Review

$\forall p \in M$, $\exists$ open $U \subset \mathbb{R}^k$
$U \cap M \xrightarrow{\phi \text{ diffeo}} \Omega \overset{\text{open}}{\subset} \mathbb{R}^m$

$M \subset \mathbb{R}^k$ smooth m-manifold $\quad p \in M$

$$T_pM = \{ \dot\gamma(0) : \gamma : \mathbb{R} \to M \text{ smooth}, \gamma(0) = p \} \subset \mathbb{R}^k$$

T_pM is an m-dimensional subspace of $\mathbb{R}^k$. (Th. 2.2.3 (iv) p. 25)

$M \subset \mathbb{R}^k$ smooth m-dim'l mfld $\quad f : M \to \mathbb{R}^\ell$ smooth

$$df(p) : T_p(M) \to \mathbb{R}^\ell$$

$$df(p)v = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) \quad \text{where } \gamma : \mathbb{R} \to M \quad \gamma(0) = p \quad \dot\gamma(0) = v$$

$df(p)$ is a linear transf & if $f(M) \subset N \subset \mathbb{R}^\ell$ (smooth n-mfld)

then $df(p)(T_p(M)) \subset T_{f(p)}N$ (Th. 2.2.13 p. 29)

2.4 Vector Fields and Flows

2.4.1 Vector Fields

Definition 2.4.1 (Vector Field). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold. A* **(smooth) vector field** *on M is a smooth map $X : M \to \mathbb{R}^k$ such that*

$$X(p) \in T_pM$$

for every $p \in M$. The set of smooth vector fields on M will be denoted by

$$\mathrm{Vect}(M) := \Big\{X : M \to \mathbb{R}^k \,|\, X \textit{ is smooth, } X(p) \in T_pM \textit{ for all } p \in M\Big\}.$$

Exercise 2.4.2. Prove that the set of smooth vector fields on M is a real vector space.

Example 2.4.3. Denote the standard cross product on $\mathbb{R}^3$ by

$$x \times y := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

For $x, y \in \mathbb{R}^3$. Fix a vector $\xi \in S^2$ and define the maps $X, Y : S^2 \to \mathbb{R}^3$ by

$$X(p) := \xi \times p, \qquad Y(p) := (\xi \times p) \times p.$$

These are vector fields with zeros $\pm\xi$. Their integral curves (see Definition 2.4.6 below) are illustrated in Figure 2.8.

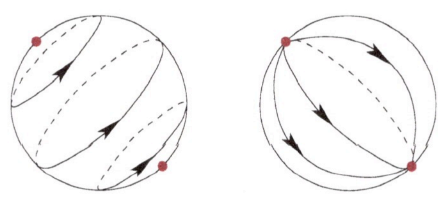


Figure 2.8: Two vector fields on the 2-sphere.

Example 2.4.4. Let $M := \mathbb{R}^2$. A vector field on M is then any smooth map $X : \mathbb{R}^2 \to \mathbb{R}^2$. As an example consider the vector field

$$X(x, y) := (x, -y).$$

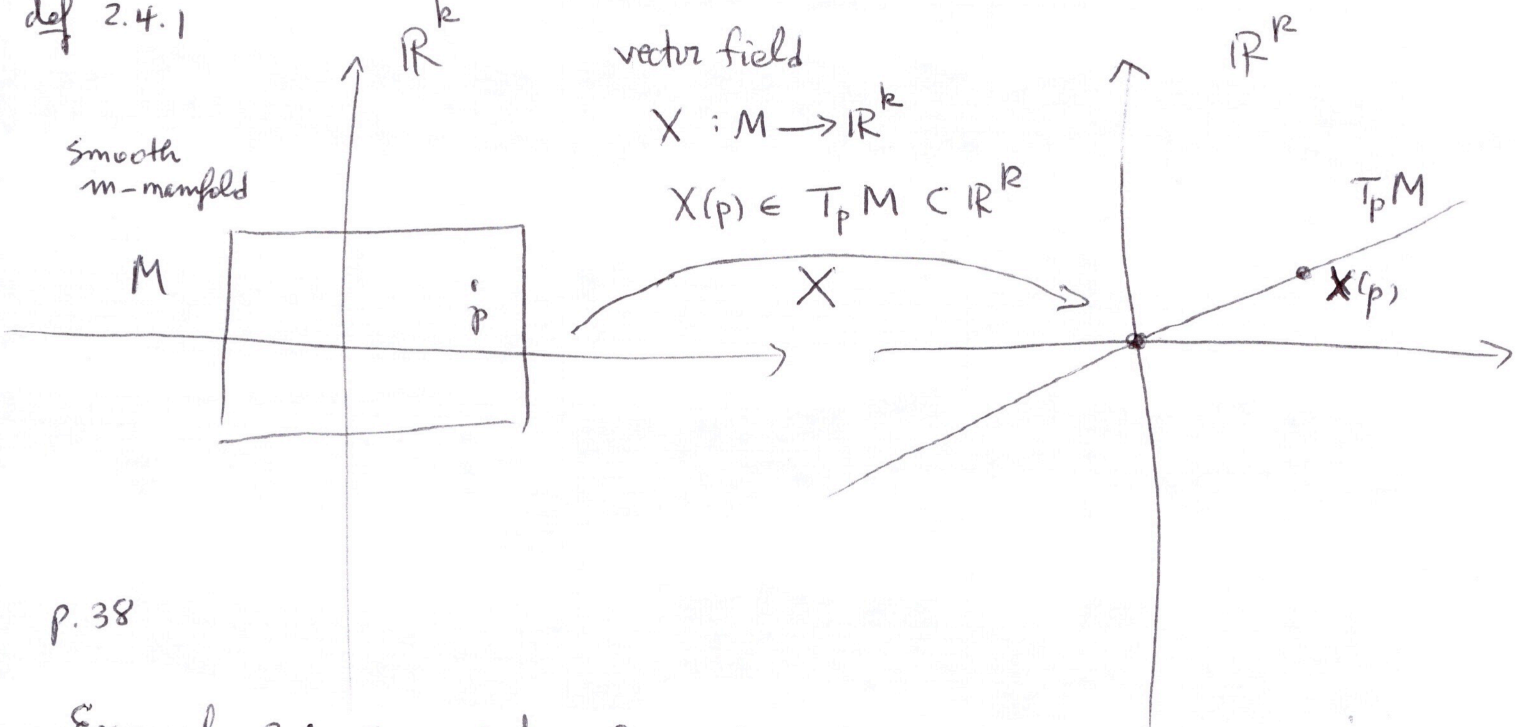
This vector field has a single zero at the origin and its integral curves are illustrated in Figure 2.9.

def 2.4.1

vector field

$X : M \to \mathbb{R}^k$

$X(p) \in T_p M \subset \mathbb{R}^k$



Example 2.4.5 let $f : \mathbb{R}^m \to \mathbb{R}$ smooth function

$M = \mathbb{R}^m$ $\quad X : M \to T_p(M)$ $\quad p = (x_1, \ldots, x_m) \in \mathbb{R}^m$

$T_p M = \mathbb{R}^m$

$$X(p) = \left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_m}(p) \right) \in \mathbb{R}^m$$

smooth vector field.

def 2.4.6 (integral curves) $M \subset \mathbb{R}^k$ smooth m-mfld

$X \in \mathrm{Vect}(M)$ $\quad X : M \longrightarrow \bigcup_{p \in M} T_p(M) \subset \mathbb{R}^k \quad X(p) \in T_p(M)$

$\gamma : I \longrightarrow M \qquad I$ open interval $\subset \mathbb{R}$

$$\dot\gamma(t) = X(\gamma(t)) \quad \forall\, t \in I.$$

Example 2.4.4 p.37 $\quad M = \mathbb{R}^2 \qquad X : \mathbb{R}^2 \to \mathbb{R}^2$

say $X(x,y) = (x,-y)$

let $\gamma(t) = (x(t), y(t)) \qquad \dot\gamma(t) = (x'(t), y'(t))$

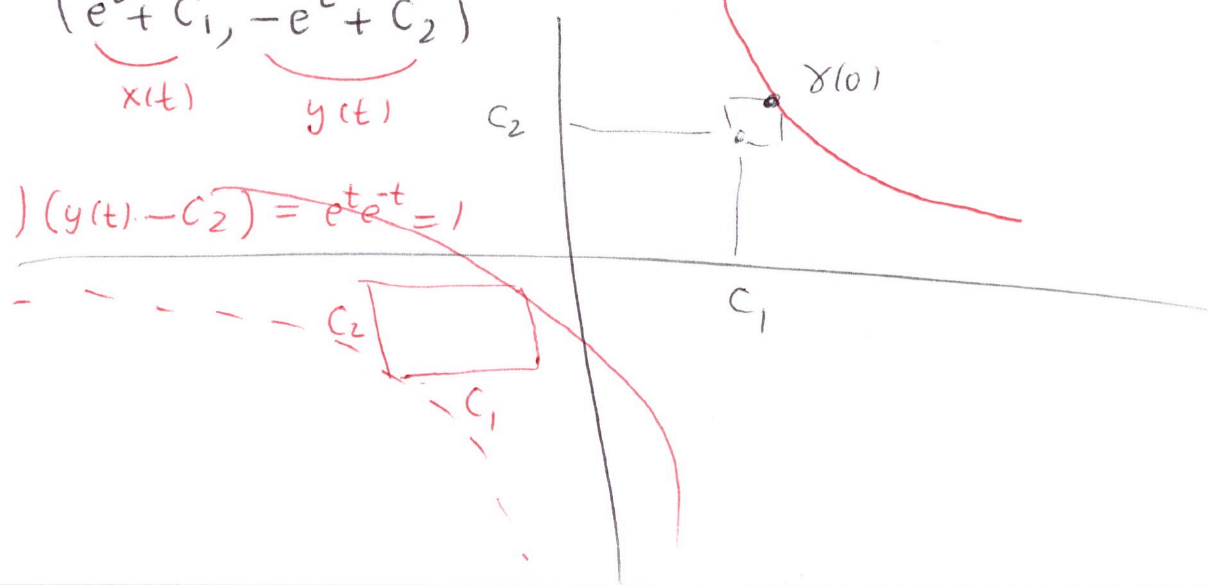
$X(\gamma(t)) = (x(t), -y(t))$

$\therefore\ X(\gamma(t)) = \dot\gamma(t) \implies x'(t) = x(t) \qquad y'(t) = -y(t)$

$x(t) = e^t + C_1 \qquad y(t) = -e^t + C_2$

$$\gamma(t) = (\underbrace{e^t + C_1}_{x(t)}, \underbrace{-e^t + C_2}_{y(t)})$$

$(x(t) - C_1)(y(t) - C_2) = e^t e^{-t} = 1$



Theorem 2.4.7 $\quad M \subset \mathbb{R}^k$ smooth m-mfld $\quad X \in \mathrm{Vect}(M)$ smooth $\quad p_0 \in M$

(i) $\exists$ open interval $I \subset \mathbb{R}$, $0 \in I$ and smooth $\gamma : I \to M$

satisfying $\quad \dot\gamma(t) = X(\gamma(t)) \quad \gamma(0) = p_0 \quad \forall t \in I$.

Proof let $U \subset \mathbb{R}^k$ open $\quad \phi_0 : (\underbrace{U \cap M}_{U_0}) \to \Omega \subset \mathbb{R}^m$ open diffeom

$U_0 \ni p_0$, $\quad \Omega = \phi_0(U_0)$

$\psi_0 : \Omega \to M$, For $x \in \Omega$ $\qquad M \xleftarrow{\psi_0 = \phi_0^{-1}} \Omega$

$d\psi_0(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective (i.e. one-to-one)

and $\quad d\psi_0(x)(\mathbb{R}^m) = T_{\psi_0(x)} M \qquad$ (Theorem 2.2.3 p.25)

Define $f : \Omega \to \mathbb{R}^m \qquad f(x) = (d\psi_0(x))^{-1}(X(\psi_0(x)))$

($\psi_0(x) \in M$, $\quad X(\psi_0(x)) \in T_{\psi_0(x)} M$, $\quad (d\psi_0(x))^{-1}(X(\psi_0(x))) \in \mathbb{R}^m$)

f is smooth (ψ_0, X, $(d\psi_0(x))^{-1}$ .... OOPS

smooth ✓ $\qquad$ linear, so obviously smooth

(BE CAREFUL!)

$$\mathbb{R}^m \supset \Omega \xrightarrow{\psi_0} M \xrightarrow{X} \mathbb{R}^k \supset T_{\psi_0(x)} M \xrightarrow{(d\psi_0(x))^{-1}} \mathbb{R}^m$$

with $M \subset \mathbb{R}^k$.

$$x \longrightarrow \psi_0(x) \longrightarrow X(\psi_0(x)) \longrightarrow (d\psi_0(x))^{-1}(X(\psi_0(x)))$$

$\psi_0(x) = (\psi_0^1(x), \dots, \psi_0^k(x))$ $\quad \psi_0 = (\psi_0^1, \dots, \psi_0^k)$ $\quad \psi_0^j : \Omega \to \mathbb{R}$ smooth $1 \le j \le k$

$X(p) = (X_1(p), \dots, X_k(p))$ $\quad X = (X_1, \dots, X_k)$ $\quad X_j : M \to \mathbb{R}$ smooth $1 \le j \le k$

$$X(\psi_0(x)) = X(\psi_0^1(x), \dots, \psi_0^k(x)) = (X_1(\psi_0^1(x), \dots, \psi_0^k(x)), \dots, X_k(\psi_0^1(x), \dots, \psi_0^k(x)))$$

$$d\psi_0(x) = \begin{bmatrix} \frac{\partial \psi_0^1}{\partial x_1}(x), & \dots & \frac{\partial \psi_0^1(x)}{\partial x_m} \\ \vdots & & \\ \frac{\partial \psi_0^k(x)}{\partial x_1} & \dots & \frac{\partial \psi_0^k(x)}{\partial x_m} \end{bmatrix}_{k \times m}$$

$d\psi_0(x)$ is injective so maps onto an m-dim'l subspace of $\mathbb{R}^k$ which we can map onto $\mathbb{R}^m \times \mathbb{R}^{k-m}$

$\therefore$ $d\psi_0(x)$ is an $m \times m$ square matrix with smooth entries which are functions of $x \in \Omega$

$$d\psi_0(x)^{-1} = \begin{bmatrix} \text{NICE functions of } \frac{\partial \psi_0^j(x)}{\partial x_\ell} \end{bmatrix}_{m \times m} = [A_{ij}(x)]$$

$\underbrace{m \times m}_{\text{square}}$

$\therefore$ $d\psi_0(x)^{-1}(X(\psi_0(x))) =$ see the next page

$$d\psi_0(x)^{-1}\left(X(\psi_0(x))\right) = \begin{bmatrix} A_{11}(x) & \cdots & A_{1m}(x) \\ \vdots & & \vdots \\ A_{m1}(x) & & A_{mm}(x) \end{bmatrix} \begin{bmatrix} X_1(\psi_0(x)) \\ \vdots \\ X_k(\psi_0(x)) \end{bmatrix}$$

$m \times m$ $\qquad$ $k \times 1$

don't match

not to worry

$X(\psi_0(x)) \in T_{\psi_0(x)} M$ ~~=~~ an m-dim'l subspace of $\mathbb{R}^k$

so the vector is really ~~$1 \times m$~~ $m \times 1$.

Consider the differential equation (actually initial value problem)

$$\begin{cases} \dot{y}(t) = f(y(t)) & t \in \mathbb{R} \\ y(0) = \varphi_0(p_0) & \boxed{y(t) \in \Omega \subset \mathbb{R}^m \quad \text{unknown function!}} \end{cases}$$

By basic existence theorem for first order differential equations $\exists$ a ~~function~~ solution $y : I \to \Omega$ $\quad I$ open interval $0 \in I$

Set $\gamma(t) = \psi_0(y(t)) \; : I \to \Omega \to U_0 \subset M$

$\gamma(0) = \psi_0(\varphi_0(p_0)) = p_0$ $\qquad \dot{\gamma}(t) = f(y(t)) = \underbrace{d\psi_0(y(t))^{-1}}_{?} X(\underbrace{\psi_0(y(t))}_{\gamma(t)})$

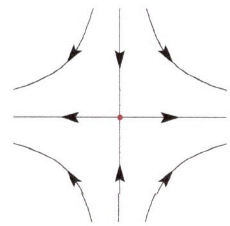


Figure 2.9: A hyperbolic fixed point.

Example 2.4.5. Every smooth function $f : \mathbb{R}^m \to \mathbb{R}$ determines a gradient vector field

$$X = \nabla f := \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_m} \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^m.$$

Definition 2.4.6 (Integral curves). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, let $X \in \mathrm{Vect}(M)$ be a smooth vector field on M, and let $I \subset \mathbb{R}$ be an open interval. A smooth map $\gamma : I \to M$ is called an* **integral curve** *of X if it satisfies the equation*

$$\dot\gamma(t) = X(\gamma(t))$$

for every $t \in I$.

Theorem 2.4.7. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. Fix a point $p_0 \in M$. Then the following holds.*

(i) *There is an open interval $I \subset \mathbb{R}$ containing 0 and a smooth curve $\gamma : I \to M$ satisfying the equation*

$$\dot\gamma(t) = X(\gamma(t)), \qquad \gamma(0) = p_0 \tag{2.4.1}$$

for every $t \in I$.

(ii) *If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are two solutions of (2.4.1) on open intervals I_1 and I_2 containing 0, then $\gamma_1(t) = \gamma_2(t)$ for every $t \in I_1 \cap I_2$.*

Proof. We prove (i). Let $\phi_0 : U_0 \to \mathbb{R}^m$ be a coordinate chart on M, defined on an M-open neighborhood $U_0 \subset M$ of p_0. The image of ϕ_0 is an open set

$$\Omega := \phi_0(U_0) \subset \mathbb{R}^m$$

and we denote the inverse map by $\psi_0 := \phi_0^{-1} : \Omega \to M$. Then, by Theorem 2.2.3, the differential $d\psi_0(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is the tangent space $T_{\psi_0(x)}M$ for every $x \in \Omega$. Define $f : \Omega \to \mathbb{R}^m$ by

$$f(x) := d\psi_0(x)^{-1} X(\psi_0(x)), \qquad x \in \Omega.$$

This map is smooth and hence, by the basic existence and uniqueness theorem for ordinary differential equations in $\mathbb{R}^m$ (see [17]), the equation

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0 := \phi_0(p_0), \tag{2.4.2}$$

has a solution $x : I \to \Omega$ on some open interval $I \subset \mathbb{R}$ containing 0. Hence the function

$$\gamma := \psi_0 \circ x : I \to U_0 \subset M$$

is a smooth solution of (2.4.1). This proves (i).

The local uniqueness theorem asserts that two solutions $\gamma_i : I_i \to M$ of (2.4.1) for $i = 1, 2$ agree on the interval $(-\varepsilon, \varepsilon) \subset I_1 \cap I_2$ for $\varepsilon > 0$ sufficiently small. This follows immediately from the standard uniqueness theorem for the solutions of (2.4.2) in [17] and the fact that $x : I \to \Omega$ is a solution of (2.4.2) if and only if $\gamma := \psi_0 \circ x : I \to U_0$ is a solution of (2.4.1).

To prove (ii) we observe that the set

$$I := I_1 \cap I_2$$

is an open interval containing zero and hence is connected. Now consider the set

$$A := \{t \in I \,|\, \gamma_1(t) = \gamma_2(t)\}.$$

This set is nonempty, because $0 \in A$. It is closed, relative to I, because the maps $\gamma_1 : I \to M$ and $\gamma_2 : I \to M$ are continuous. Namely, if $t_i \in I$ is a sequence converging to $t \in I$ then $\gamma_1(t_i) = \gamma_2(t_i)$ for every i and, taking the limit $i \to \infty$, we obtain $\gamma_1(t) = \gamma_2(t)$ and hence $t \in A$. The set A is also open by the local uniqueness theorem. Since I is connected it follows that $A = I$. This proves (ii) and Theorem 2.4.7. $\square$