Stability (§5)

A vertex x_j of a weighted digraph is <u>pulse stable</u> under a pulse process if $\{|p_j(t)| : t = 0, 1, 2, ..\}$ is bounded.

It is value stable if $\{|v_j(t)| : t = 0, 1, 2, ..\}$ is bounded.

If each vertex is pulse (resp. value) stable we say the pulse process is pulse (resp. value) stable.

<u>Remark</u> Value stable $\Longrightarrow$ pulse stable

Since $$|p_j(t)| = |v_j(t) - v_j(t-1)| \leq |v_j(t)| + |v_j(t-1)|$$

A Necessary condition for pulse stability

Theorem 4 If a weighted digraph D has an eigenvalue greater than 1 in absolute value, then D is pulse unstable under some simple pulse process.

$A = [w_{ij}]$ adjacency matrix. λ eigenvalue of A

$|\lambda| > 1$ $\quad U$ eigenvector $\|U\| = 1 = \left(\sum_{i=1}^{n} \alpha_i^2\right)^{1/2}$ $\therefore |\alpha_i| \le 1$

$\therefore AU = \lambda U \qquad U = \sum \alpha_i E_i \qquad E_i = (0, 0, 1, 0 \ldots 0)$

If $t > 0$ $\quad (AU = \lambda U,\ A^2 U = \lambda A U = \lambda^2 U, \ldots \quad A^t U = \lambda^t U)$

$$|\lambda|^t = \|A^t U\| = \|A^t (\sum_{i=1}^{n} \alpha_i E_i)\| \le \sum_{i=1}^{n} |\alpha_i| \, \|A^t E_i\| \le \sum_{i=1}^{n} \|A^t E_i\|.$$

For any $t > 0$ $\exists\, i = i_t$ $\|A^t E_i\| \ge \frac{1}{n} |\lambda|^t$

$(\|A^t E_i\| \ge \|A^t E_k\| \quad \forall k = 1, \ldots, n)$

$\left(n \|A^t E_i\| \ge \sum_{k=1}^{n} \|A^t E_k\| \ge |\lambda^t|\right)$

$t = 1, 2, 3, 4, \ldots$ infinite sequence

$i = i_1, i_2, i_3, i_4, \ldots$ infinite sequence only finitely many distinct terms

So $\exists\, i$ such that $\|A^t E_i\| \ge \frac{1}{n} |\lambda|^t$ for infinitely many t

Define a pulse process with starting $P(0) = E_i$

$\| P(t) \| = \| P(0) A^t \| \geq \frac{1}{n} |\lambda|^t$ for infinitely many $t > 0$

Since $|\lambda| > 1$ $\quad \| P(t) \|$ is unbounded; $\sup_{t>0} \| P(t) \| = \infty$

So D is pulse unstable under any simple pulse process starting at vertex x_i.

<u>Corollary</u> If an integer-valued digraph D is pulse stable under all simple pulse processes then each non-zero eigenvalue has absolute value $\underline{1}$.

The constant term in the characteristic polynomial of A is the product of the eigenvalues (± 1). It is an integer since the digraph has only integer weights. By pulse stability of all simple pulse processes, each eigenvalue has absolute value ≤ 1. The product is a positive integer so all non-zero eigenvalues have abs. value $= 1$.

(this assumes all eigenvalues are $\neq 0$ — can adjust for this)

(Camille)

Jordan Canonical Form (not the same Jordan (Pascal) as in Jordan algebras)

Every $n\times n$ matrix A can be written in the form

$$J = \begin{bmatrix} B_1 & 0 & & \\ 0 & B_2 & & 0 \\ & & \ddots & \\ 0 & & & B_r \end{bmatrix}$$

$A = S^{-1} J S$

S non-singular

eigenvalues $\lambda_1, \dots, \lambda_n$ (not nec. distinct)

$$B_j(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ 0 & \lambda & 1 & \\ 0 & 0 & \lambda & \ddots \\ & & & 1 \\ 0 & 0 & \cdots 0 & \lambda \end{bmatrix}$$

$B_j = [\lambda]$, $B_j = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $B_j = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \cdots$ $(\lambda = \lambda_j)$

λ_j = eigenvalues of A.

We have
$$J^t = \begin{bmatrix} B_1^t & & 0 \\ & B_2^t & \\ 0 & \ddots & B_r^t \end{bmatrix}$$
(block matrix multiplication)

$n=2$ example $B_j = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

$$B_j^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$B_j^3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 2\lambda^2 + \lambda^2 \\ 0 & \lambda^3 \end{pmatrix}, \quad B_j^4 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix} = \begin{pmatrix} \lambda^4 & 4\lambda^2 \\ 0 & \lambda^4 \end{pmatrix}$$

You can believe: (for any $n\times n$)

$$b^{j,t}_{k,l} = \begin{cases} 0 & k>l \\ \binom{t}{l-k} \lambda^{t-l+k} & \end{cases}$$

$$B_j^t = \begin{bmatrix} b^{j,t}_{11} & b^{j,t}_{12} \\ b^{j,t}_{21} & b^{j,t}_{22} \end{bmatrix} \text{ etc.}$$

Lemma 1 (H) A $n \times n$ matrix, J Jordan canonical form

(C) $\{\|A^t\| : t = 0,1,2,\ldots\}$ is bounded

$\Longleftrightarrow \{\|J^t\| : t = 0,1,2,\ldots\}$ is bounded.

If $B = CD$ then $b_{ij} = \sum_k c_{ik} d_{kj}$

$$\|B\|^2 = \sum_{i,j} |b_{ij}|^2 = \sum_{i,j} \left| \sum_k c_{ik} d_{kj} \right|^2 \leq \sum_{i,j} \left(\sum_k |c_{ik}|^2 \right) \left(\sum_k |d_{kj}|^2 \right)$$

(Schwarz inequality)

$$= \left(\sum_{i,k} |c_{ik}|^2 \right) \left(\sum_{j,k} |d_{kj}|^2 \right)$$

$$= \|C\|^2 \cdot \|D\|^2$$

$$A = S^{-1} J S \Rightarrow A^t = (S^{-1} J S)(S^{-1} J S) \cdots (S^{-1} J S) = S^{-1} J^t S$$

$$\Rightarrow \|A^t\| \leq \|S^{-1}\| \, \|J^t\| \, \|S\| \qquad \therefore \|J^t\| \text{ bounded} \Rightarrow \|A^t\| \text{ bounded}$$

$$J = S A S^{-1} \Rightarrow J^t = S A^t S^{-1} \Rightarrow \|J^t\| \leq \|S\| \, \|A^t\| \, \|S^{-1}\|$$

$$\therefore \|A^t\| \text{ bounded} \Rightarrow \|J^t\| \text{ bounded}$$

Lemma 2 If a weighted digraph D is pulse stable under all simple pulse processes, then $\{\|J^t\| : t = 0,1,2,\ldots\}$ is bounded. Conversely, if $\{\|J^t\| : t = 0,1,2,\ldots\}$ is bounded, then D is stable under all autonomous pulse processes.

By Theorem 3 $\quad P(t) = P(0)A^t$ for all autonomous pulse processes. We want $\|A^t\|$ bounded

if $\|P(0)A^t\| = \|P(t)\|$ is bounded (pulse stable!)

then so is $\|P(0)A^t\|$, but $\|P(0)A^t\| \leq \|P(0)\|\,\|A^t\|$

NO GO!

<u>need more detail</u>: $\quad P(t) = (p_1(t), \ldots, p_n(t))$

$P(0) = (p_1(0), \ldots, p_n(0))$

Let
$$A^t = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \\ a_{n1}(t) & & a_{nn}(t) \end{bmatrix}$$

so
$$\begin{aligned} p_1(t) &= p_1(0)a_{11}(t) + \cdots + p_n(0)a_{n1}(t) \\ &\cdots \\ p_j(t) &= p_1(0)a_{1j}(t) + \cdots + p_n(0)a_{nj}(t) \\ &\cdots \\ p_n(t) &= p_1(0)a_{1n}(t) + \cdots + p_n(0)a_{nn}(t) \end{aligned}$$

each $p_j(t)$ is bounded

so $\quad p_1(0)a_{1j}(t) + \cdots + p_n(0)a_{nj}(t)$ is bounded $\forall\, t = 0, 1, 2 \ldots$

and all <u>simple pulse processes</u>

Pick $P(0) = (0, \ldots, \underbrace{1}_{i}, 0 \ldots 0)$ then $\underbrace{p_i(0)}_{=1} a_{ij}(t)$ is bounded

So $a_{ij}(t)$ is bounded $\forall\, i, j$

and $\|A^t\| = \sum_{i,j} |a_{ij}(t)|^2$ is bounded.

By Lemma 1, $\|J^t\|$ is bounded, proving the first statement.

Conversely, if $\|J^t\|$ is bounded, then by Lemma 1 again $\|A^t\|$ is bounded and by Theorem 3, D is pulse stable under all autonomous pulse processes. $\blacksquare$

A necessary and sufficient condition for pulse stability

Theorem 5 If D is a weighted digraph and J is the canonical Jordan form of its adjacency matrix A, then the following are equivalent.

(a) D is pulse stable under all autonomous pulse processes

(b) D is pulse stable under all simple pulse processes.

(c) every eigenvalue of D has absolute value ≤ 1 and if $B_j(\lambda) = [\lambda]$, then $|\lambda| < 1$.

Next:

- Proof of Theorem 5
- Necessary and sufficient condition for value stability (Theorem 6)