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p. 57

$\mathfrak{g}, \mathfrak{h}$ are Lie algebras of the Lie groups G, H respectively

$\mathfrak{g} = \mathrm{Lie}(G)$ $\qquad \mathfrak{h} = \mathrm{Lie}(H)$

defs Lie group homomorphism

$$G \xrightarrow{\rho} H$$

Lie group

smooth a group homomorphism $\rho : G \to H$ is a Lie group homomorphism

Lie algebra homomorphism

a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism

if $\phi[\xi, \eta]_{\mathfrak{g}} = [\phi(\xi), \phi(\eta)]_{\mathfrak{h}}$

> lemma 2.5.14 If $\rho : G \to H$ is a Lie group homo.
> then $\dot{\rho} := d\rho(\mathbb{1}) : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homo.

($d\rho$ = derivative of ρ $\qquad d\rho : G \to \{\text{linear transf } \mathfrak{g} \to \mathfrak{h}\}$

$\mathbb{1}$ = identity elt of G $\qquad \dot{\rho} = d\rho(\mathbb{1}) : \mathfrak{g} \to \mathfrak{h}$)

< $\dot{\rho}$ usually denotes $d\rho$ — here it denotes $d\rho(\mathbb{1})$ >

Proof

step 1 $\quad \rho(\exp t\xi) = \exp(t\dot{\rho}(\xi)) \qquad (\xi \in \mathfrak{g}, t \in \mathbb{R})$

step 2 $\quad \dot{\rho}(g\eta g^{-1}) = \rho(g)\dot{\rho}(\eta)\rho(g^{-1}) \qquad (g \in G, \eta \in \mathfrak{g})$

Step 3 $\quad \dot{\rho}[\xi, \eta] = [\dot{\rho}(\xi), \dot{\rho}(\eta)] \qquad (\xi, \eta \in G)$

see the diagrams on the next page

proofs are on pp 57-58

$G \subset GL(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$ $\qquad H \subset GL(m, \mathbb{R}) \subset \mathbb{R}^{m\times m}$

$$\begin{array}{ccc}
\mathbb{R}^{n\times n} & & \mathbb{R}^{m\times m} \\
\cup & & \cup \\
T_{\mathbb{1}}(G) = \mathfrak{g} & \xrightarrow{\dot{\rho} = d\rho(\mathbb{1})} & \mathfrak{h} = T_{\mathbb{1}}(H) \\
 & & \\
G & \xrightarrow{\rho} & H
\end{array}$$

note $\rho(\mathbb{1}) = \mathbb{1}$

step 1

$t\xi \qquad \dot\rho(t\xi) = t\dot\rho(\xi)$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\dot\rho"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\rho"'] & H \end{tikzcd}$$

in other words

$$\rho \circ \exp = \exp \circ \dot\rho$$

$$\exp(t\xi) \longrightarrow \boxed{\exp(t\dot\rho\xi) = \rho(\exp t\xi)}$$

step 2

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^n \\ \mathbb{R}^n \arrow[u, "\eta"] & \mathbb{R}^n \arrow[l, "g^{-1}"] \arrow[u] \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{R}^m & \mathbb{R}^m \arrow[l, "\rho(g)"'] \\ \mathbb{R}^m \arrow[u] \arrow[r, "\rho(g^{-1})"'] & \mathbb{R}^m \arrow[u, "\dot\rho(\eta)"'] \end{tikzcd}$$

$$g\eta g^{-1} \longrightarrow \boxed{\dot\rho(g\eta g^{-1}) = \rho(g)\dot\rho(\eta)\rho(g^{-1})}$$

$\in \mathfrak{g}$ by Lemma 2.59

step 3

$$\begin{tikzcd} \mathbb{R}^n \arrow[d] \\ \mathbb{R}^n \end{tikzcd}
\qquad
\begin{aligned}
\xi &\in \mathfrak{g} \longrightarrow \dot\rho(\xi) \in \mathfrak{h} \\
\eta &\in \mathfrak{g} \longrightarrow \dot\rho(\eta) \in \mathfrak{h} \\
[\xi,\eta] &\in \mathfrak{g} \longrightarrow \boxed{\dot\rho[\xi,\eta] = [\dot\rho(\xi), \dot\rho(\eta)]} \in \mathfrak{h}
\end{aligned}
\qquad
\begin{tikzcd} \mathbb{R}^m \arrow[d] \\ \mathbb{R}^m \end{tikzcd}$$

p. 58 ③

<u>Example 2.5.15</u> $G = U(n)$ is a Lie group (Example 2.5.2 p. 51) (unitary matrices $g \in \mathbb{C}^{n \times n}$

$g^* g = \mathbb{1}_n$

$H = S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ is a Lie group (see Example 2.2.5 p. 26 for S^1 is a manifold)

$g^* = \bar{g}^T$ (conjugate transform)

$\rho(g) = \det g$ is a

Lie group homo: $\rho : U(n) \longrightarrow S^1$

(recall that $\underbrace{\det(g_1 g_2)}_{\rho(g_1 g_2) =} = (\det g_1)(\det g_2) = \rho(g_1)\rho(g_2)$

and $1 = \det(\mathbb{1}_n) = \det(g^*)\det(g) = \overline{\det g}\,\det g = |\det g|^2$)

The associated Lie algebra homomorphism is

$\dot{\rho}(\xi) = \text{trace}(\xi) \qquad \xi \in \text{Lie}(G)$

proof From Exercise 2.5.12 (p. 56) $\text{Lie}(U(n)) = u(n) = \{\xi \in \mathbb{C}^{n \times n} : \xi + \xi^* = 0\}$

By Example 2.2.5 (p. 26) $\text{Lie}(S^1) = \{(0,y) : y \in \mathbb{R}\} = i\mathbb{R}$ in (complex notation)

For $\xi \in u(n)$, $\dot{\rho}(\xi) = d\rho(\mathbb{1})(\xi)$

$$= \lim_{t \to 0} \frac{\rho(\mathbb{1} + t\xi) - \rho(\mathbb{1})}{t} = \lim_{t \to 0} \frac{\det(\mathbb{1} + t\xi) - 1}{t}$$

For example, with $n = 2$, $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\xi^* = \begin{pmatrix} -\bar{a} & -\bar{c} \\ -\bar{b} & -\bar{d} \end{pmatrix}$

and $\mathbb{1} + t\xi = \begin{pmatrix} 1+ta & tb \\ tc & 1+td \end{pmatrix}$

$$\therefore \dot{\rho}(\xi) = \lim_{t \to 0} \frac{(1+ta)(1+td) - 1}{t} = \lim_{t \to 0} (a + d + t\,ad)$$

$= a + d = \text{trace}(\xi)$ ▨

$a + d \in i\mathbb{R}$ since $a = -\bar{a}$ & $d = -\bar{d}$

Skip p. 59 for now

(p. 60)

<u>Example 2.5.19</u> Let $\mathfrak{g}$ be a finite dim'l Lie algebra and define $\mathrm{Aut}(\mathfrak{g}) = \{ \Phi : \mathfrak{g} \to \mathfrak{g} \mid \Phi$ is linear & bijective and $\Phi([\xi,\eta]) = [\Phi(\xi), \Phi(\eta)]\ \forall \xi, \eta \in \mathfrak{g} \}$

which is the set of Lie algebra <u>automorphisms</u> of $\mathfrak{g}$

<u>Claim</u> • $\mathrm{Aut}(\mathfrak{g})$ is a Lie group (not proved now)

• $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g}) := \{ A : \mathfrak{g} \to \mathfrak{g} \mid A$ is linear, $A[x,y] = [Ax,y] + [x,Ay] \}$

(proved on pp (5)-(6) which follow)

Proof If $\dim \mathfrak{g} = n$, then $\mathrm{Aut}(\mathfrak{g})$ is a subgroup of $GL(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$

We need to prove it is a <u>submanifold</u> of $\mathbb{R}^{n\times n} = \mathbb{R}^{n^2}$

This is not so obvious! We can come back to this at a later time.

Let's prove the second bullet on p. 4

$$(**) \qquad \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) = \mathrm{Der}(\mathfrak{g})$$

First, let $A \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$ We shall prove $A \in \mathrm{Der}(\mathfrak{g})$.

By Lemma 2.5.9 (i) (p. 55), $\exp tA \in \mathrm{Aut}(\mathfrak{g})\ \forall t \in \mathbb{R}$

i.e.

$$\exp tA([\xi,\eta]) = [(\exp tA)\xi, (\exp tA)\eta]$$
$$= (\exp tA)\xi \circ (\exp tA)\eta - (\exp tA)\eta \cdot (\exp tA)\xi$$

take derivative w/R t at $t=0$

$\frac{d}{dt}$: $$A \exp tA([\xi,\eta]) = (\exp tA)\xi \cdot (A(\exp tA)\eta) + A(\exp tA)\xi \cdot (\exp tA)\eta$$
$$- (\exp tA)\eta \cdot (A(\exp tA)\xi) + A(\exp tA)\eta \cdot (\exp tA)\xi$$

Set $t=0$

$\left.\frac{d}{dt}\right|_{t=0}$: $$A([\xi,\eta]) = \xi(A\eta) + (A\xi)\eta - \eta(A\xi) - (A\eta)\xi$$
$$= [A\xi,\eta] + [\xi, A\eta]$$ so A is a derivation.

$$\boxed{\therefore \quad \mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{Der}\,\mathfrak{g}}$$

Now let's prove $\mathrm{Der}(\mathfrak{g}) \subset \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$

So let $A \in \mathrm{Der}(\mathfrak{g})$ we first show that

$$\exp tA \in \mathrm{Aut}(\mathfrak{g})$$

let $\gamma : \mathbb{R} \to \mathfrak{g}$ $\gamma(t) = \exp tA\,([\xi,\eta])$

let $\beta : \mathbb{R} \to \mathfrak{g}$ $\beta(t) = (\exp tA)\xi \cdot (\exp tA)\eta - (\exp tA)\eta \cdot (\exp tA)\xi$

$\gamma'(t) = A \exp t([\xi,\eta])$

$\gamma'(0) = A([\xi,\eta])$

$\gamma(0) = [\xi,\eta]$

$$\beta'(t) = (\exp tA)\xi \cdot A(\exp tA)\eta + A(\exp tA)\xi \cdot (\exp tA)\eta - (\exp tA)\eta \cdot A(\exp tA)\xi - A(\exp tA)\eta \cdot \exp tA(\xi)$$

$\beta(0) = [\xi,\eta]$

$$\beta'(0) = \xi(A\eta) + (A\xi)\eta - \eta(A\xi) - (A\eta)\xi = [A\xi,\eta] + [\xi,A\eta]$$

Since $A \in \mathrm{Der}(\mathfrak{g})$ $\gamma'(0) = \beta'(0)$

so by uniqueness of initial value problem $\gamma = \beta$

which says that $\exp tA \in \mathrm{Aut}(\mathfrak{g})$.

Finally, with $\Gamma(t) = \exp tA \in \mathrm{Aut}(\mathfrak{g})$, we have

$\gamma(0) = \mathbb{1}$ and $\dot{\gamma}(0) = A \exp tA\big|_{t=0} = A$

so $A \in T_{\mathbb{1}}(\mathrm{Aut}\,\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}\,\mathfrak{g})$ $\blacksquare$

This proves $\mathrm{Der}(\mathfrak{g}) \subset \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$ proving (**) on p. (5)

Further remarks on (p. 60), let G be a Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

Define $\mathrm{ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ by $\mathrm{ad}(g)\eta = g\eta g^{-1}$.

By Lemma 2.5.9 (ii) $\mathrm{ad}(g)$ is a linear map on $\mathfrak{g}$ (it is obviously linear — 2.5.9 (ii) claims it map into $\mathfrak{g}$)

For $g \in G$,

- The map $\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism

proof: $\mathrm{ad}(g)[\xi,\eta] = g(\xi\eta - \eta\xi)g^{-1} = (g\xi g^{-1})(g\eta g^{-1}) - (g\eta g^{-1})(g\xi g^{-1})$

$$= [\mathrm{ad}(g)\xi, \mathrm{ad}(g)\eta]$$

Thus $\mathrm{ad}(g) \in \mathrm{Aut}(\mathfrak{g})$ since $\mathrm{ad}(g)$ is invertible (with inverse $\xi \to g\eta g^{-1}$)

- The map $\mathrm{ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$ ($\mathfrak{g} = \mathrm{Lie}(G)$) is a Lie group homomorphism

proof: $\mathrm{ad}(gh)\eta = (gh)\eta(gh)^{-1} = g(h\eta h^{-1})g^{-1} = \mathrm{ad}(g)\,\mathrm{ad}(h)(\eta)$

- Let $\mathrm{Ad} := \dot{\mathrm{ad}}$ be the derivative of ad, so $\mathrm{Ad} : \mathrm{Lie}(\mathfrak{g}) \to \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$

Then $\mathrm{Ad}(\xi)(\eta) = [\xi, \eta]$ (recall $\mathrm{ad} : G \to \mathrm{Aut}(\mathfrak{g})$)

proof

$$\dot{\mathrm{ad}}(g)(\xi) = \lim_{t\to 0} \frac{\mathrm{ad}(g+t\xi) - \mathrm{ad}\, g}{t}$$

$$\dot{\mathrm{ad}}(\mathbb{1})(\xi) = \lim_{t\to 0} \frac{\mathrm{ad}(\mathbb{1}+t\xi) - \mathrm{ad}(\mathbb{1})}{t} \in \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$$

so $\mathring{ad}(\mathbb{1})(\xi)(\eta) = \lim_{t\to 0} \dfrac{ad(\mathbb{1}+t\xi)(\eta) - \eta}{t}$ $\quad (ad\,\mathbb{1} = Id_{\mathfrak{g}})$

$$= \lim_{t\to 0} \frac{(\mathbb{1}+t\xi)\,\eta\,(\mathbb{1}+t\xi)^{-1} - \eta}{t}$$

but $(\mathbb{1}+t\xi)^{-1}$ "$=$" $\mathbb{1} - t\xi + t^2\xi^2 - t^3\xi^3 + \cdots$ if $|t|$ small

(this is calculus!!)

the series converges for small $|t|$

So we have $\mathring{ad}(\mathbb{1})(\xi)(\eta) =$

$$\lim_{t\to 0} \frac{(\eta + t\xi\eta)(\mathbb{1} - t\xi + t^2\xi^2 - t^3\xi^3 + \cdots)}{t}$$

$$= \lim_{t\to 0} \left(\frac{\begin{matrix} \eta - t\eta\xi + t^2\eta\xi^2 - t^3\eta\xi^3 + \cdots \\ + t\xi\eta - t^2\xi\eta\xi + t^3\xi\eta\xi^2 - \cdots \end{matrix} - \eta}{t} \right)$$

$$= \lim_{t\to 0} \left(\xi\eta - \eta\xi + t(\eta\xi^2 - t^2\eta\xi^3 + \cdots - t(\xi\eta\xi - t^2\xi\eta\xi^2 + \cdots \right)$$

$$= \xi\eta - \eta\xi$$

$\blacksquare$