Evolution Algebras

Math 199C-Spring 2021

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April 21, 2021

An **algebra** is a vector space A over a field K, provided with a bilinear map $A \times A \to A$ given by $(a, b) \mapsto ab$, called the **multiplication** or the product of A. An algebra A such that $ab = ba$ every $a, b \in A$ will be called **commutative**. If $(ab)c = a(bc)$ for every $a, b, c \in A$, then we say that A is **associative**.

An **evolution algebra** (of dimension $n < \infty$) over a field K is a K-algebra A provided with a basis $B = \{e_i : i \leq 1 \leq n\}$ such that $e_i e_j = 0$ whenever $\neq j$. Such a basis B is called a **natural basis**. For a fixed natural basis B in A, the scalars $\omega_{ki} \in K$ such that $e_i^2 = \sum_k \omega_{ki} e_k$ will be called the **structure constants** of A relative to B, and the matrix $M_B = (\omega_{ki})$ is said to be the **structure matrix** of A relative to B. We will write $M_B(A)$ emphasize the evolution algebra we refer to.

Every evolution algebra is uniquely determined by its structure matrix: if A is an evolution algebra and B a natural basis of A, there is a matrix, M_B, associated to B which represents the product of the elements in this basis. Conversely, for a fixed basis $B = \{e_i : 1 \leq i \leq n\}$ of a K-vector space A, each matrix in $M_n(K)$ defines a product in A under which A is an evolution algebra and B is a natural basis.

Let A be an evolution algebra and $B = \{e_i : 1 \leq i \leq n\}$ a natural basis. Consider elements $a = \sum \alpha_i e_i$ and $b = \sum_i \beta_i e_i$ in A, with $\alpha_i, \beta_i \in K$. Then

$$ab = \sum_i \alpha_i \beta_i e_i^2 = \sum_i \alpha_i \beta_i \left(\sum_j \omega_{ji} e_j \right) = \sum_{i,j} \alpha_i \beta_i \omega_{ji} e_j.$$

An **evolution subalgebra** of an evolution algebra A is a subalgebra $A' \subset A$ such that A' is an evolution algebra, i.e., A' has a natural basis.

HW2 Exercise 3

Let A be an evolution algebra with natural basis e_1, e_2, e_3, with $e_1^2 = e_1 + e_2,\ e_2^2 = -e_1 - e_2,\ e_3^2 = -e_2 + e_3$. Let A' be the linear span of the two vectors $u_1 = e_1 + e_2$ and $u_2 = e_1 + e_3$. Show that A' is a two dimensional subalgebra of A but it is not an evolution algebra, that is, it does not have a natural basis. Hence it is not an evolution subalgebra.

An **evolution ideal** of an evolution algebra A is an ideal I of A such that I has a natural basis.

HW2 Exercise 4

Let A be an evolution algebra with natural basis e_1, e_2, e_3 with $e_1^2 = e_2 + e_3,\ e_2^2 = e_1 + e_2,\ e_3^2 = -e_1 - e_2$. Let I be the linear span of the two vectors $u_1 = e_1^2 = e_2 + e_3$ and $u_2 = e_2^2 = e_1 + e_2$. Show that I is a two dimensional ideal of A, but it is not an evolution algebra, that is, it does not have a natural basis. Hence it is not an evolution ideal.

We say that an evolution subalgebra A' has the **extension property** if there exists a natural basis B' of A' which is contained in a natural basis of A.

HW2 Exercise 5

Let A be an evolution algebra with natural basis e_1, e_2, e_3, with $e_1^2 = e_3,\ e_2^2 = e_1 + e_2,\ e_3^2 = e_3$. Let I be the linear span of the two vectors $e_1 + e_2$ and e_3. Show that I is an evolution ideal but that it does not have the extension property, that is, no natural basis of I can be extended to a natural basis of A. Explicitly, if $u = \alpha(e_1 + e_2) + \beta e_3$ and $v = \gamma(e_1 + e_2) + \delta e_3$ is a natural basis for I, then $\{u, v, w\}$, where $w = \lambda e_1 + \mu e_2 + e_3$ cannot be a natural basis for A.