(skipping p. 61 for now) 199B 031618_2.5bisbis.pdf 3/14/18

p.62 Let $M \subset \mathbb{R}^k$ be a smooth manifold. $\mathcal{F}(M) = C^\infty(M, \mathbb{R})$

Theorem 2.5.23 (p.62) Every *unital* algebra homomorphism $\varepsilon : \mathcal{F}(M) \longrightarrow \mathbb{R}$ is of the form $\varepsilon = \varepsilon_p$ for some $p \in M$

$\varepsilon(1) = 1$ $\quad \varepsilon(fg) = \varepsilon(f)\varepsilon(g)$ $\quad \varepsilon(f+g) = \varepsilon(f) + \varepsilon(g)$

the function identically 1 on M

$\mathcal{F}(M)$ is an (real) algebra under pointwise multiplication and addition:

- $(f+g)(p) = f(p) + g(p)$
- $(fg)(p) = f(p)\,g(p)$

$\Big\}\ f, g \in \mathcal{F}(M)$

- $(\alpha f)(p) = \alpha f(p)$ $\quad \alpha \in \mathbb{R},\ f \in \mathcal{F}(M)$

Proof is on p.62 (omitted)

Notation

Let M, N be smooth manifolds. Since $\mathcal{F}(M), \mathcal{F}(N)$ are algebras it is clear what we mean by a homomorphism $\Phi : \mathcal{F}(N) \to \mathcal{F}(M)$,

- Φ is linear : $\Phi(f+g) = \Phi(f) + \Phi(g)$ $\quad \Phi(\alpha f) = \alpha \Phi(f)$
- $\Phi(fg) = \Phi(f)\Phi(g)$
- $\Phi(1) = 1$

(unital algebra homomorphism!)

$\mathrm{Hom}(\mathcal{F}(N), \mathcal{F}(M))$ denotes the set of homos

$\mathrm{Aut}(\mathcal{F}(M)) = \{$ bijective homos : $\mathcal{F}(M) \to \mathcal{F}(M) \}$

$\mathrm{Aut}(\mathcal{F}(M))$ is a group. Think of it as a Lie group with Lie algebra $\mathrm{Der}(\mathcal{F}(M))$ (see p. 3)

$\mathrm{Diff}(M)$ is also a group. Think of it as a Lie group with Lie algebra $\mathrm{Vect}(M) \left(= T_{id}\,\mathrm{Diff}(M)\right)$

skip p. 63 for now

(p.64) (recall that!) this page is not at all clear!

def A <u>derivation</u> of $\mathcal{F}(M)$ is a linear map $\delta : \mathcal{F}(M) \to \mathcal{F}(M)$

satisfying $\delta(fg) = f\,\delta(g) + \delta(f)\,g$

$\mathrm{Der}(\mathcal{F}(M))$ is a lie algebra (we know that)

Consider the pullback operation

$$\phi \longrightarrow \phi^* \leftarrow$$ (this is misleading on p. 64)

This is what is going on $\mathrm{Diff}(M) \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{F}(M))$

skip this

$\phi : M \to M \qquad X \in \mathrm{Vect}(M),\ q \in M \qquad \phi^* : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$

$$(\phi^* X)(q) = d\phi(q)^{-1}\, X(\phi(q)) \qquad d\phi(q) : T_q M \to T_{\phi(q)} M$$

$$X : M \to \mathbb{R}^n \qquad X(p) \in T_p(M)$$

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ & {\scriptstyle f\circ\phi}\searrow & \downarrow f \\ & & \mathbb{R} \end{array}$$

$\Phi : \mathcal{F}(M) \to \mathcal{F}(M) \qquad \Phi \in \mathrm{Aut}(\mathcal{F}(M))$

$\Phi(f) = f \circ \phi$

$\alpha : \mathrm{Diff}(M) \to \mathrm{Aut}(\mathcal{F}(M))$

Then $\phi \xrightarrow{\alpha} \Phi$ is a group anti-isomorphism (see p. 4)

i.e. $\alpha(\phi \circ \psi) = \alpha(\psi) \circ \alpha(\phi) \qquad \alpha(\phi)(f) = f \circ \phi$

$$M \xrightarrow{\psi} M \xrightarrow{\phi} M$$

$$\alpha(\phi\circ\psi)(f) = f\circ(\phi\circ\psi)^{\cancel{-1}}\ \cancel{= f\circ\psi^{-1}\circ\phi^{-1}}$$

$$\alpha(\psi)\circ\alpha(\phi)(f) = \alpha(\psi)\big(\alpha(\phi)(f)\big) = \alpha(\psi)\big(f\circ\phi^{\cancel{-1}}\big)$$

$$= f\circ\phi^{\cancel{-1}}\circ\psi^{\cancel{-1}}$$

$$\mathrm{Diff}(M) \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{F}(M))$$

$$\mathrm{Vect}(M) = \mathrm{Lie}(\mathrm{Diff}(M)) \xrightarrow{d\alpha} \mathrm{Der}(\mathcal{F}(M)) = \mathrm{Lie}(\mathrm{Aut}(\mathcal{F}(M)))$$

$$d\alpha(\mathrm{id}_M) : \mathrm{Vect}(M) \longrightarrow \mathrm{Der}(\mathcal{F}(M))$$
$$X \longrightarrow \mathcal{L}_X$$

etc. leave this for now

we will figure this out later