(p. 51) 3-1-18

def 2.5.1 (p. 51) Lie group

$$\begin{array}{c} \mathbb{R}^{n\times n} = \mathbb{R}^{n^2} \\ \cup \\ G \end{array}$$

- submanifold of $\mathbb{R}^{n^2}$
- subgroup of $GL(n,\mathbb{R})$ (= invertible matrices)

$$g,h \in G \Rightarrow gh \in G$$
$$g \in G \Rightarrow \det(g) \neq 0,\ g^{-1} \in G$$

Example 2.5.2 $GL(n,\mathbb{R})$ is $\boxed{\text{open set}}$ not obvious $\therefore$ smooth sub mfld $\subset \mathbb{R}^{n\times n}$ $\therefore$ Lie group

$$GL(n,\mathbb{R}) = \{ g \in \mathbb{R}^{n\times n} : \det(g) \neq 0 \}$$

Exercise 2.1.18 (p. 23) $SL(n,\mathbb{R})$ is a smooth sub. mfld $\subset \mathbb{R}^{n\times n}$ $\therefore$ Lie gp

$$SL(n,\mathbb{R}) = \{ g \in \mathbb{R}^{n\times n} : \det(g) = 1 \}$$

Example 2.1.9 (p. 23) $O(n)$ is a smooth sub-mfld $\subset \mathbb{R}^{n\times n}$ $\therefore$ Lie gp.

$$O(n) = \{ g \in \mathbb{R}^{n\times n} : g^T g = \mathbb{1}_n \}$$

Exercise 2.5.3 $SL(n,\mathbb{C})$, $U(n)$, $SU(n)$ are Lie gps $\subset \mathbb{C}^{n\times n} = \mathbb{R}^{2n^2}$

$\det(g)=1$ $\quad g^* g = \mathbb{1}_n$ $\quad$ both

$$g^* = \bar{g}^T$$

⟨ skip p. 52 for now ⟩

$\boxed{\text{multiplication of matrices is smooth}}$ ②

$\boxed{\text{this is stated but not proved at the top of p.53}}$

Proof $g = (x_{ij}) \quad h = (y_{ij}) \quad gh = (z_{ij})$

$$z_{ij} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$

$$z_{ij}\big(\underbrace{(x_{11},\dots,x_{nn}),(y_{11},\dots,y_{nn})}_{2n^2 \text{ coordinates}}\big) = \sum_{k=1}^{n} x_{ik}y_{kj}$$

$(g,h) \longrightarrow gh$ is smooth if $\dfrac{\partial z_{ij}}{\partial x_{pq}}, \dfrac{\partial z_{ij}}{\partial y_{pq}}$ and all higher partial derivatives exist.

$$\frac{\partial z_{ij}}{\partial x_{pq}} = \frac{\partial\left(\sum_{k=1}^{n} x_{ik}y_{kj}\right)}{\partial x_{pq}} = \begin{cases} 0 & \text{if } i \neq p \\ y_{qj} & \text{if } i = p \end{cases}$$

$$\frac{\partial z_{ij}}{\partial y_{pq}} = \frac{\partial\left(\sum_{k=1}^{n} x_{ik}y_{kj}\right)}{\partial y_{pq}} = \begin{cases} 0 & \text{if } j \neq q \\ x_{ip} & \text{if } j = q \end{cases}$$

so $\dfrac{\partial^2(z_{ij})}{\partial y_{pq}^2} = 0 \text{ or } 1 \quad \dfrac{\partial^2(z_{ij})}{\partial x_{pq}^2} = 0 \text{ or } 1 \quad \dfrac{\partial^2(z_{ij})}{\partial x_{pq}\,\partial y_{r,s}} = 0 \text{ or } 1$

Formally

Let $\alpha = (\alpha_{11},\dots,\alpha_{nn},\beta_{11},\dots,\beta_{nn})$ then

$$\frac{\partial^\alpha (z_{ij})}{\partial x_{11}^{\alpha_{11}} \cdots \partial x_{nn}^{\alpha_{nn}}\, \partial y_{11}^{\beta_{11}} \cdots \partial y_{nn}^{\beta_{nn}}} = 0 \quad \text{if } \alpha_{11}+\dots+\alpha_{nn}+\beta_{11}+\dots+\beta_{nn} \geq 3$$

$\boxed{\text{all 3rd order derivatives are zero, so multiplication of matrices is smooth}}$

(p. 53 – continued) This is proved on p. 53 by a different method

<u>Proof</u> Fix $h = (y_{ij})$ (constant) let $g = (x_{ij})_{i,j}$ (variable)

let $R_h : g \longrightarrow gh$

$(x_{ij}) \longrightarrow (z_{pq})$ $\quad Z_{pq}(g) = \sum_{k=1}^{n} x_{pk} y_{kq}$ multi-linear in x_{pk}

$G \longrightarrow G$ $\quad R_h(g) = (Z_{pq}(g))_{p,q}$

The derivative of R_h at g is the matrix $R_h'(g) = \left(\frac{\partial z_{pq}}{\partial x_{ij}}(g) \right)_{pq,ij} : \mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n^2}$

Recall $T_gG = \mathbb{R}^{n^2}$ $\quad T_{gh}G = \mathbb{R}^{n^2}$ $\quad$ ↑ n^2 by n^2 matrix

let $A \in \mathbb{R}^{n^2}$ $\quad A = (a_{ij})_{ij}$

$$R_h'(g) \cdot A = \lim_{t \to 0} \frac{R_h(g + tA) - R_h(g)}{t}$$

$$g + tA = (x_{ij} + ta_{ij})_{ij} \qquad R_h(g) = (z_{pq}(g))_{p,q}$$

$$R_h(g + tA) = (z_{pq}(g + tA))_{pq} = \left(\sum_{k=1}^{n} (x_{pk} + ta_{pk}) y_{kq} \right)_{p,q}$$

$$R_h'(g) \cdot A = \lim_{t \to 0} \frac{\left(\sum_{k=1}^{n} \left((x_{pk} + ta_{pk}) y_{kq} - x_{pk} y_{kq} \right) \right)_{p,q}}{t}$$

$$= \left(\lim_{t \to 0} \frac{\sum_{k=1}^{n} t a_{pk} y_{kq}}{t} \right)_{p,q}$$

$$= \left(\sum_{k=1}^{n} a_{pk} y_{kq} \right)_{p,q} = A \cdot h$$ (matrix multiplication by h on the right)

Thus $R_h'(g) : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{n \times n}$ is the map $A \longrightarrow Ah$ independant of g!

Similarly $L_g'(h) : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{n \times n}$ is $A \longrightarrow gA$ independent of h!

<u>Summary of p. 3</u> Let $G \subset \mathbb{R}^{n\times n}$ be a Lie group

Fix h, let $R_h : G \to G$ be right multiplication by h: $R_h(g) = gh$

Then for $g \in G$ $\quad R_h'(g) = \tilde{R}_h$

where $\tilde{R}_h$ is right multiplication by h on all of $\mathbb{R}^{n\times n}$

To emphasize: we defined R_h only on a subset $G \subset \mathbb{R}^{n\times n}$
Its derivative $R_h'(g)$ is the same as R_h but defined on all of $\mathbb{R}^{n\times n}$. Also $R_h'(g_1) = R_h'(g_2)$; $R_h'(g)$ does not depend on g.

Exercise 2.5.7 If G is a Lie group, then the map $g \longrightarrow g^{-1}$ is a diffeomorphism with derivative

$$T_g(G) \longrightarrow T_{g^{-1}}(G) \qquad v \longrightarrow -g^{-1} v g^{-1}$$

The Lie algebra of a Lie group

(p. 54) $G \subset GL(n, \mathbb{R})$ a Lie group.

$\mathbb{1} \in G$ The Lie algebra of G is $T_{\mathbb{1}}(G) = \mathbb{R}^m \subseteq \mathbb{R}^{n^2}$

$\mathfrak{g} = \text{Lie}(G) = T_{\mathbb{1}}(G)$

(p. 55)

Lemma 2.5.9

$$\boxed{(i) \quad \xi \in \mathfrak{g} \Rightarrow \exp t\xi \in G \quad \forall t \in \mathbb{R}}$$

$$\mathfrak{g} = T_{\mathbb{1}}G \to T_gG$$

$\xi \to \xi g$ vector space isomorphism

$\therefore$ $\xi \in T_{\mathbb{1}}G$ determines a vector field $X_\xi \in \text{Vect}(G)$

$$X_\xi(g) = \xi g \in T_gG$$

$\exists$ integral curve $\gamma : (-\varepsilon, \varepsilon) \to G$ $\quad \dot\gamma(t) = X_\xi(\gamma(t)) = \xi\gamma(t)$

$\gamma(0) = \mathbb{1}$

Since (2.5.5 on p 54): $\frac{d}{dt}\exp(t\xi) = \xi \exp t\xi = (\exp t\xi)\xi$

$\exp(t\xi) = \sum_{k=0}^{\infty} \frac{t^k \xi^k}{k!}$

just like in Freshman Calculus except here ξ is a matrix, not just a number

see p. 54

Let $\beta : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ $\quad \beta(t) = \exp t\xi$ ~~set~~ Then

$\beta(0) = \mathbb{1}$ and $\dot\beta(t) = \xi \exp t\xi = \xi\beta(t)$

By uniqueness $\gamma(t) = \exp t\xi \in G \ \forall \ |t| < \varepsilon$

If $t \in \mathbb{R}$ is arbitrary, $\frac{t}{N} \in (-\varepsilon, \varepsilon)$ for some N so $\exp \frac{t}{N}\xi \in G$

$$\left(2.5.6 \text{ on p 54 } \exp(s+t)\xi = \exp s\xi \exp t\xi\right) \Rightarrow \exp t\xi = \exp\Big(\underbrace{\tfrac{t}{N}\xi + \tfrac{t\xi}{N} + \dots + \tfrac{t\xi}{N}}_{N \text{ terms}}\Big)$$

$$= \underbrace{\exp\tfrac{t\xi}{N} \cdots \exp\tfrac{t\xi}{N}}_{N \text{ terms}} = \left(\exp\tfrac{t\xi}{N}\right)^N \in G \quad \text{for (i)}$$

(ii) $\begin{matrix} g \in G \\ \eta \in \mathfrak{g} \end{matrix} \Rightarrow g\eta g^{-1} \in \mathfrak{g}$

Let $\gamma : \mathbb{R} \to \mathbb{R}^{n\times n}$ $\quad \gamma(t) = g \exp t\eta\, g^{-1}$ $\quad (\in G$ by (i))

$\gamma(0) = \mathbb{1} \quad \therefore \quad \dot\gamma(0) \in T_{\mathbb{1}}G = \mathfrak{g}$

By the (non-commutative) product rule

$$\dot\gamma(t) = g(\exp t\xi)\eta g^{-1} \qquad \dot\gamma(0) = g\eta g^{-1}.$$

$\boxed{///}$ for (ii)

(iii) $\xi, \eta \in \mathfrak{g} \Rightarrow [\xi,\eta] = \xi\eta - \eta\xi \in \mathfrak{g}$

Let $\eta(t) = \underbrace{(\exp t\xi)}_{\in G}\underbrace{\eta}_{\in \mathfrak{g}}\underbrace{(\exp(-t\xi))}_{\in G}$ $\qquad \eta : \mathbb{R} \to \mathbb{R}^{n\times n}$

$\exp(-t\xi) = (\exp t\xi)^{-1}$

Thus $\dot\eta(0) = [\xi, \eta]$ $\qquad$ But $\eta(0) = \eta$ $\quad$ so $\quad [\xi,\eta] \in \mathfrak{g}$.

$$\dot\eta(t) = \overbrace{(\exp t\xi)}^{1^{st}}\overbrace{\eta \exp(-t\xi)(-\xi)}^{\text{derivative of } 2^{nd}} + \overbrace{\xi \exp t\xi}^{\text{derivative of } 1^{st}}\underbrace{\eta \exp(-t\xi)}_{2^{nd}}$$

$$\dot\eta(0) = -\eta\xi + \xi\eta$$

$\boxed{///}$ for (iii)

Lemma 2.5.9 on p. 55 (see the previous pages 5 and 6)

shows that the tangent space of a Lie group at the identity matrix is a Lie algebra

This fact was the goal of Math 199B Winter 2018

(bottom of p. 55)

- The curve $\gamma : \mathbb{R} \to G$ defined by $\gamma(t) = (\exp t\xi) g$ for $g \in G$, $t \in \mathbb{R}$, $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$ is the integral curve of the vector field X_ξ, $(X_\xi h = \xi h \text{ for } h \in G)$ and $\gamma(0) = g$ Thus X_ξ is a complete vector field $\forall \xi \in \mathfrak{g}$

(p. 56)

Lemma 2.5.10 If $\xi \in \mathfrak{g}$ and $\boxed{\gamma : \mathbb{R} \to G}$ is smooth satisfying

γ is a group homomorphism

$$\boxed{\gamma(s+t) = \gamma(s)\gamma(t) \qquad \gamma(0) = \mathbb{1}} \qquad \dot{\gamma}(0) = \xi$$

then $\gamma(t) = \exp(t\xi) \quad \forall t \in \mathbb{R}$

Example 2.5.11

$$\mathrm{Lie}(GL(n,\mathbb{R})) = \mathbb{R}^{n \times n}$$

$$\mathrm{Lie}(SL(n,\mathbb{R})) = \{ \xi \in \mathbb{R}^{n\times n} : \mathrm{trace}(\xi) = 0 \}$$

$$\mathrm{Lie}(SO(n)) = \{ \xi \in \mathbb{R}^{n\times n} : \xi^T = -\xi \}$$

(skew-symmetric matrices)