

(skipping p. 61 for now)

199B 031618 - 2.5 his his. pdf 3/14/18 (1)

p. 62 Let $M \subset \mathbb{R}^k$ be a smooth manifold. $\mathcal{F}(M) = C^\infty(M, \mathbb{R})$

Theorem 2.5.23 (p. 62) Every unital algebra homomorphism

$\varepsilon : \mathcal{F}(M) \rightarrow \mathbb{R}$ is of the form $\varepsilon = \varepsilon_p$ for some $p \in M$

$$\varepsilon(1) = 1 \quad \varepsilon(fg) = \varepsilon(f)\varepsilon(g)$$

↑
the function identically 1 on M

$$\varepsilon(f+g) = \varepsilon(f) + \varepsilon(g)$$

$\mathcal{F}(M)$ is a (real) algebra under

pointwise multiplication and addition:

$$\left. \begin{aligned} \bullet (f+g)(p) &= f(p) + g(p) \\ \bullet (fg)(p) &= f(p)g(p) \end{aligned} \right\} f, g \in \mathcal{F}(M)$$

$$\bullet (\alpha f)(p) = \alpha f(p)$$

$$\alpha \in \mathbb{R}, f \in \mathcal{F}(M)$$

Proof on p. 62 (omitted)

Notation

Let M, N be smooth manifolds. Since $\mathcal{F}(M), \mathcal{F}(N)$ are algebras it is clear what we mean by a

homomorphism $\Phi: \mathcal{F}(N) \rightarrow \mathcal{F}(M)$,

- Φ is linear: $\Phi(f+g) = \Phi(f) + \Phi(g)$ $\Phi(\alpha f) = \alpha \Phi(f)$
- $\Phi(fg) = \Phi(f)\Phi(g)$

- $\Phi(1) = 1$

(unital
algebra homomorphism!)

$\text{Hom}(\mathcal{F}(N), \mathcal{F}(M))$ denotes the set of homos

$$\text{Aut}(\mathcal{F}(M)) = \{ \text{bijective homos: } \mathcal{F}(M) \rightarrow \mathcal{F}(M) \}$$

$\text{Aut}(\mathcal{F}(M))$ is a group. Think of it as a lie
group with lie algebra $\text{Der}(\mathcal{F}(M))$ (see p. 3)

$\text{Diff}(M)$ is also a group. Think of it as a

lie group with lie algebra $\text{Vect}(M) (= T_{id} \text{Diff}(M))$

skip p. 63 for now

(p. 64) (recall that!) this page is not at all clear!

def A derivation of $\mathcal{F}(M)$ is a linear map $\delta: \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ satisfying $\delta(fg) = f\delta(g) + \delta f \cdot g$

$\text{Der}(\mathcal{F}(M))$ is a Lie algebra (we know that)

Consider the pullback operation

this is

$$\phi \longrightarrow \phi^* \leftarrow (\text{misleading on p. 64})$$

This is
what is going on

$$\text{Diff}(M) \xrightarrow{\alpha} \text{Aut}(\mathcal{F}(M))$$

skip this $\downarrow$

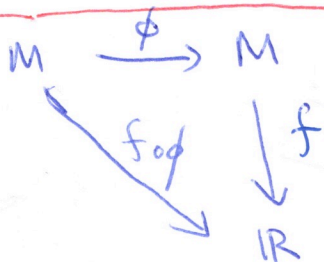
$$\phi: M \rightarrow M \quad X \in \text{Vect}(M), \quad g \in M$$

$$\phi^*: \text{Vect}(M) \rightarrow \text{Vect}(M)$$

$$(\phi^* X)(g) = d\phi(g)^{-1} X(\phi(g))$$

$$d\phi(g): T_g M \rightarrow T_{\phi(g)} M$$

$$X: M \rightarrow \mathbb{R}^n \quad X(p) \in T_p(M)$$



$$\Phi: \mathcal{F}(M) \rightarrow \mathcal{F}(M)$$

$$\Phi \in \text{Aut}(\mathcal{F}(M))$$

$$\Phi(f) = f \circ \phi$$

$$\alpha: \text{Diff}(M) \rightarrow \text{Aut}(\mathcal{F}(M))$$

Then $\phi \xrightarrow{\alpha} \Phi$ is a group anti-isomorphism (see p. 4)

$$\text{i.e. } \alpha(\phi \circ \psi) = \alpha(\psi) \circ \alpha(\phi) \quad \alpha(\phi)(f) = f \circ \phi$$

$$M \xrightarrow{\psi} M \xrightarrow{\phi} M$$

$$\alpha(\phi \circ \psi)(f) = f \circ (\phi \circ \psi) \quad \cancel{f \circ \psi \circ \phi}$$

$$\begin{aligned} \alpha(\psi) \circ \alpha(\phi)(f) &= \alpha(\psi)(\alpha(\phi)(f)) = \alpha(\psi)(f \circ \phi) \\ &= f \circ \phi \circ \psi \end{aligned}$$

$$\text{Diff}(M) \xrightarrow{\alpha} \text{Aut}(\mathcal{F}(M))$$

$$\text{Vect}(M) = \text{Lie}(\text{Diff}(M)) \xrightarrow{d\alpha} \text{Der}(\mathcal{F}(M)) = \text{Lie}(\text{Aut}(\mathcal{F}(M)))$$

$$d\alpha(\text{id}_M) : \text{Vect}(M) \longrightarrow \text{Der}(\mathcal{F}(M))$$

$$X \longrightarrow \mathcal{L}X$$

etc. leave this for now

we will figure this out later