The Lie algebra of vector fields – overview

$\boxed{\text{pp } 40\text{–}50}$

2.4.3 Lie bracket $\boxed{\text{p. } 45}$

$$\mathbb{R}^k$$
$$\cup$$
$$M \quad \text{smooth}$$

$X, Y \in \operatorname{Vect} M$ smooth maps $M \longrightarrow \mathbb{R}^k$ $\quad X(p), Y(p) \in T_pM \subset \mathbb{R}^k$

$dX(p)$ is a linear map : $T_p(M) \longrightarrow \mathbb{R}^k = T_{X(p)}\mathbb{R}^k$

lemma 2.4.18 $\boxed{\text{p. } 46}$ If X and Y are <u>complete</u> vector fields then $\underbrace{dX(p)Y(p) - dY(p)X(p)}$ $\in T_p(M)$

Def 2.4.19 $\boxed{\text{p. } 48}$ The <u>Lie bracket</u> of complete vector fields X, Y

$[X, Y]$ is $\quad [X,Y](p) = \underbrace{dX(p)Y(p) - dY(p)X(p)}$

$[X,Y]$ is a smooth vector field

lemma 2.4.20 $\boxed{\text{p. } 48}$ For vector fields X, Y, Z on M

$$[X,Y] = -[Y,X]$$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Example 2.4.22 $\boxed{\text{p. } 49}$

$(\operatorname{Vect} M, [\cdot,\cdot])$ is a Lie algebra.

2.4.2 Flow of a vector field

$M \subset \mathbb{R}^k$ smooth m-manifold, $X \in \mathrm{Vect}\, M$ smooth vector field

$p_0 \in M$ By Theorem 2.4.7(ii) (p. 38) $\exists$ open interval

$0 \in I \subset \mathbb{R}$ and smooth map $\gamma : I \to M$ (curve)

$$\boxed{(2.4.1) \longrightarrow \quad \dot\gamma(t) = X(\gamma(t)), \quad \gamma(0) = p_0, \quad t \in I.}$$

let $I(p_0) :=$ the union of all such intervals I

Then $\exists !$ smooth $\Gamma : I(p_0) \to M$ satisfying the initial value

problem (2.4.1) : $\Gamma(0) = p_0, \quad \dot\Gamma(t) = X(\Gamma(t)) \quad \forall t \in I(p_0)$

" $\Gamma = \Gamma_{p_0}$ "

(see the next page (3) for details)

Definition 2.4.8 p. 40 (The flow of a vector field)

Let $\mathcal{D} = \{(t, p_0) : p_0 \in M, t \in I(p_0)\} \subseteq \mathbb{R} \times M$

and define $\phi : \mathcal{D} \to M$ by $\phi(t, p_0) = \Gamma(t)$

ϕ is the flow of the vector field X

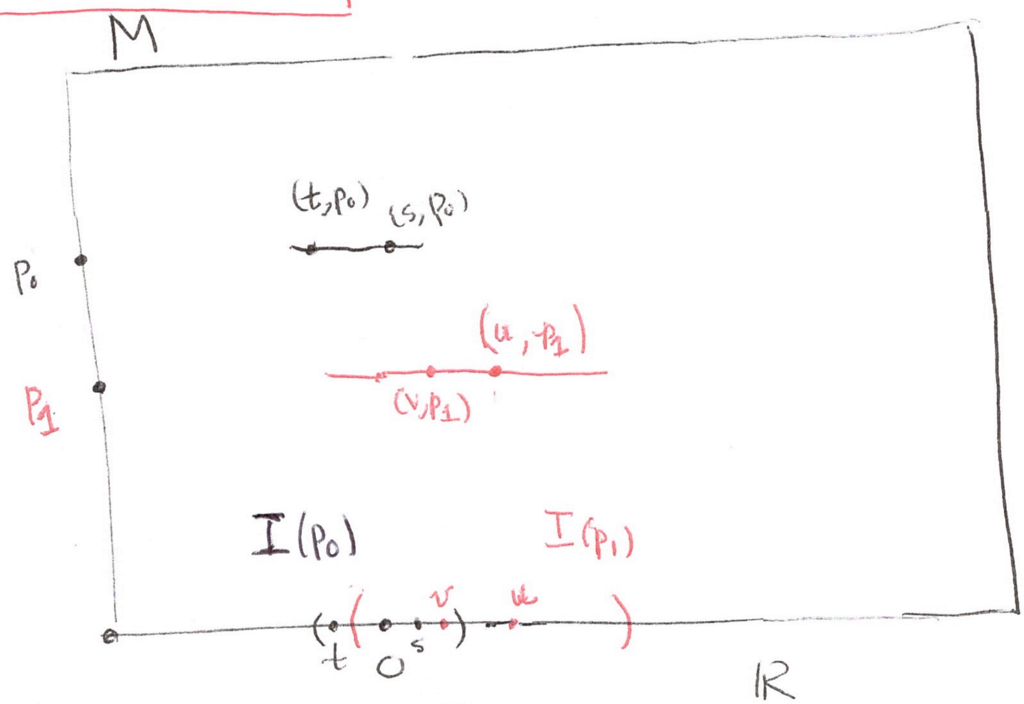
$\frac{d}{dt}\phi(t,p_0) = \dot\Gamma(t) = X(\Gamma(t)) = X(\phi(t,p_0))$

$\phi(0, p_0) = \Gamma(0) = p_0.$

$\phi(\cdot, p_0)$ is the solution of the initial value problem

$u'(t) = X(u(t)), \quad u(0) = p_0$ on $I(p_0)$

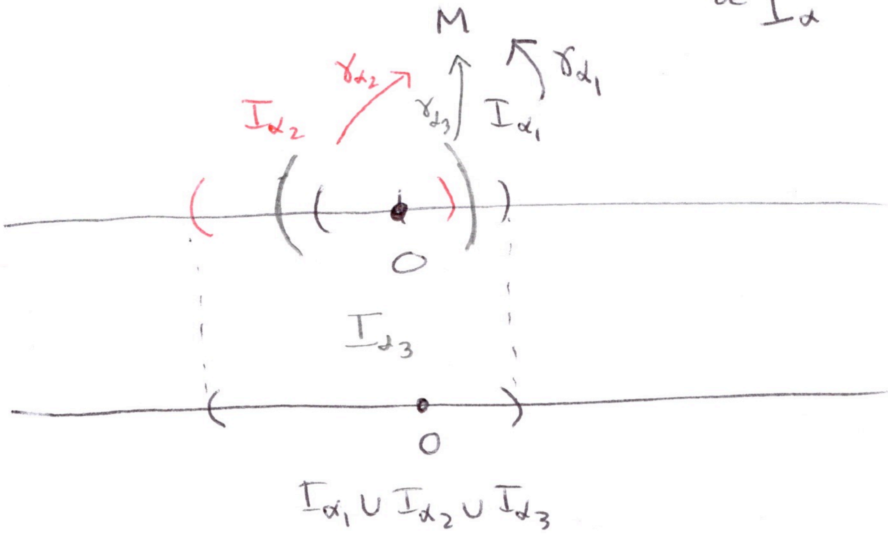
$$\mathcal{D} = \bigcup_{p \in M} I(p) \times \{p\}$$



For a fixed vector field X,

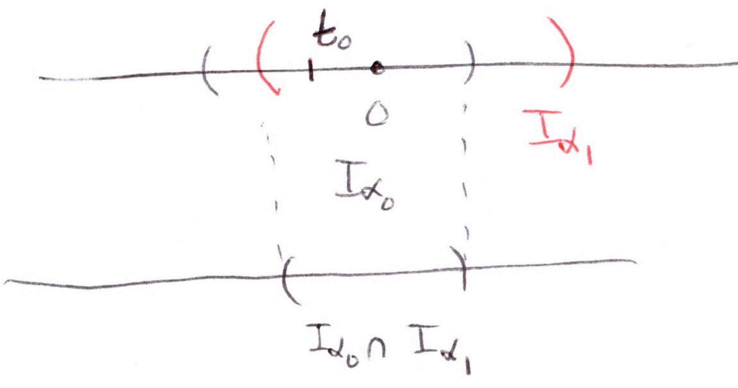
$$I(p_0) = \bigcup_{\alpha \in A} \left\{ I_\alpha : \ \exists\, \gamma_\alpha^{\text{smooth}} : I_\alpha \to M \quad \gamma_\alpha(0) = p_0,\ \dot\gamma_\alpha(t) = X(\gamma_\alpha(t)) \ \ \forall t \in I \right\}$$

$$0 \in I_\alpha$$



Define $\Gamma : I(p_0) \to M$ as follows. If $t_0 \in I(p_0)$, then $t_0 \in I_{\alpha_0}$ for some I_{α_0} so define $\Gamma(t_0) = \gamma_{\alpha_0}(t_0)$

Is Γ well defined? Let $t_0 \in I_{\alpha_1}$ also



we need $\gamma_{\alpha_0}(t_0) = \gamma_{\alpha_1}(t_0)$

This follows from part (ii) of Theorem 2.4.7 which states that $\gamma_{\alpha_0}(t) = \gamma_{\alpha_1}(t)$ $\forall t \in I_{\alpha_0} \cap I_{\alpha_1}$

Theorem 2.4.9 p. 40 $M \subset \mathbb{R}^k$ m-mfld, $X \in \mathrm{Vect}\, M$, $\phi : \mathcal{D} \to M$ flow of X

$\Longrightarrow$ (i) $\mathcal{D}$ is an open subset of $\mathbb{R} \times M \subseteq \mathbb{R}^{k+1}$

(ii) $\phi : \mathcal{D} \longrightarrow M$ is smooth map

(iii) let $p_0 \in M$, $s \in I(p_0)$. Then

Proof of Th 2.4.9 on p. 41

(2.4.3) $I(\phi(s, p_0)) = I(p_0) - s$

(2.4.4) $\phi(s+t, p_0) = \phi(t, \phi(s, p_0))$, $\forall t \in \mathbb{R}$ with $s+t \in I(p_0)$

Definition 2.4.11 (p 41) X is a complete vector field if $I(p_0) = \mathbb{R}$ $\forall p_0 \in M$

Lemma 2.4.12 (p. 42) If $M \subset \mathbb{R}^k$ is a compact manifold then every vector field on M is complete.

Remark (p. 42) let $M \subset \mathbb{R}^k$ be a smooth mfld, $X \in \mathrm{Vect}\, M$

$$\boxed{X \text{ is complete} \iff I(p) = \mathbb{R}\ \forall p \in M \iff \mathcal{D} = \mathbb{R} \times M}$$

If X is complete, with flow $\phi : \mathcal{D} = \mathbb{R} \times M \longrightarrow M$, write, for $t \in \mathbb{R}$

$$\phi^t : M \longrightarrow M, \quad \phi^t(p) = \phi(t, p). \quad \text{Then}$$

- ϕ^t is smooth, $\forall t \in \mathbb{R}$ by (Th 2.4.9(ii))

$$\boxed{\mathrm{Diff}(M) := \{\phi : M \to M : \phi \text{ is a diffeomorphism}\}}$$

(2.4.5) • $\phi^{s+t} = \phi^s \circ \phi^t$, $\phi^0 = \mathrm{id}_M$ by Th 2.4.9 (iii)

[proof $\phi^{s+t}(p) = \phi(s+t, p) = \phi(t, \phi(s,p)) = \phi^t(\phi(s,p)) = \phi^t(\phi^s(p)) = \phi^t \circ \phi^s(p)$]

- ϕ^t is bijective, $(\phi^t)^{-1} = \phi^{-t}$, so ϕ^t is a diffeomorphism.

[proof $\phi^0(p) = \phi(0,p) = p$ so $\phi^0 = \mathrm{id}_M$ $\phi^t \circ \phi^{-t} = \phi^0 = \mathrm{id}$.]

(smooth?) what is the manifold structure of Diff(M)?

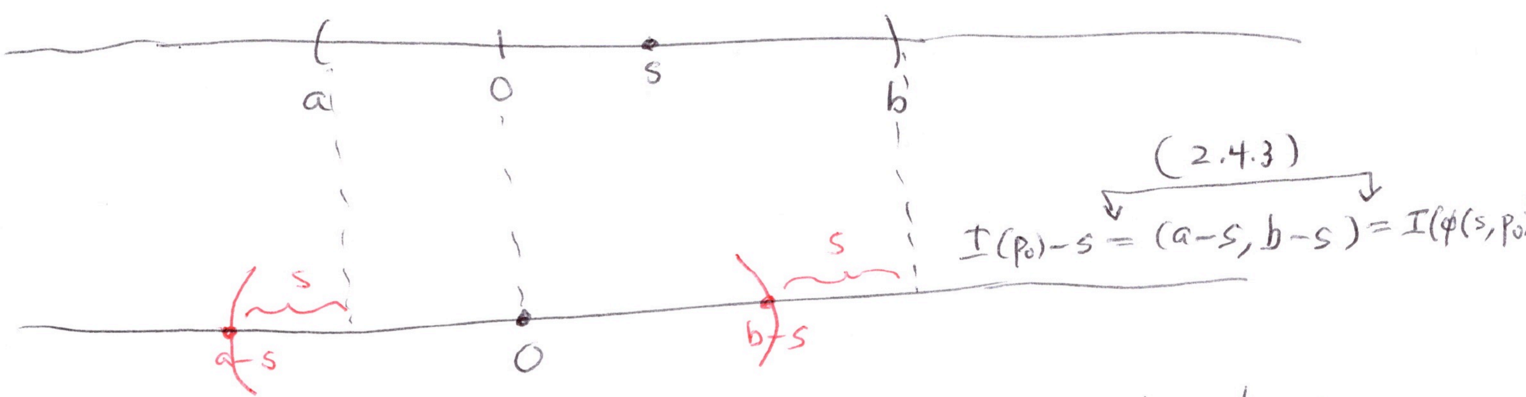
- $\mathbb{R} \ni t \longrightarrow \phi^t \in \mathrm{Diff}(M)$ is a group homomorphism

$(\mathbb{R}, +)$ $(\mathrm{Diff}(M), \circ)$

$$\boxed{\frac{d}{dt}\phi^t(p) = X(\phi^t(p)), \quad \phi^0(p) = p}$$

proof of (2.4.4) assumming (2.4.3)

$I(p_0) = (a, b)$



$a - s < t < b - s$

$s+t \in (a,b) \iff a < s+t < b \iff$

$\phi(\cdot, p_0)$ is the unique solution of $u'(t) = X(u(t))$, $u(0) = p_0$, t inside $I(p_0)$

$\phi(\cdot, \phi(s,p_0))$ is the unique solution of $u'(t) = X(u(t))$, $u(0) = \phi(s, p_0)$, t inside $I(\phi(s,p_0))$

Fix $s \in I(p_0)$. Consider $\gamma(t) = \phi(s+t, p_0)$ for $t \in I(p_0) - s$

$\gamma(0) = \phi(s, p_0)$ and $\dot\gamma(t) = \frac{d}{dt}\phi(s+t, p_0) = \frac{d}{dt}\phi(r(t), p_0)$

$r(t) = s+t \quad \frac{dr}{dt} = 1$

So $\dot\gamma(t) = \frac{d}{dr}\phi(r, p_0) \cdot \frac{dr}{dt} = \frac{d}{dr}\phi(r, p_0) = X(\phi(r, p_0)) = X(\phi(s+t, p_0)) = X(\gamma(t))$

Thus γ is a solution of $u'(t) = X(u(t))$, $\gamma(0) = \phi(s, p_0)$ for t in the interval $I(p_0) \cap (I(p_0) - s) = (a, b-s)$ in the diagram

By uniqueness of solutions $\gamma(t) = \phi(t, \phi(s, p_0))$ for $t \in (a, b-s)$

$\shortparallel$

$\phi(s+t, p_0)$

$\blacksquare$

2.4.3 p. 45

$$\begin{array}{ccc} \mathbb{R}^k & & \mathbb{R}^l \\ \cup & & \cup \\ M & \underset{\Psi \text{ diffeo}}{\overset{\phi \text{ diffeo}}{\rightleftarrows}} & N \end{array}$$

$\phi_* X(q) = d\phi(\phi^{-1}(q))X(\phi^{-1}(q))$

$\phi_* X \in \operatorname{Vect} N$

Def: $\phi_* X$ = pushforward of X under ϕ

given $\operatorname{Vect} M \ni X$

Def $\Psi^* X$ = pullback of X under Ψ

$\Psi^* X(q) = d\Psi(q)^{-1}(X(\Psi(q)))$

$\Psi^* X \in \operatorname{Vect} N$

$\Psi : N \longrightarrow M$	$\phi : M \longrightarrow N$
$d\Psi(q) : T_q N \longrightarrow T_{\Psi(q)} M$	$d\phi(p) : T_p M \longrightarrow T_{\phi(p)} N$
$d\Psi(q)^{-1} : T_{\Psi(q)} M \longrightarrow T_q(N)$	$d\phi(\phi^{-1}(q)) : T_{\phi^{-1}(q)} M \longrightarrow T_q N$

$$N \xrightarrow{\Psi} M \xrightarrow{X} \bigcup_{p \in M} T_p(M)$$

$$\begin{array}{ccccc} & & \cup & & \\ & & T_{\Psi(q)} M & \xrightarrow{d\Psi(q)^{-1}} & T_q(N) \\ \Psi & \Psi & \Psi & & \\ q & \Psi(q) & X(\Psi(q)) & & \end{array}$$

$$N \xrightarrow{\phi^{-1}} M \xrightarrow{X} \bigcup_{p \in M} T_p M$$

$$\begin{array}{ccc} \cup & & \\ T_{\phi^{-1}(q)} M & \xrightarrow{d\phi(\phi^{-1}(q))} & T_q N \end{array}$$

Lemma 2.4.18 p. 46 $M \subset \mathbb{R}^k$ m-mfld $X, Y \in \mathrm{Vect}\, M$ (complete)

Let $t \to \phi^t$ be the flow of X, $t \to \psi^t$ the flow of Y
$\mathbb{R} \to \mathrm{Diff}(M)$ $\mathbb{R} \to \mathrm{Diff}(M)$

Fix $p \in M$ & let $\gamma : \mathbb{R} \longrightarrow M$ $\gamma(t) = \phi^t \circ \psi^t \circ \phi^{-t} \circ \psi^{-t}(p)$

Then

- $\dot{\gamma}(0) = 0$
- $$\frac{1}{2}\ddot{\gamma}(0) = \frac{d}{ds}\Big|_{s=0} \left((\phi^s)_* Y\right)(p)$$
$$= \frac{d}{dt}\Big|_{t=0} \left((\psi^t)^* X\right)(p)$$
$$= dX(p)\, Y(p) - dY(p)\, X(p) \in T_pM$$

Definition 2.4.19 $[X,Y](p) = dX(p)\,Y(p) - dY(p)\,X(p)$

is the <u>Lie bracket</u> of the vector fields X and Y

Lemma 2.4.20 Let $M \subset \mathbb{R}^k$ be a smooth manifold.

(2.4.18) • For $X, Y \in \mathrm{Vect}\, M$, $[X,Y] + [Y,X] = 0$

(2.4.19) • For $X, Y, Z \in \mathrm{Vect}\, M$, $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

Let $\phi : M \longrightarrow N$ be a diffeomorphism of smooth manifolds M, N
Then for $X, Y \in \mathrm{Vect}\, M$,

(2.4.17) • $\phi^*[X,Y] = [\phi^* X, \phi^* Y]$