§2.1 Submanifolds of Euclidean Space

Recall §1.4 >:

- Smooth maps were only defined on open sets in $\mathbb{R}^n$ (since limits need to exist in all directions of approach)

- recall: if M is any abstract set, a chart is a pair (ϕ, U), $U \subset M$, $\phi: U \to \phi(U) \subset \mathbb{R}^m$ bijection and $\phi(U)$ is an open set in $\mathbb{R}^m$.

- recall: Charts (ϕ_1, U_1), (ϕ_2, U_2) are smoothly compatible if either $U_1 \cap U_2 = \phi$ and whenever $U_1 \cap U_2 \neq \phi$, we have $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are open subsets of $\mathbb{R}^m$

  I forgot to mention this last time

  and $\phi_{21} \doteq \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$

  is a diffeomorphism (i.e. ϕ_{21} is bijection and ϕ_{21} and ϕ_{21}^{-1} are each C^∞ functions (on open sets!))

- recall: Smooth manifold is a set M with a smooth maximal atlas (atlas = a collection $\mathcal{A}$ of charts covering M, smooth atlas if each pair $\mathcal{A} = \{(\phi_\alpha, U_\alpha) : \alpha \in A\}$, $M = \bigcup_{\alpha \in A} U_\alpha$ $(\phi_\alpha, U_\alpha), (\phi_\beta, U_\beta)$ is compatible smoothly, and every chart (ϕ, U) which is compatible with all the (ϕ_α, U_α) is already in $\mathcal{A}$ (i.e. maximal).

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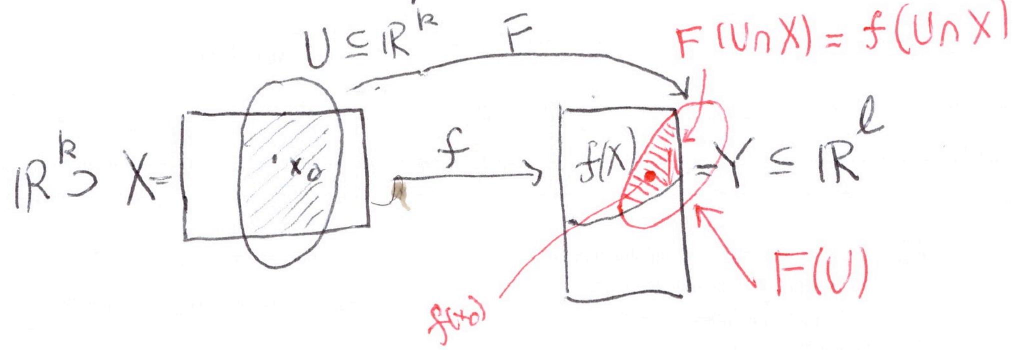
2.1 Submanifolds of Euclidean Space

X not necessarily an open set

If $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^l$, and $f : X \to Y$, then f is <u>smooth</u> if $\forall x_0 \in X$ $\exists$ open $U \subset \mathbb{R}^k$, $x_0 \in U$ and a smooth map $F : U \to \mathbb{R}^l$ with $F|U \cap X = f|U \cap X$



f is a <u>diffeomorphism</u> if f is bijective and $f : X \to Y$ and $f^{-1} : Y \to X$ are smooth in the above sense.

Exercise 2.1.1 (p. 16) (chain rule – <u>see the next page</u>)

Exercise 2.1.2 (p. 16) (<u>see the page after that</u>)

"Fake News"

$$\begin{array}{ccc} \mathbb{R}^m & & \mathbb{R}^k \\ \cup & & \cup \\ \mathbb{R}^m & \longrightarrow & E \\ (x_1, \ldots, x_m) & \longrightarrow & \sum_{i=1}^{m} x_i v_i \end{array}$$

Extend $\{v_1, \ldots, v_m\}$ to a basis $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_k\}$ of $\mathbb{R}^k$

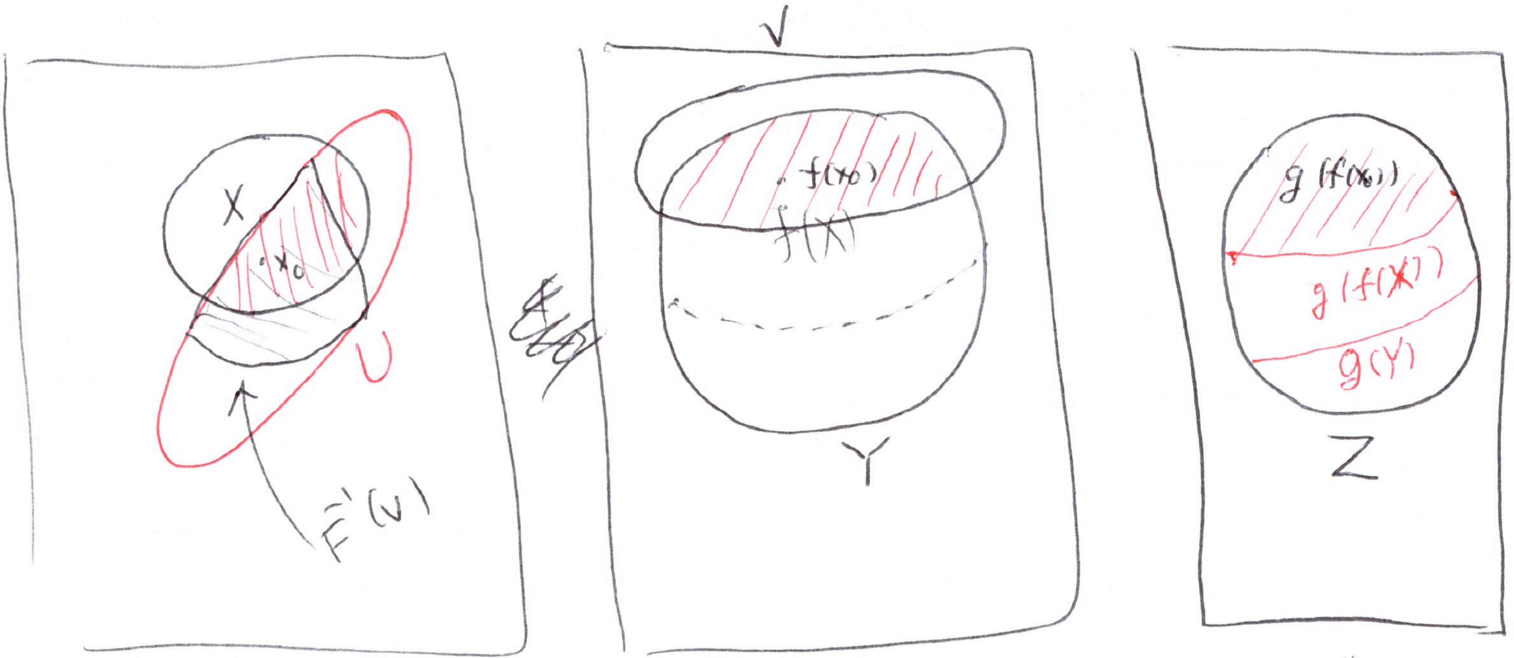
define $F : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ by

$$F(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i v_i + \sum_{i=m+1}^{k} 0 \cdot v_i$$

Take $U = \mathbb{R}^m$

F is smooth (the coordinate functions are smooth – BE CAREFUL)

Exercise 2.1.1 Chain Rule



$$\begin{array}{ccccccc} \| & & & \| & & & \| \\ \mathbb{R}^k & \mathbb{R}^k & & \mathbb{R}^\ell & \mathbb{R}^\ell & & \mathbb{R}^m \\ \cup & \cup & \xrightarrow{F} & \cup & \cup & \xrightarrow{G} & \\ X & U & F = f \text{ on } U\cap X & Y & V & G = g \text{ on } V\cap Y & \\ \ni & \cup & & \cup & \ni & & \\ x_0 & F^{-1}(V) & & f(X) & f(x_0) & & \\ & & & \ni & & & \\ & & & f(x_0) & & & \end{array}$$

$F = f$ on $F^{-1}(V)\cap X$

$G\circ F$ is smooth : $F^{-1}(V) \longrightarrow \mathbb{R}^m$

$G\circ F = g\circ f$ on $F^{-1}(V)\cap X$

let $x \in F^{-1}(V)\cap X \subset U\cap X$ $\quad F(x) = f(x) \in Y$ and $f(x) \in V\cap Y$

$G\circ F(x) = G(F(x)) = G(\underbrace{f(x)}_{\in V\cap Y}) = g(f(x))$

∎

Exercise 2.1.2

$$\begin{array}{ccc}
\mathbb{R}^3 \ (x_1, x_2, x_3) & \xrightarrow{F} & \mathbb{R}^3 = sp\{v_1, v_2, v_3\} \\
\cup & & \cup \\
\cancel{E = sp\{v_1, v_2\}} & \longrightarrow & E = sp\{v_1, v_2\} \\
\mathbb{R}^2 \quad \cancel{x_1 v_1 + x_2 v_2} & & \\
(x_1, x_2) & \xrightarrow[f]{} & x_1 v_1 + x_2 v_2
\end{array}$$

basis — not the usual basis $\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}$

define $F : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad e_1 \to v_1 \quad e_2 \to v_2 \quad e_3 \to v_3$

$$F(x_1, x_2, x_3) = x_1 v_1 + x_2 v_2 + x_3 v_3$$

$F(e_1) = F(1,0,0) = v_1$ etc.

$$= x_1 (v_{11}, v_{12}, v_{13}) + x_2 (v_{21}, v_{22}, v_{23}) + x_3 (v_{31}, v_{32}, v_{33})$$

$F \mid \mathbb{R}^2 = f$

$$= \Big(\underbrace{x_1 v_{11} + x_2 v_{21} + x_3 v_{31}}_{f_1(x)}, \underbrace{x_1 v_{12} + x_2 v_{22} + x_3 v_{32}}_{f_2(x)}, \underbrace{x_1 v_{13} + x_2 v_{23} + x_3 v_{33}}_{f_3(x)}\Big)$$

$$F = (f_1, f_2, f_3) \qquad f_1(x_1, x_2, x_3) = v_{11} x_1 + v_{21} x_2 + v_{31} x_3$$

$$\frac{\partial f_1}{\partial x_1} = v_{11} \qquad \frac{\partial f_1}{\partial x_2} = v_{21} \qquad \frac{\partial f_1}{\partial x_3} = v_{31}$$

etc.

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Exercise 2.1.1 (Chain Rule). Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^\ell$, $Z \subset \mathbb{R}^m$ be arbitrary subsets. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps then so is the composition $g \circ f : X \to Z$. The identity map $\mathrm{id} : X \to X$ is smooth.

Exercise 2.1.2. Let $E \subset \mathbb{R}^k$ be an m-dimensional linear subspace and let $v_1, \dots, v_m$ be a basis of E. Then the map $f : \mathbb{R}^m \to E$ defined by $f(x) := \sum_{i=1}^m x_i v_i$ is a diffeomorphism.

This is extrinsic: $M \subset \mathbb{R}^k$ and charts have domains $M \cap U$ where U is open in $\mathbb{R}^k$

intrinsic means M is any set which has a maximal smooth atlas.

Definition 2.1.3. *Let $k, m \in \mathbb{N}_0$. A subset $M \subset \mathbb{R}^k$ is called a* **smooth m-dimensional submanifold of $\mathbb{R}^k$** *iff every point $p \in M$ has an open neighborhood $U \subset \mathbb{R}^k$ such that $U \cap M$ is diffeomorphic to an open subset $\Omega \subset \mathbb{R}^m$. A diffeomorphism*

$$\phi : U \cap M \to \Omega$$

is called a **coordinate chart** *of M and its inverse*

$$\psi := \phi^{-1} : \Omega \to U \cap M$$

is called a **(smooth) parametrization** *of $U \cap M$.*

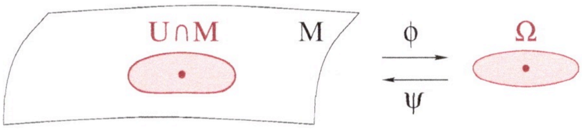


Figure 2.1: A coordinate chart $\phi : U \cap M \to \Omega$.

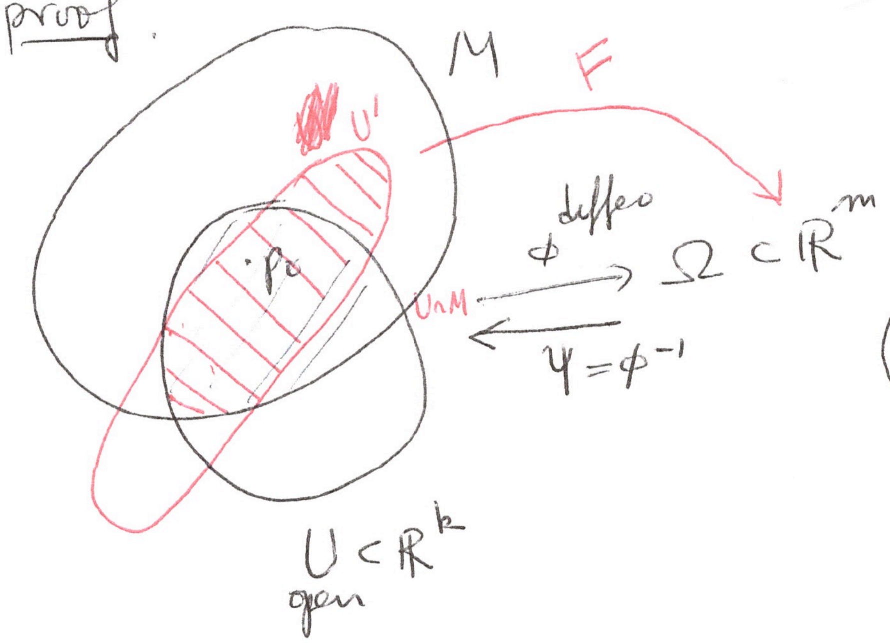
In Definition 2.1.3 we have used the fact that the domain of a smooth map can be an arbitrary subset of Euclidean space and need not be open (see page 15). The term *m-manifold in $\mathbb{R}^k$* is short for *m-dimensional submanifold of $\mathbb{R}^k$*. In keeping with the Master Plan §1.5 we will sometimes say *manifold* rather than *submanifold of $\mathbb{R}^k$* to indicate that the context holds in both the intrinsic and extrinsic settings.

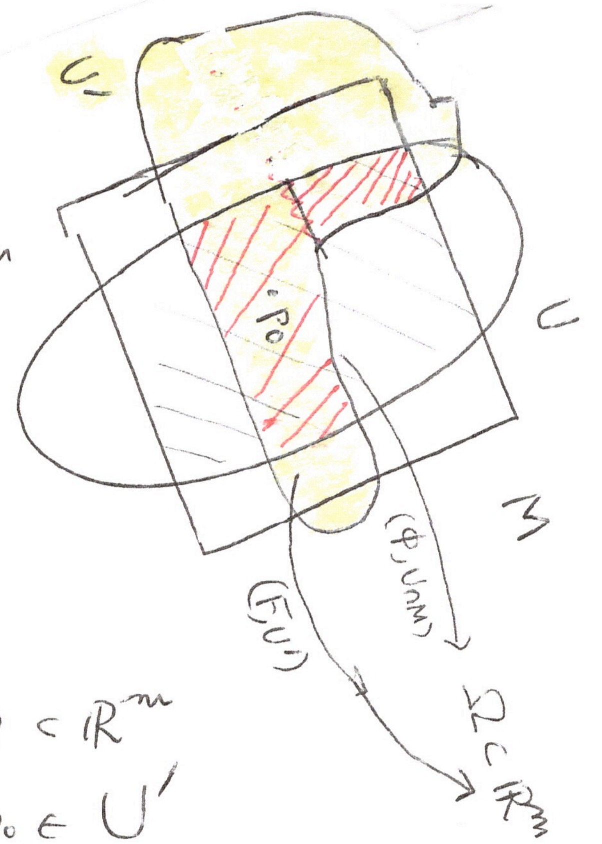
Lemma 2.1.4. *If $M \subset \mathbb{R}^k$ is a nonempty smooth m-manifold then $m \le k$.*

Proof. Fix an element $p_0 \in M$, choose a coordinate chart $\phi : U \cap M \to \Omega$ with $p_0 \in U$ and values in an open subset $\Omega \subset \mathbb{R}^m$, and denote its inverse by $\psi := \phi^{-1} : \Omega \to U \cap M$. Shrinking U, if necessary, we may assume that ϕ extends to a smooth map $\Phi : U \to \mathbb{R}^m$. This extension satisfies $\Phi(\psi(x)) = \phi(\psi(x)) = x$ and hence $d\Phi(\psi(x))d\psi(x) = \mathrm{id} : \mathbb{R}^m \to \mathbb{R}^m$ for all $x \in \Omega$, by the chain rule. Hence the derivative $d\psi(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective for all $x \in \Omega$, and hence $m \le k$ because Ω is nonempty. This proves Lemma 2.1.4. □

Lemma 2.1.4 $\quad M \subset \mathbb{R}^k$ smooth m manifold $\Rightarrow m \le k$

<u>proof</u>.





ϕ is a smooth map $U \cap M \longrightarrow \Omega \subset \mathbb{R}^m$

which means $\exists\ U'$ open $\subset \mathbb{R}^k$ s.t. $p_0 \in U'$

and a smooth map $F : U' \to \mathbb{R}^m$ with

$F \mid U' \cap (U \cap M) = \phi \mid U' \cap (U \cap M)$ $\qquad \Omega' \subset \Omega \subset \mathbb{R}^m$

Then ϕ extends to a smooth map $\overset{=F!}{\Phi} : U \cap U' \to \mathbb{R}^m$

We have $\Phi(\psi(x)) = \phi(\psi(x)) = x$ $\quad$ for $x \in \Phi(U \cap U') = \Omega'$

chain rule $\quad \underbrace{d\Phi(\psi(x))}_{\mathbb{R}^k \to \mathbb{R}^m}\ \underbrace{d\psi(x)}_{\mathbb{R}^m \to \mathbb{R}^k} = \mathrm{id} : \mathbb{R}^m \to \mathbb{R}^m \quad \forall\ x \in \Omega'$

If $d\psi(x)\xi = 0$, $\xi \in \mathbb{R}^m$ then $0 = d\Phi(\psi(x))\, d\psi(x)\, \xi \overset{= \mathrm{id}(\xi)}{} $ so $\xi = 0$ & $d\psi(x)$ is injective

so $\mathbb{R}^m$ is isomorphic to a subspace of $\mathbb{R}^k$

So $m \le k$.

we are skipping pp 17–23 at this time

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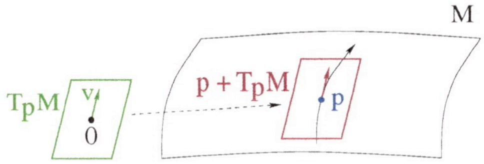

Figure 2.3: The tangent space T_pM and the translated tangent space $p+T_pM$.

2.2 Tangent Spaces and Derivatives

The main reason for first discussing the extrinsic notion of embedded manifolds in Euclidean space as explained in the Master Plan §1.5 is that the concept of a tangent vector is much easier to digest in the embedded case: it is simply the derivative of a curve in M, understood as a vector in the ambient Euclidean space in which M is embedded.

2.2.1 Tangent Space

$\gamma : \mathbb{R} \to M$

$\dot{\gamma} = d\gamma : \mathbb{R} \to \mathbb{R}^k$

is a linear transformation

Definition 2.2.1. *Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and fix a point $p \in M$. A vector $v \in \mathbb{R}^k$ is called a* **tangent vector** *of M at p if there exists a smooth curve $\gamma : \mathbb{R} \to M$ such that*

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = v.$$

The set

$$T_pM := \{\dot{\gamma}(0) \,|\, \gamma : \mathbb{R} \to M \text{ is smooth, } \gamma(0) = p\}$$

of tangent vectors of M at p is called the **tangent space** *of M at p.*

Theorem 2.2.3 below shows that T_pM is a linear subspace of $\mathbb{R}^k$. As does any linear subspace it contains the origin; it need not actually intersect M. Its translate $p + T_pM$ touches M at p; this is what you should visualize for T_pM (see Figure 2.3).

Remark 2.2.2. Let $p \in M$ be as in Definition 2.2.1 and let $v \in \mathbb{R}^k$. Then

$$v \in T_pM \qquad \iff \qquad \left\{ \begin{array}{l} \exists \varepsilon > 0 \;\; \exists \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that} \\ \gamma \text{ is smooth, } \gamma(0) = p,\ \dot{\gamma}(0) = v. \end{array} \right.$$

To see this suppose that $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Define $\widetilde{\gamma} : \mathbb{R} \to M$ by

$$\widetilde{\gamma}(t) := \gamma\left(\frac{\varepsilon t}{\sqrt{\varepsilon^2 + t^2}}\right), \qquad t \in \mathbb{R}.$$

Then $\widetilde{\gamma}$ is smooth and satisfies $\widetilde{\gamma}(0) = p$ and $\dot{\widetilde{\gamma}}(0) = v$. Hence $v \in T_pM$.

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Theorem 2.2.3 (**Tangent spaces**). *Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and fix a point $p \in M$. Then the following holds.*

(i) *Let $U_0 \subset M$ be an M-open set with $p \in U_0$ and let $\phi_0 : U_0 \to \Omega_0$ be a diffeomorphism onto an open subset $\Omega_0 \subset \mathbb{R}^m$. Let $x_0 := \phi_0(p)$ and let $\psi_0 := \phi_0^{-1} : \Omega_0 \to U_0$ be the inverse map. Then*

$$T_pM = \operatorname{im}\left(d\psi_0(x_0) : \mathbb{R}^m \to \mathbb{R}^k\right).$$

(ii) *Let $U, \Omega \subset \mathbb{R}^k$ be open sets and $\phi : U \to \Omega$ be a diffeomorphism such that $p \in U$ and $\phi(U \cap M) = \Omega \cap (\mathbb{R}^m \times \{0\})$. Then*

$$T_pM = d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right).$$

(iii) *Let $U \subset \mathbb{R}^k$ be an open neighborhood of p and $f : U \to \mathbb{R}^{k-m}$ be a smooth map such that 0 is a regular value of f and $U \cap M = f^{-1}(0)$. Then*

$$T_pM = \ker df(p).$$

(iv) *T_pM is an m-dimensional linear subspace of $\mathbb{R}^k$.*

Proof. Let $\psi_0 : \Omega_0 \to U_0$ and $x_0 \in \Omega_0$ be as in (i) and let $\phi : U \to \Omega$ be as in (ii). We prove that

→ $d\phi(p)$ is invertible linear transf $\mathbb{R}^k \to \mathbb{R}^k$

$$\operatorname{im} d\psi_0(x_0) \subset T_pM \subset d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right). \tag{2.2.1}$$

To prove the first inclusion in (2.2.1), choose a constant $r > 0$ such that

$$B_r(x_0) := \{x \in \mathbb{R}^m \mid |x - x_0| < r\} \subset \Omega_0.$$

Now let $\xi \in \mathbb{R}^m$ and choose $\varepsilon > 0$ so small that

$$\varepsilon\,|\xi| \le r.$$

Then $x_0 + t\xi \in \Omega_0$ for all $t \in \mathbb{R}$ with $|t| < \varepsilon$. Define $\gamma : (-\varepsilon, \varepsilon) \to M$ by

$$\gamma(t) := \psi_0(x_0 + t\xi) \qquad \text{for } -\varepsilon < t < \varepsilon.$$

Then γ is a smooth curve in M satisfying

$$\gamma(0) = \psi_0(x_0) = p, \qquad \dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \psi_0(x_0 + t\xi) = d\psi_0(x_0)\xi.$$

Hence it follows from Remark 2.2.2 that $d\psi_0(x_0)\xi \in T_pM$, as claimed.

proof:

$\phi : U \to \Omega$

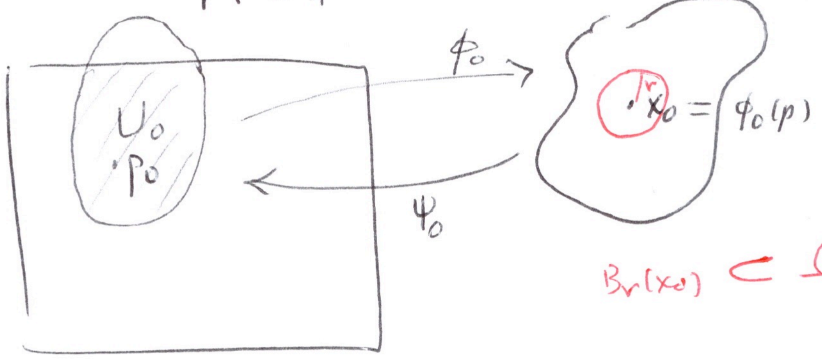
$\phi^{-1} \circ \phi\,(p) = p$

$\phi^{-1} \circ \phi = \mathrm{id}_U$

$\underbrace{d\phi^{-1}(\phi(p))}_{\text{this is the inverse of } d\phi(p)}\, d\phi(p) = \mathrm{id}_{\mathbb{R}^k}$

Proof of Th 2.2.3 p. 25

$M \subset \mathbb{R}^k$ $\qquad$ $\Omega_0 \subset \mathbb{R}^m \subset \mathbb{R}^k$



$B_r(x_0) \subset \Omega_0$ $\qquad$ Ω_0 is an open set

~~Claim~~ $\xi \in \mathbb{R}^m$ $\quad \varepsilon|\xi| \le r$ $\quad$ (Choose $\varepsilon \le \frac{r}{\|\xi\|}$)

$|t|<\varepsilon,\ t\in\mathbb{R} \Rightarrow x_0 - t\xi \in \Omega_0$

$\gamma: (-\varepsilon,\varepsilon) \longrightarrow M \qquad \gamma(t) = \psi_0(x_0 + t\xi)$

$\gamma(0) = \psi_0(x_0) = p \qquad \dot\gamma(0) = \frac{d}{dt}\Big|_{t=0} \psi_0(x_0 + t\xi) = d\psi_0(x_0)\,\xi$

By Remark 2.2.2 $\quad d\psi_0(x_0)\xi \in T_pM$ $\quad$ (you only need $\gamma: (-\varepsilon,\varepsilon) \longrightarrow M$ not necessarily $\gamma: \mathbb{R} \longrightarrow M$)

This proves $\quad \operatorname{im} d\psi_0(x_0) \subset T_pM$ $\quad$ (*)

Next we prove $\quad T_pM \subset d\phi(p)^{-1}(\mathbb{R}^m \times \{0\})$ $\quad$ (**)

where $\phi : U \to \Omega$ is a diffeomorphism with $p \in U$, U, Ω open in $\mathbb{R}^k$

$$\phi(U \cap M) = \Omega \cap (\mathbb{R}^m \times \underbrace{\{0\}}_{\in \mathbb{R}^{k-m}})$$

Fix $v \in T_p(M)$ so $\exists$ smooth $\gamma: \mathbb{R} \to M$ $\quad \gamma(0) = p \quad \dot\gamma(0) = v$.

$\exists\ \varepsilon > 0$ with $\gamma(|t|<\varepsilon) \subset U$ and $\phi(\gamma(|t|<\varepsilon)) \subseteq \phi(U \cap M) \subset \mathbb{R}^m \times \{0\}$ $\quad$ by assumption

$$\therefore \quad d\phi(p)v = d\phi(\gamma(0))\,\dot\gamma(0) = \frac{d}{dt}\Big|_{t=0} \underbrace{\phi(\gamma(t))}_{\in \mathbb{R}^m \times \{0\}} \in \mathbb{R}^m \times \{0\}$$

proving (**)

To prove the second inclusion in (2.2.1) we fix a vector $v \in T_pM$. Then, by definition of the tangent space, there exists a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $U \subset \mathbb{R}^k$ be as in (ii) and choose $\varepsilon > 0$ so small that $\gamma(t) \in U$ for $|t| < \varepsilon$. Then

$$\phi(\gamma(t)) \in \phi(U \cap M) \subset \mathbb{R}^m \times \{0\}$$

for $|t| < \varepsilon$ and hence

$$d\phi(p)v = d\phi(\gamma(0))\dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \phi(\gamma(t)) \in \mathbb{R}^m \times \{0\}.$$

This shows that $v \in d\phi(p)^{-1}(\mathbb{R}^m \times \{0\})$ and thus we have proved (2.2.1).

page 9 continues here

Now the sets $\operatorname{im} d\psi_0(x_0)$ and $d\phi(p)^{-1}(\mathbb{R}^m \times \{0\})$ are both m-dimensional linear subspaces of $\mathbb{R}^k$. Hence it follows from (2.2.1) that these subspaces agree and that they both agree with T_pM. Thus we have proved assertions (i), (ii), and (iv).

skip this at this time

We prove (iii). If $v \in T_pM$ then there is a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. For t sufficiently small we have $\gamma(t) \in U$, where $U \subset \mathbb{R}^k$ is the open set in (iii), and $f(\gamma(t)) = 0$. Hence

$$df(p)v = df(\gamma(0))\dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0$$

and this implies $T_pM \subset \ker df(p)$. Since T_pM and the kernel of $df(p)$ are both m-dimensional linear subspaces of $\mathbb{R}^k$ we deduce that $T_pM = \ker df(p)$. This proves part (iii) and Theorem 2.2.3. □

Example 2.2.4. Let $A = A^\mathsf{T} \in \mathbb{R}^{k\times k}$ be a nonzero matrix as in Example 2.1.12 and let $c \neq 0$. Then, by Theorem 2.2.3 (iii), the tangent space of the manifold

$$M = \left\{x \in \mathbb{R}^k \,|\, x^\mathsf{T}Ax = c\right\}$$

at a point $x \in M$ is the $k-1$-dimensional linear subspace

$$T_xM = \left\{\xi \in \mathbb{R}^k \,|\, x^\mathsf{T}A\xi = 0\right\}.$$

Example 2.2.5. As a special case of Example 2.2.4 with $A = \mathbb{1}$ and $c = 1$ we find that the tangent space of the unit sphere $S^m \subset \mathbb{R}^{m+1}$ at a point $x \in S^m$ is the orthogonal complement of x:

$$T_xS^m = x^\perp = \left\{\xi \in \mathbb{R}^{m+1} \,|\, \langle x, \xi\rangle = 0\right\}.$$

Here $\langle x, \xi\rangle = \sum_{i=0}^m x_i\xi_i$ denotes the standard inner product on $\mathbb{R}^{m+1}$.

(11)

Exercise 2.2.6. What is the tangent space of the 5-manifold

$$M := \left\{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, |x - y| = r\right\}$$

at a point $(x, y) \in M$? (See Exercise 2.1.14.)

Example 2.2.7. Let $H(x, y) := \frac{1}{2}|y|^2 + V(x)$ be as in Exercise 2.1.17 and let c be a regular value of H. If $(x, y) \in M := H^{-1}(c)$ Then

$$T_{(x,y)}M = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \langle y, \eta\rangle + \langle \nabla V(x), \xi\rangle = 0\}.$$

Here $\nabla V := (\partial V/\partial x_1, \ldots, \partial V/\partial x_n) : \mathbb{R}^n \to \mathbb{R}^n$ denotes the gradient of V.

Exercise 2.2.8. The tangent space of $\mathrm{SL}(n, \mathbb{R})$ at the identity matrix is the space

$$\mathfrak{sl}(n, \mathbb{R}) := T_{\mathbb{1}}\mathrm{SL}(n, \mathbb{R}) = \left\{\xi \in \mathbb{R}^{n\times n} \,|\, \mathrm{trace}(\xi) = 0\right\}$$

of traceless matrices. (Prove this, using Exercise 2.1.18.)

Example 2.2.9. The tangent space of $\mathrm{O}(n)$ at g is

$$T_g\mathrm{O}(n) = \left\{v \in \mathbb{R}^{n\times n} \,|\, g^\mathsf{T}v + v^\mathsf{T}g = 0\right\}.$$

In particular, the tangent space of $\mathrm{O}(n)$ at the identity matrix is the space of skew-symmetric matrices

$$\mathfrak{o}(n) := T_{\mathbb{1}}\mathrm{O}(n) = \left\{\xi \in \mathbb{R}^{n\times n} \,|\, \xi^\mathsf{T} + \xi = 0\right\}$$

To see this, choose a smooth curve $\mathbb{R} \to \mathrm{O}(n) : t \mapsto g(t)$. Then $g(t)^\mathsf{T}g(t) = \mathbb{1}$ for all $t \in \mathbb{R}$ and, differentiating this identity with respect to t, we obtain $g(t)^\mathsf{T}\dot{g}(t) + \dot{g}(t)^\mathsf{T}g(t) = 0$ for every t. Hence every matrix $v \in T_g\mathrm{O}(n)$ satisfies the equation $g^\mathsf{T}v + v^\mathsf{T}g = 0$. With this understood, the claim follows from the fact that $g^\mathsf{T}v + v^\mathsf{T}g = 0$ if and only if the matrix $\xi := g^{-1}v$ is skew-symmetric and that the space of skew-symmetric matrices in $\mathbb{R}^{n\times n}$ has dimension $n(n-1)/2$.

Exercise 2.2.10. Let $\Omega \subset \mathbb{R}^m$ be an open set and $h : \Omega \to \mathbb{R}^{k-m}$ be a smooth map. Prove that the tangent space of the graph of h at a point $(x, h(x))$ is the graph of the differential $dh(x) : \mathbb{R}^m \to \mathbb{R}^{k-m}$:

$$M = \{(x, h(x)) \,|\, x \in \Omega\}, \qquad T_{(x,h(x))}M = \{(\xi, dh(x)\xi) \,|\, \xi \in \mathbb{R}^m\}.$$