Evolution Algebras

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Bernard Russo

University of California, Irvine

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An **algebra** is a vector space A over a field K, provided with a bilinear map $A \times A \to A$ given by $(a, b) \mapsto ab$, called the **multiplication** or the product of A. An algebra A such that $ab = ba$ for every $a, b \in A$ will be called **commutative**. If $(ab)c = a(bc)$ for every $a, b, c \in A$, then we say that A is **associative**.

An **evolution algebra** (of dimension $n < \infty$) over a field K is a K-algebra A provided with a basis $B = \{e_i : i \leq 1 \leq n\}$ such that $e_i e_j = 0$ whenever $i \neq j$. Such a basis B is called a **natural basis**. For a fixed natural basis B in A, the scalars $\omega_{ki} \in K$ such that $e_i^2 = \sum_k \omega_{ki} e_k$ will be called the **structure constants** of A relative to B, and the matrix $M_B = (\omega_{ki})$ is said to be the **structure matrix** of A relative to B. We will write $M_B(A)$ to emphasize the evolution algebra we refer to.

Every evolution algebra is uniquely determined by its structure matrix: if A is an evolution algebra and B a natural basis of A, there is a matrix, M_B, associated to B which represents the product of the elements in this basis. Conversely, for a fixed basis $B = \{e_i : 1 \leq i \leq n\}$ of a K-vector space A, each matrix in $M_n(K)$ defines a product in A under which A is an evolution algebra and B is a natural basis.

Let A be an evolution algebra and $B = \{e_i : 1 \leq i \leq n\}$ a natural basis. Consider elements $a = \sum \alpha_i e_i$ and $b = \sum_i \beta_i e_i$ in A, with $\alpha_i, \beta_i \in K$. Then

$$ab = \sum_i \alpha_i \beta_i e_i^2 = \sum_i \alpha_i \beta_i \left(\sum_j \omega_{ji} e_j \right) = \sum_{i,j} \alpha_i \beta_i \omega_{ji} e_j.$$