} R_z = 0$$

act on w

(3'') $\quad R_{zy} R_x - R_z R_y R_x + R_{xy} R_z$ (1) $- R_x R_y R_z + R_{xz} R_y - R_{y(xz)} = 0$

by R_y

interchange x, y

$$R_z R_x R_y + R_{yx} R_z - R_y R_x R_z + R_{yz} R_x - R_{x(yz)} = 0$$

$$R_{zx} R_y - R_y R_x R_z + R_z R_{yx} - R_z R_{xy} + R_x R_{yz} - R_{x(yz)} = 0$$

$$[R_z, [R_x, R_y]]$$

$$= R_z (R_x R_y - R_y R_x) - (R_x R_y - R_y R_x) R_z$$

$$= R_z R_x R_y - R_z R_y R_x - R_x R_y R_z + R_y R_x R_z$$

subtract

$$R_{zy} R_x - R_{zx} R_y - R_z R_y R_x + R_z R_x R_y + R_x R_y R_z - R_y R_x R_z$$
$$+ R_{xz} R_y - R_{yz} R_x - R_{y(xz)} + R_{x(yz)} = 0$$

$$R_z [R_x, R_y] + [R_x, R_y] R_z - R_{(x,z,y)} = 0$$

↑ should be a minus sign

$$(x,z,y) = (xz)y - x(zy)$$
$$= (R_y R_x - R_y R_x) z$$
$$= [R_y, R_x] z$$

- Question October 16

  I'm still confused with several things in Myberg's notes.

  1. Page 2: Can you give me some further explanation to "subalgebra" and "ideal"? I think it is pretty weird to say "If U*U belongs to U, then U is a subalgebra." Also, I think I need some examples for the definition of "ideal".

  2. Page 2: What does "rsp." stands for?

  3. What is the formal way to prove Congruence?

- ANSWERS to Question October 16: (sent to entire class on October 18)

  1. Page 2: Can you give me some further explanation to "subalgebra" and "ideal"? I think it is pretty weird to say "If U*U belongs to U, then U is a subalgebra." Also, I think I need some examples for the definition of "ideal.

  ANSWER:
  A subalgebra of an algebra A is a subset B of A with the following properties:

  First,   B is a linear subspace of A: if x,y are in B and c is a constant, then cx+y belongs to B

  Second,   If x,y are in B, then xy belongs to B.
  Recall that BB means the set of all finite sums of products xy of elements x,y of B

  So saying that BB is a subset of B, is the same as the second point above.
  it is assumed also that
  B+B is a subset of B, so that B is a linear subspace of A.

  An ideal is a subalgebra but instead of the second condition, we have the stronger conditions
  BA subset of B   and   AB subset of B
  which can also be stated more compactly as
  AB+BA is a subset of B.

  Example 1 of ideal: A=continuous functions on unit interval.
  B=continuous functions on unit interval which vanish at the origin.

  Example 2 of ideal: A is any associative algebra, x is any element of A,
  B=AxA (which is the set of all sums of products of the form   axb where a and b are elements of A).

2. Page 2: What does "rsp." stands for?

ANSWER: rsp. is an abbreviation for respectively and is usually abbreviated resp. and includes parentheses

The meaning is that instead of saying

we use the notation U+V for the submodule (for us subspace) generated by all u+v, for u in U and v in V
and we use the notation UV for the submodule (for us subspace) generated by all uv, for u in U and v in V

it is shorter to say

we use the notations U+V (resp. UV) for the submodules (subspaces for us) generated by all u+v (resp. uv), for u in U and v in V

3. What is the formal way to prove Congruence?

ANSWER:

You mean isomorphism" instead of congruence An isomorphism of an algebra A onto another algebra B is a one-to-one and onto mapping f which is linear and multiplicative, that is,

f(cx+y)=cf(x)+f(y) and f(xy)=f(x)f(y)

The formal way to prove that f is an isomorphism is to show four things;

it is one to one
it is onto
it is linear
it is multiplicative.