4/29/21

Brown/Roberts

signed digraph

Theorem 1 simple pulse process starting at vertex x_i

$\Rightarrow$

(i) $p_j(t) = \boxed{\text{the signed number of sequences}}$ from x_i to x_j of length $= t$

(positive means the product of the signs is positive)

i.e. the difference between the number of positive sequences from x_i to x_j of length t and the number of negative sequences from x_i to x_j of length t.

(ii) $v_j(t) = v_j(\text{start}) + p_j^0(0) +$ the signed number of sequences of length $\leq t$

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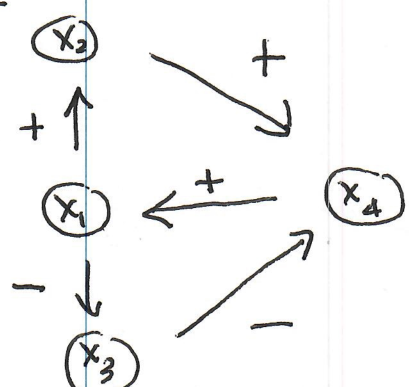
example



start at x_1

some sequences x_1 to x_2	length
x_1, x_2	1
$\overbrace{x_1, x_2, x_4, x_1}^{\text{cycle}}, x_2, x_4, x_1, x_2, x_4$	1, 4, 7, 10, ...
$\overbrace{x_1, x_3, x_4, x_1}^{\text{cycle}}, x_2, x_4, x_1, x_2, x_4$	4, 7, 10
$\overbrace{x_1, x_3, x_4, x_1}, x_3, x_4, x_1, x_2, x_4, x_1$	7, 10
$\overbrace{x_1, x_3, x_4, x_1}, x_2, x_4, x_1, x_3, x_4, x_1, x_2$	4, 10
$\overbrace{x_1, x_2, x_4, x_1}, x_3, x_4, x_1, x_2, x_4, x_1$ → x_2	10
x_3	13
x_4
x_1
x_2	16

1, 7, 10, 13

see p. 4 of notes for April 24

according to Th 1

$P(1) = (0, 1, -1, 0)$

$p_2(1) = 1$

$p_3(1) = -1$

$p_4(1) = 0$

$p_1(1) = 0$

check

$p_2(2) = 0$

$p_3(2) = 0$

$p_4(2) = 1 + 1 = 2$

$p_1(2) = 0$

check

$p(2) = (0, 0, 0, 2)$

$p_2(3) = 0$

$p_3(3) = 0$

$p_4(3) = 0$

$p_1(3) = 1 + 1 = 2$

check

$p(3) = (2, 0, 0, 0)$

according to Th 1

$v(\text{start}) = (0, 0, 0, 0)$

$P^\circ(0) = (1, 0, 0, 0)$

$V(1) = (1, 1, -1, 0)$

$v_2(1) = 0 + 0 + 1 = 1$

$v_3(1) = 0 + 0 + (-1) = -1$

$v_4(1) = 0 + 0 + 0 = 0$

$v_1(1) = 0 + 1 + 0 = 1$

check

1+

$V(2) = (1, 1, -1, 2)$

$v_2(2) = 0 + 0 + 0 = 1$

$v_3(2) = \overset{-1+}{0} + 0 + 0 = -1$

$v_4(2) = \overset{0+}{0} + 0 + \overset{(1+1)}{2} = 2$

$v_1(2) = \overset{0+}{0} + 1 + 0 = 1$

check

(length ≤ 2 !)

$V(3) = (3, 1, -1, 2)$

$v_2(3) = \overset{1+}{0} + 0 + 0 = 1$

$v_3(3) = \overset{-1+}{0} + 0 + 0 = -1$

$v_4(3) = \overset{2+}{0} + 0 + 0 = 2$

$v_1(3) = \overset{1+}{0} + 1 + 2 = 3$

check

(length ≤ 3)

def. of a weighted digraph.

$\boxed{\text{adjacency matrix}}$: if $w(x_i, x_j)$ is the weight assigned to the arc (x_i, x_j) then the adjacency matrix is $A = (a_{ij})$ $a_{ij} = w(x_i, x_j)$

in the example with weights 1, -1, 0

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} w(x_1,x_1) & w(x_1,x_2) & w(x_1,x_3) & w(x_1,x_4) \\ w(x_2,x_1) & w(x_2,x_2) & - & \\ w(x_3,x_1) & \cdot & - & - \\ w(x_4,x_1) & - & \cdot & w(x_4,x_4) \end{bmatrix}$$

Theorem 2 In a simple pulse process starting at vertex x_i, $p_j(t)$ = the (i,j) entry of A^t and $v_j(t) = v_j(\text{start})$ + the (i,j) entry of $I + A + A^2 + \cdots + A^t$

For $i=1$
According to Th 2,

$$\left.\begin{aligned} p_2(1) &= \text{the } (1,2) \text{ entry of } A && = 1 \\ p_3(1) &= \text{the } (1,3) \text{ entry of } A && = -1 \\ p_4(1) &= \text{the } (1,4) \text{ entry of } A && = 0 \\ p_1(1) &= \text{the } (1,1) \text{ entry of } A && = 0 \end{aligned}\right\} \text{check}$$

$$\left.\begin{aligned} V_2(1) &= 0 + (1,2) \text{ entry of } I + A && = 1 \\ V_3(1) &= 0 + (1,3) \text{ entry of } I + A && = -1 \\ V_4(1) &= 0 + (1,4) \cdots\cdots && = 0 \\ V_1(1) &= 0 + (1,1) \cdots\cdots && = 1 \end{aligned}\right\} \text{check}$$

$$A^2 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$I + A + A^2 = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} + \begin{matrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{matrix} + \begin{matrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$$\left.\begin{array}{ll} p_2(2) = \text{the } (1,2) \text{ entry of } A^2 & = 0 \\ p_3(2) = \text{the } (1,3) \text{ --- ---} & = 0 \\ p_4(2) = \quad (1,4) & = 2 \\ p_1(2) = \quad (1,1) & = 0 \end{array}\right\} \text{check}$$

$$\left.\begin{aligned} v_2(2) &= (1,2) \text{ entry of } I + A + A^2 = 1 \\ v_3(2) &= -1 \\ v_4(2) &= 2 \\ v_1(2) &= 1 \end{aligned}\right\} \text{ check}$$

$$\left.\begin{aligned} p_2(3) &= (1,2) \text{ entry of } A^3 = 0 \\ p_3(3) &= 0 \\ p_4(3) &= 0 \\ p_1(3) &= 2 \end{aligned}\right\} \text{ check}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$I + A + A^2 + A^3 = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 & 2 \\ 1 & 2 & -1 & 1 \\ -1 & -1 & 2 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

$$\left.\begin{aligned} V_2(3) &= (1,2) \text{ entry of } I + A + A^2 + A^3 = 1 \\ V_3(3) &= -1 \\ V_4(3) &= 2 \\ V_1(3) &= 3 \end{aligned}\right\} \text{check}$$

Theorem 3 In an autonomous pulse process on a weighted digraph, $P(t) = P(0)A^t$

$P(t) = (p_1(t), p_2(t), \dots, p_n(t))$

$p(0) = (p_1(0), p_2(0), \dots, p_n(0))$

$$\underset{1\times n}{(p_1(t), \dots, p_n(t))} = \underset{1\times n}{(p_1(0), \dots, p_n(0))} \underset{n\times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}^t}$$

$$\boxed{\begin{aligned} P(0) &= P^0(0) \\ P(t) &= V(t) - V(t-1) \\ &\quad (t > 0) \end{aligned}}$$

If $t = 0$ $A^0 = I$ $P(0) = P(0)A^0$ check

If $t = 1$ $P(1) \stackrel{?}{=} P(0)A$

In the example $P^0(0) = (1,0,0,0)$

$$\left.\begin{aligned} P(1) &= (0,1,-1,0) \\ P(0) &= (1,0,0,0) \\ A &= \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}\right\} \quad P(0)A = (1,0,0,0)\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$= (0,1,-1,0)$ check

$= P(1)$

if $t=2$ $\quad P(2) \overset{?}{=} P(0)A^2$

$P(2) = (0,0,0,2)$

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$P(0)A^2 = (1,0,0,0)\begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$= (0,0,0,2) = P(2)$ check

if $t=3$ $\quad P(3) = P(0)A^3$

$P(3) = (2,0,0,0)$

$$P(0)A^3 = (1,0,0,0)\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$ check