199B 2-2-18

Exercise 2.2.11 (Monge coordinates). Let M be a smooth m-manifold in $\mathbb{R}^k$ and suppose that $p \in M$ is such that the projection $T_pM \to \mathbb{R}^m \times \{0\}$ is invertible. Prove that there exists an open set $\Omega \subset \mathbb{R}^m$ and a smooth map $h : \Omega \to \mathbb{R}^{k-m}$ such that the graph of h is an M-open neighborhood of p (see Example 2.1.6). Of course, the projection $T_pM \to \mathbb{R}^m \times \{0\}$ need not be invertible, but it must be invertible for at least one of the $\binom{k}{m}$ choices of the m dimensional coordinate plane. Hence every point of M has an M-open neighborhood which may be expressed as a graph of a function of some of the coordinates in terms of the others as in e.g. Example 2.1.5.

2.2.2 Derivative

SEE THE DIAGRAM ON THE NEXT PAGE

A key purpose behind the concept of a smooth manifold is to carry over the notion of a smooth map and its derivatives from the realm of first year analysis to the present geometric setting. Here is the basic definition. It appeals to the notion of a smooth map between arbitrary subsets of Euclidean spaces as introduced on page 15.

Definition 2.2.12. *Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold and*

$$f : M \to \mathbb{R}^\ell$$

be a smooth map. The **derivative** *of f at a point $p \in M$ is the map*

$$df(p) : T_pM \to \mathbb{R}^\ell$$

defined as follows. Given a tangent vector $v \in T_pM$ choose a smooth curve

$$\gamma : \mathbb{R} \to M$$

satisfying

$$\gamma(0) = p, \qquad \dot\gamma(0) = v.$$

Now define the vector

$$df(p)v \in \mathbb{R}^\ell$$

by

$$df(p)v := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \lim_{h\to 0} \frac{f(\gamma(h)) - f(p)}{h}. \tag{2.2.2}$$

That the limit on the right in equation (2.2.2) exists follows from our assumptions. We must prove, however, that the derivative is well defined, i.e. that the right hand side of (2.2.2) depends only on the tangent vector v and not on the choice of the curve γ used in the definition. This is the content of the first assertion in the next theorem.

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def 2.2.12

$\mathbb{R}^k$

U

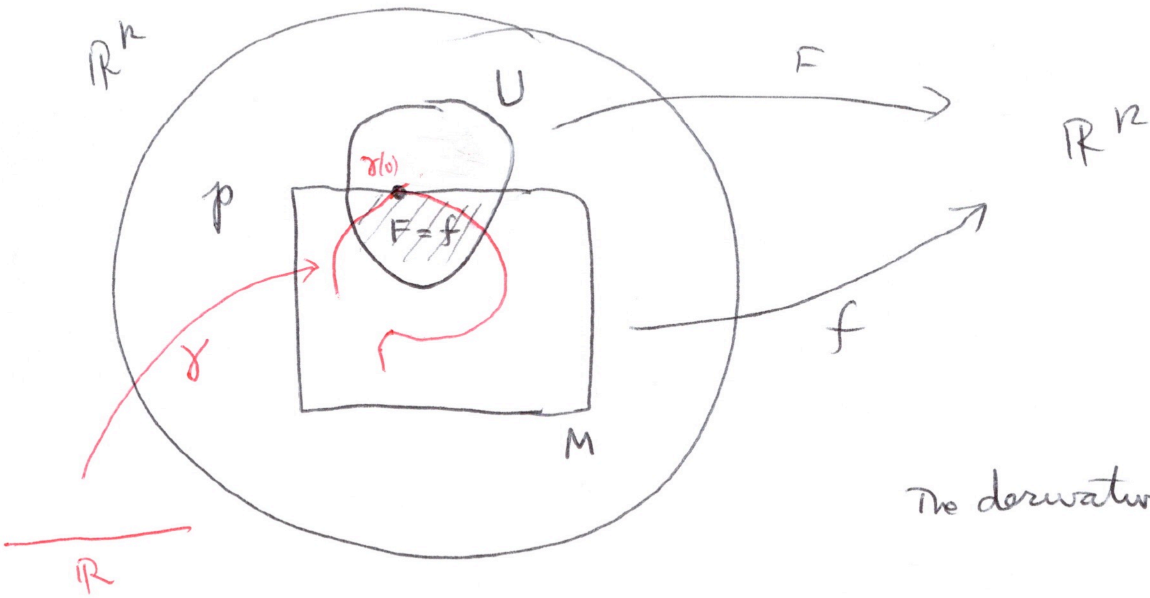
$M \xrightarrow[f \text{ smooth map}]{} \mathbb{R}^l$

m-dim'l smooth mfld

f smooth means $\forall \; p \in M \;\; \exists$ open set $U \subset \mathbb{R}^k$ with $p \in U$

and $\exists$ a smooth map $F : U \to \mathbb{R}^l$ with

$F|U \cap M = f|U \cap M$



The derivative of f at p is

$df(p) : T_pM \longrightarrow \mathbb{R}^l$ defined as follows

if $v \in T_pM$ & $\gamma : \mathbb{R} \to M$ $\gamma(0) = p$ $\dot{\gamma}(0) = v$ define

$df(p)v \in \mathbb{R}^l$ by $df(p)v = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$

$= \lim_{h \to 0} \frac{f(\gamma(h)) - f(p)}{h}$

Theorem 2.2.13 (Derivatives). *Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold and $f : M \to \mathbb{R}^\ell$ be a smooth map. Fix a point $p \in M$. Then the following holds.*

(i) *The right hand side of (2.2.2) is independent of γ.*

(ii) *The map $df(p) : T_pM \to \mathbb{R}^\ell$ is linear.*

see the notes on the next page

(iii) *If $N \subset \mathbb{R}^\ell$ is a smooth n-manifold and $f(M) \subset N$ then*

$$df(p)T_pM \subset T_{f(p)}N.$$

(iv) (Chain Rule) *Let N be as in (iii), suppose that $f(M) \subset N$, and let $g : N \to \mathbb{R}^d$ be a smooth map. Then*

$$d(g \circ f)(p) = dg(f(p)) \circ df(p) : T_pM \to \mathbb{R}^d.$$

(v) *If $f = \mathrm{id} : M \to M$ then $df(p) = \mathrm{id} : T_pM \to T_pM$.*

Proof. We prove (i). Let $v \in T_pM$ and $\gamma : \mathbb{R} \to M$ be as in Definition 2.2.12. By definition there is an open neighborhood $U \subset \mathbb{R}^k$ of p and a smooth map $F : U \to \mathbb{R}^\ell$ such that

$$F(p') = f(p') \qquad \text{for all } p' \in U \cap M.$$

Let $dF(p) \in \mathbb{R}^{\ell \times k}$ denote the Jacobian matrix (i.e. the matrix of all first partial derivatives) of F at p. Then, since $\gamma(t) \in U \cap M$ for t sufficiently small, we have

$$\begin{aligned} dF(p)v &= dF(\gamma(0))\dot\gamma(0) \\ &= \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)). \end{aligned}$$

The right hand side of this identity is independent of the choice of F while the left hand side is independent of the choice of γ. Hence the right hand side is also independent of the choice of γ and this proves (i). Assertion (ii) follows immediately from the identity

$$df(p)v = dF(p)v$$

just established.

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Theorem 2.2.13

(i) $df(p)v$ is well-defined, i.e. doesn't depend on the choice of γ (with $\gamma(0)=p$ $\gamma'(0)=v$)

$\exists$ open $U \subset \mathbb{R}^k$ & smooth $F: U \to \mathbb{R}^l$, $F=(f_1, .., f_l)$ with $F=f$ on $U \cap M$. F has a derivative at p $p \in U$

$dF(p): \mathbb{R}^k \to \mathbb{R}^l$ which is a linear transformation given by the Jacobian matrix ($l \times k$)

$$L = dF(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p), \dots, \frac{\partial f_1}{\partial x_k}(p) \\ \vdots \\ \frac{\partial f_l(p)}{\partial x_1} \dots \frac{\partial f_l(p)}{\partial x_k} \end{bmatrix}$$

So that for $v \in \mathbb{R}^k$, $v=(v_1, .., v_k)$

$$dF(p)v = Lv = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \dots & \frac{\partial f_1}{\partial x_k}(p) \\ \vdots & & \vdots \\ \frac{\partial f_l}{\partial x_1}(p) & \dots & \frac{\partial f_l}{\partial x_k}(p) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} \nabla f_1(p) \cdot v \\ \vdots \\ \nabla f_l(p) \cdot v \end{bmatrix}$$

Since γ is continuous $\gamma(t) \in U$ for all $|t| < \varepsilon$

so $\underbrace{dF(p)v}_{\text{doesn't depend on } \gamma} = dF(\gamma(0))\gamma'(0) \overset{\text{chain rule}}{=} \frac{d}{dt}\Big|_{t=0} F(\gamma(t)) = \underbrace{\frac{d}{dt}\Big|_{t=0} f(\gamma(t))}_{\text{doesn't depend on } F}$

This also proves (ii)

(ii) $df(p)$ is a linear transformation.

Assertion (iii) follows directly from the definitions. Namely, if γ is as in Definition 2.2.12 then

$$\beta := f \circ \gamma : \mathbb{R} \to N$$

is a smooth curve in N satisfying

$$\beta(0) = f(\gamma(0)) = f(p) =: q, \qquad \dot{\beta}(0) = df(p)v =: w.$$

Hence $w \in T_qN$. Assertion (iv) also follows directly from the definitions. If $g : N \to \mathbb{R}^d$ is a smooth map and β, q, w are as above then

$$\begin{aligned} d(g \circ f)(p)v &= \left.\frac{d}{dt}\right|_{t=0} g(f(\gamma(t))) \\ &= \left.\frac{d}{dt}\right|_{t=0} g(\beta(t)) \\ &= dg(q)w \\ &= dg(f(p))df(p)v. \end{aligned}$$

and this proves (iv). Assertion (v) follows directly from the definitions and this proves Theorem 2.2.13. □

Corollary 2.2.14 (Diffeomorphisms). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $N \subset \mathbb{R}^\ell$ be a smooth n-manifold and let $f : M \to N$ be a diffeomorphism. Then $m = n$ and the differential $df(p) : T_pM \to T_{f(p)}N$ is a vector space isomorphism with inverse*

$$df(p)^{-1} = df^{-1}(f(p)) : T_{f(p)}N \to T_pM$$

for all $p \in M$.

Proof. Define $g := f^{-1} : N \to M$ so that

$$g \circ f = \mathrm{id}_M, \qquad f \circ g = \mathrm{id}_N.$$

Then it follows from Theorem 2.2.13 that, for $p \in M$ and $q := f(p) \in N$, we have

$$dg(q) \circ df(p) = \mathrm{id} : T_pM \to T_pM, \qquad df(p) \circ dg(q) = \mathrm{id} : T_qN \to T_qN.$$

Hence $df(p) : T_pM \to T_qN$ is a vector space isomorphism with inverse

$$dg(q) = df(p)^{-1} : T_qN \to T_pM.$$

Hence $m = n$ and this proves Corollary 2.2.14. □