§1.2 Coordinates

def 1.2.1 **chart** (on a set M) is a pair (ϕ, U) where $U \subset M$ and $\phi : U \to \phi(U)$ is a bijection from U onto an open set $\phi(U)$ in $\mathbb{R}^m$

atlas (on M) is a collection of charts $\mathcal{A} = \{(\phi_\alpha, U_\alpha) : \alpha \in A\}$ such that $\{U_\alpha\}_{\alpha \in A}$ is a cover of M, i.e.

$$M = \bigcup_{\alpha \in A} U_\alpha$$

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if $\phi(p) = (x_1(p), x_2(p), \ldots, x_m(p))$ for $p \in U$,

the functions $x_i(\cdot)$ form a system of local coordinates on U

Example 1.2.2 $U \overset{\text{open}}{\subset} \mathbb{R}^m$, $\mathcal{A} = \{(\mathrm{Id}_U, U)\}$ is an atlas on U.

Example 1.2.3 (graph of a function)

e.g paraboloids (see Figure 1.3 on p. 6)

* Example 1.2.4 (The m-sphere)

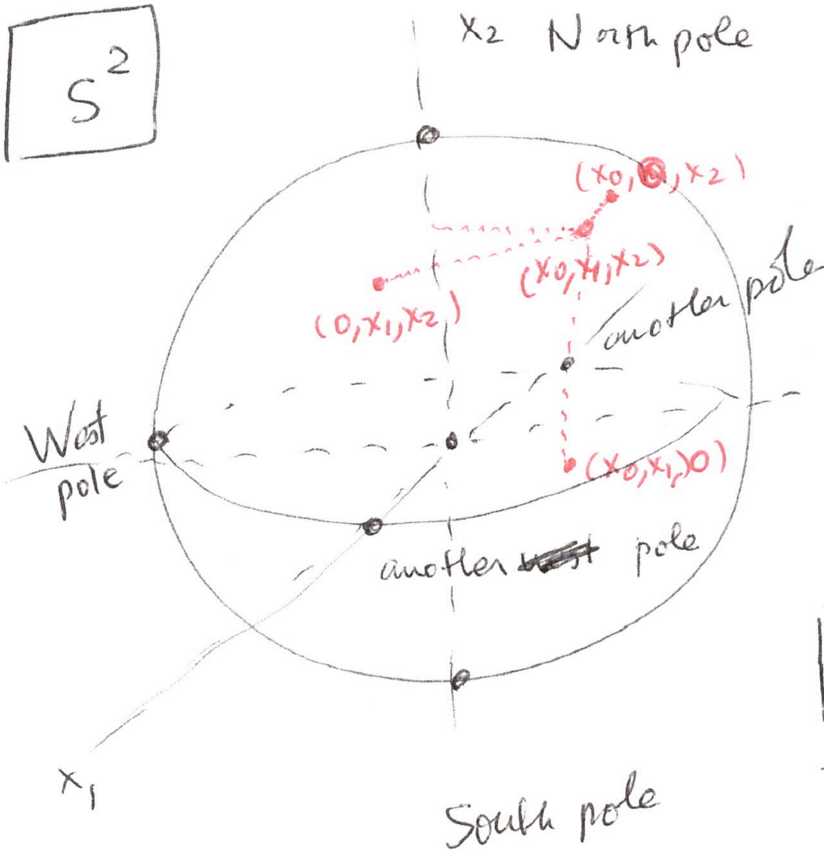
$$S^m = \{p = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} : x_0^2 + x_1^2 + \cdots + x_m^2 = 1\}$$

Example 1.2.5 (quadratic form) e.g. ellipsoid hyperboloids (Figure 1.3)

Example 1.2.6 (projective space) = set of lines

Example 1.2.7 (Grassman manifold) = set of k-planes

~~Example~~ Exercise 1.2.8 An atlas for Grassman Manifold



U_{0+} = ~~right~~ eastern hemisphere

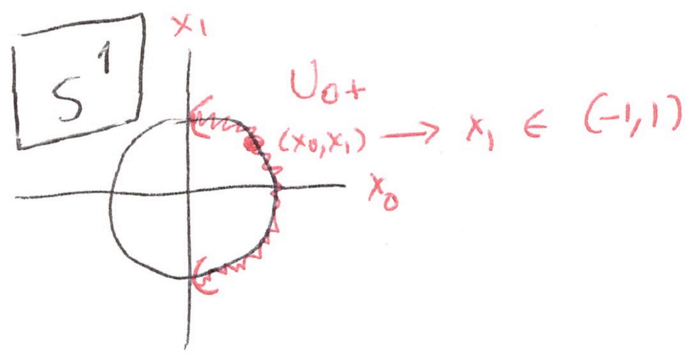
U_{0-} = ~~left~~ westen "

U_{2+} = northen "

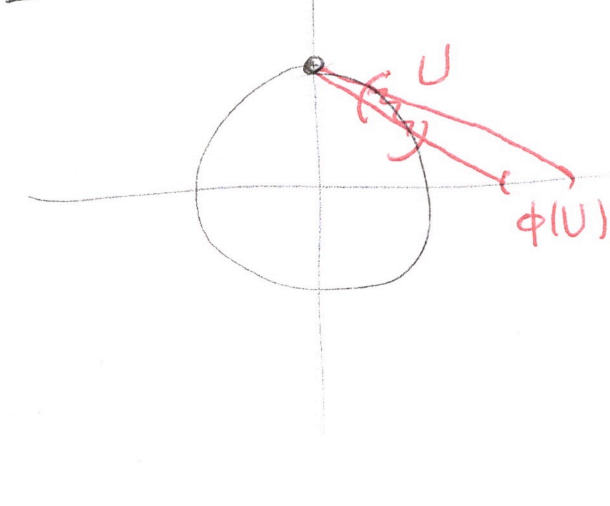
U_{2-} = southen "

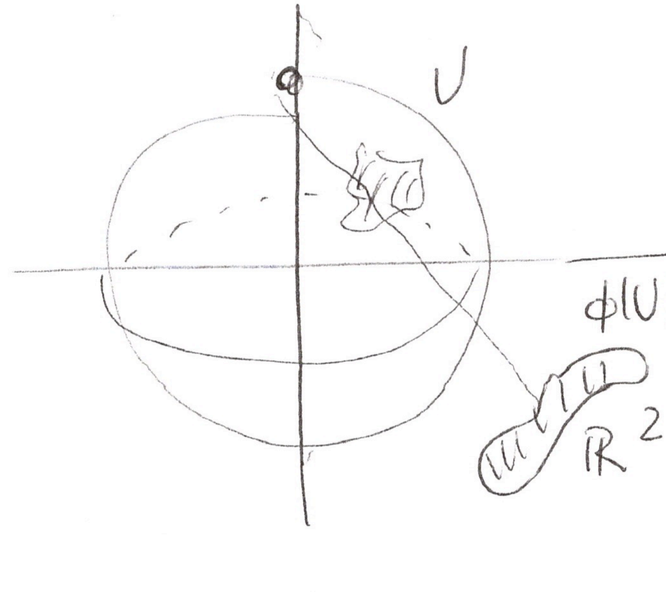
x_0

U_{1+} =



stereographic projection





§1.4 Smooth Manifolds

$U \subset \mathbb{R}^n \quad V \subset \mathbb{R}^m$ open sets $\quad f: U \to V$ $= (f_1, \dots, f_m)$ misleading in the book

is **smooth** if $\quad \partial^\alpha f_i = \dfrac{\partial^{\alpha_1 + \dots + \alpha_n} f_i}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ exists and

is continuous $\quad \forall \, \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n \quad 1 \le i \le m$

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$

$C^\infty(U, V)$ = all smooth functions from U to V

~~Def 1.4.1~~ If $f: U \to V$, U, V open $\quad f = (f_1, \dots, f_m)$

$$f(x_1, \dots, x_n) = \big(f_1(x_1, \dots, x_n), f_2(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\big)$$

$$f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \qquad f_i : \mathbb{R}^n \to \mathbb{R} \quad 1 \le i \le m$$

let $f = (f_1, \dots, f_m) : U \to V \qquad U \subset \mathbb{R}^n,\ V \subset \mathbb{R}^m$

Fix $x \in U$ and let $L\xi(x) = \left.\frac{d}{dt}\right|_{t=0} f(x + t\xi)$

$$= \lim_{t \to 0} \frac{f(x+t\xi) - f(x)}{t} \qquad \xi \in \mathbb{R}^n$$

L is a function $L : \mathbb{R}^n \to \mathbb{R}^m$

$L(a\xi) \stackrel{?}{=} aL(\xi) \qquad$ if $a = 0$ both sides are 0

$$L(a\xi)(x) = \lim_{t \to 0} \frac{f(x + at\xi) - f(x)}{t} = \lim_{s \to 0} \frac{f(x+s\xi) - f(x)}{s/a}$$

$$= aL(\xi)(x)$$

let $s = at \qquad t = \frac{s}{a}$

$L(\xi + \eta) \stackrel{?}{=} L(\xi) + L(\eta)$

$$L(\xi+\eta)(x) = \lim_{t \to 0} \frac{f(x + t(\xi+\eta)) - f(x)}{t}$$

$$L(\xi)(x) = \lim_{s \to 0} \frac{f(x+s\xi) - f(x)}{s}$$

NO GO

$$L(\xi+\eta)(x) - L(\xi)(x) - L\eta(x) = \lim_{t \to 0} \frac{f(x+t(\xi+\eta)) - f(x+t\xi) - f(x+t\eta) + f(x)}{t}$$

$$= \lim_{t\to 0} \frac{f(x+t\xi+t\eta) - f(x+t\xi)}{t} + \lim_{t\to 0} \frac{f(x+t\eta)-f(x)}{t}$$

Math 140C to the rescue

$$L\xi = \frac{\partial f_i}{\partial x_j}(x)\xi = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} ; \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$L\xi \overset{?}{=} \lim_{t\to 0} \frac{f(x+t\xi)-f(x)}{t}$$

LOOK at First coordinate

$$\boxed{\frac{\partial f_1}{\partial x_1}\xi_1 + \cdots + \frac{\partial f_1}{\partial x_n}\xi_n \overset{?}{=} \lim_{t\to 0} \frac{f_1(x+t\xi)-f_1(x)}{t}}$$

(*) Theorem $\underbrace{D_\xi f_1(x)}_{\text{directional derivative of } f_1 \text{ at } x \text{ in the direction of } \xi} = \underbrace{\nabla f_1(x)\cdot\xi}_{\text{scalar or inner product}}$

$$\left(\frac{\partial f_1}{\partial x_1}, \ldots, \frac{\partial f_1}{\partial x_n}\right) \cdot \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$$

(*) is proved on the next page using Theorem (15.3) of 140C

Theorem 15.3 (p. 28 of 140C notes) $f : D \to \mathbb{R}$, $\overset{\text{open}}{D} \subset \mathbb{R}^n$

$$\lim_{p \to p_0} \frac{|f(p) - f(p_0) - \nabla f(p_0) \cdot (p - p_0)|}{|p - p_0|} = 0$$

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write $p = p_0 + t\xi$

$$\lim_{t \to 0} \frac{|f(p_0 + t\xi) - f(p_0) - \nabla f(p_0) \cdot t\xi|}{|t\xi|} = 0$$

$|t\xi| = |t|\,|\xi|$

$$\lim_{t \to 0} \frac{\left| \dfrac{f(p_0 + t\xi) - f(p_0)}{t} - \nabla f(p_0) \cdot \xi \right|}{|\xi|} = 0$$

proving (*) on the previous page.

<u>chain rule</u>

$$\begin{tikzcd}
 & \mathbb{R}^n \arrow[r, "df(x)"] & \mathbb{R}^m \arrow[r, "dg(f(x))"] & \mathbb{R}^l \\
 & \cup & \cup & \cup \\
x \in & U \arrow[r, "f"'] \arrow[rr, bend right, "g\circ f"'] & V \arrow[r, "g"'] & W \\
 & \cap & & \cap \\
 & \mathbb{R}^n \arrow[rr, "d(g\circ f)(x)"'] & & \mathbb{R}^l
\end{tikzcd}$$

<u>Def</u> diffeomorphism is a 1:1 map which is C^∞ and its inverse is C^∞

Def 1.4.2 <u>smooth manifold</u> Recall from def 1.2.1

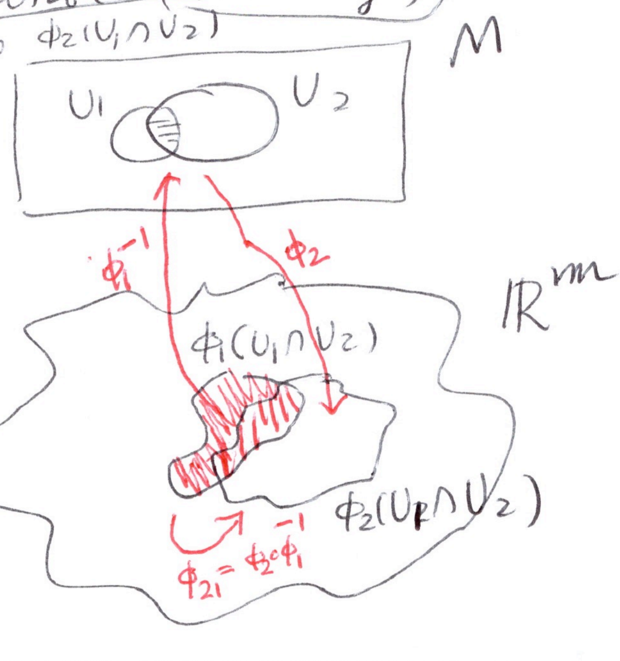
that a $\boxed{\text{chart}}$ on a set M ← just a set is a pair (ϕ, U)

where $U \subset M$, $\phi : U \to \phi(U) \subset \mathbb{R}^m$ is a bijection

$\phi(U)$ being an open set in $\mathbb{R}^m$.

charts (ϕ_1, U_1) (ϕ_2, U_2) are compatible (smoothly) if ϕ_{21} is diffeomorphism of $\phi_1(U_1 \cap U_2)$ onto $\phi_2(U_1 \cap U_2)$

$$\begin{tikzcd}
 & M & \\
U_1 \cap U_2 \subset U_1 \arrow[d, "\phi_1"'] \arrow[ur, hook] & & U_2 \supset U_1 \cap U_2 \arrow[ul, hook'] \arrow[d, "\phi_2"] \\
\phi_1(U_1) \subset \mathbb{R}^m \supset \phi_2(U_2) & & \\
\phi_1(U_1 \cap U_2) \arrow[r] & \phi_2(U_1 \cap U_2) &
\end{tikzcd}$$



recall that an $\boxed{\text{atlas}}$ on M is a collection of charts

$$\mathcal{A} = \{ (\phi_\alpha, U_\alpha) : \alpha \in A, \bigcup_{\alpha \in A} U_\alpha = M \}$$

$\mathcal{A}$ is a $\boxed{\text{smooth atlas}}$ if every pair $(\phi_1, U_1), (\phi_2, U_2) \in \mathcal{A}$ such that $U_1 \cap U_2 \neq \phi$ is smoothly compatible.

$(M, \mathcal{A})$ is a smooth manifold if $\mathcal{A}$ is a maximal smooth atlas

(maximal means: if $\mathcal{A} = \{ (\phi_\alpha, U_\alpha) : \alpha \in A \}$ and (ϕ, U) is a chart which is smoothly compatible with (ϕ_α, U_α) $\forall \alpha \in A$ then $(\phi, U) \in \mathcal{A}$ (i.e. $(\phi, U) = (\phi_{\alpha_0}, U_{\alpha_0})$ for some $\alpha_0 \in A$))

Lemma 1.4.3 Every atlas extends uniquely to a maximal atlas.

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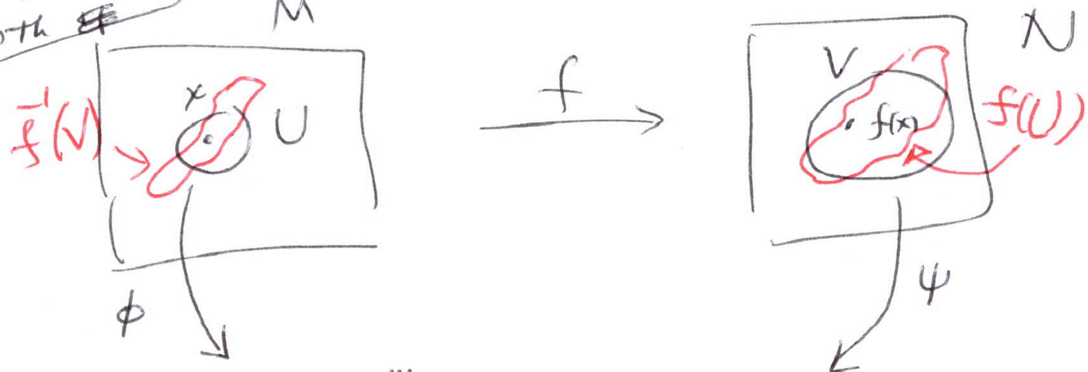
preparation for Exercise 1.4.5

what is a smooth map $f : M \to N$ between smooth manifolds M, N ?

Def $f: M \to N$ is smooth



$\phi(f^{-1}(V) \cap U) \subseteq \phi(U) \subseteq \mathbb{R}^m \longrightarrow \psi(V) \subseteq \mathbb{R}^m$

$\underbrace{\psi \circ f \circ \phi^{-1}}$ should be smooth

$\forall$ charts (ϕ, U) $x \in U$

(ψ, V) $f(x) \in V$

what is the domain of this function?

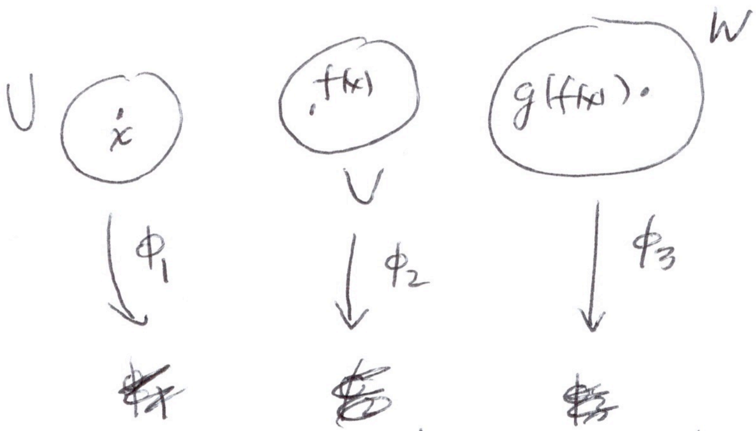
ANS: $\phi(f^{-1}(V) \cap U)$

Its range is $\psi(V \cap f(U))$

(since $f(f^{-1}(V) \cap U) = V \cap f(U)$)

Exercise 1.4.5. (a) $f: M \to M$ $f(x) = x$ is smooth

(b) $M \xrightarrow{f} N \xrightarrow{g} P$



$\phi_1(f^{-1}(V) \cap U)$ $\quad$ $\phi_2(g^{-1}(W) \cap V)$ $\quad$ ϕ_3

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & P \\ U & & V & & W \\ x & & f(x) & & g(f(x)) \end{array}$$

$$\phi_1(f^{-1}(V) \cap U) \longrightarrow \phi_2(V \cap f(U))$$

$$F = \phi_2 \circ f \circ \phi_1^{-1}$$

$$\phi_2(g^{-1}(W) \cap V) \longrightarrow \phi_3(W \cap g(V))$$

$$G = \phi_3 \circ g \circ \phi_2^{-1}$$

$$\phi_1((g \circ f)^{-1}(W) \cap U) \longrightarrow \phi_3(W \cap (g \circ f)(U))$$

$$H = \phi_3 \circ (g \circ f) \circ \phi_1$$

$$G \circ F = \left(\phi_3 \circ g \circ \phi_2^{-1}\right) \circ \left(\phi_2 \circ f \circ \phi_1^{-1}\right)$$

$$= \phi_3 \circ g \circ f \circ \phi_1^{-1} \qquad \blacksquare$$