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5-7-21 ① I

Proof of Th. 5

Need to prove (b) ⇒ (c) ⇒ (a)

$\boxed{b \Rightarrow c}$ assume D is pulse stable under all simple pulse processes. By theorem 4 every eigenvalue has absolute value ≤ 1. We need to show that if λ_0 is an eigenvalue with $B_j(\lambda_0) \neq [\lambda_0]$ i.e. $B_j(\lambda_0)$ is $n \times n$ with $n \geq 2$ then $|\lambda_0| < 1$. From Lemma 2 $\|J^t\|$ is bdd $(t = 0, 1, 2, \ldots)$

Look at $B_{j_0}(\lambda_0)$ Then $b_{1,2}^{j,t} = t\lambda_0^{t-1}$

$\left(\text{recall } b_{k,l}^{j,t} = \begin{cases} 0 & k > l \\ \binom{t}{l-k} \lambda^{t-l+k} \end{cases} \right)$

If $|\lambda_0| > 1$, As $t \to \infty$ $b_{1,2}^{j_0,t}(\lambda_0) \to \infty$

$$\infty > \|J^t\|^2 = \|\sum_j B_j^t(\lambda_j)\|^2 = \sum_j \|B_j^t(\lambda_j)\|^2$$

$$\geq \|B_{j_0}^t(\lambda_0)\|^2 \geq |b_{1,2}^{j_0,t}(\lambda_0)|^2 \to \infty$$

$$J^t = \begin{bmatrix} B_1^t(\lambda_1) & & & 0 \\ & B_2^t(\lambda_2) & & \\ & & \ddots & \\ 0 & & & B_r^t(\lambda_r) \end{bmatrix}$$

Contradiction, proving $|\lambda_0| < 1$ and $\boxed{b \Rightarrow c}$

$\boxed{c \Rightarrow a}$ From Lemma 2 it suffices to prove $\|J^t\|$ bounded.

It suffices to prove each $\|B_j^t\|$ is bounded.

$$B_j(\lambda)^t = \begin{bmatrix} & & \\ & b_{k,l}^{j,t}(\lambda) & \\ & & \end{bmatrix} \qquad |\lambda| \leq 1$$

if $B_j(\lambda) = [\lambda]$ then $B_j(\lambda)^t = [\lambda^t]$ $\|B_j(\lambda)^t\|^2 = |\lambda|^{2t} \leq 1 < \infty$

if $B_j(\lambda) \neq [\lambda]$ then by hypothesis $|\lambda| < 1$ It suffices to show that if $k \leq l$ $\{|b_{k,l}^{j,t}|\}$ is bounded $(t = 0, 1, 2, \ldots)$

$\left(\|B_j^t\|^2 = \sum_{k,l} |b_{k,l}^{j,t}|^2 \right)$

Recall that $b_{k,l}^{j,t} = \binom{t}{l-k} \lambda^{t-l+k}$ if $k \leq l$

LEMMA If $b > a \geq 1$ a, b integers

then $\binom{t}{a} \leq \binom{t}{b}$ for all sufficiently large t $\quad \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

$= \dfrac{n(n-1)\cdots(n-k+1)}{k!}$

At first let $t > 2b$, say $t = 2b + s$ $\quad s > 0$

$$\binom{2b+s}{a} \overset{?}{\leq} \binom{2b+s}{b}$$

$$\frac{(2b+s)(2b+s-1)\cdots(2b+s-a+1)}{a!} \overset{?}{\leq} \frac{(2b+s)\,(2b+s-1)\cdots(2b+s-b+1)}{b!}$$

$$\frac{b!}{a!} \overset{?}{\leq} ((2b+s-a+1)-1)((2b+s-a+1)-2)((2b+s-a+1)-3)\cdots\cdots(2b+s-b+2)(2b+s-b+1)$$

(*) $$\frac{b!}{a!} \overset{?}{\leq} (2b+s-a)(2b+s-a-1)(2b+s-a-2)\cdots\cdots(b+s+2)(b+s+1)$$

This is true for all sufficiently large s

Thus $\binom{t}{a} \leq \binom{t}{b}$ for all sufficiently large t

let $k = b - a > 0$

(*) says $\dfrac{b(b-1)\cdots 2\cdot 1}{a(a-1)\cdots 2\cdot 1} \overset{?}{\leq} (b+s+k)(b+s+k-1)\cdots\cdots(b+s+2)(b+s+1)$

(k terms each involving s)

let $B_j(\lambda)$ be $n \times n$ with $n \geq 2$

Then $\ell - k \leq n-1$ so by LEMMA

$$\binom{t}{\ell-k} \leq \binom{t}{n-1} \quad \text{for all sufficiently large } t$$

$$\binom{t}{n-1} = \frac{t!}{(t-n+1)!\,(n-1)!} = \frac{t(t-1)\cdots(t-n+2)}{(n-1)!} \leq \frac{t^{n-1}}{(n-1)!}$$

$$|\lambda^{t-\ell+k}| = |\lambda|^{t-\ell+k} \leq |\lambda|^{t-(n-1)} \qquad (\ell-k \leq n-1)$$

$$|b_{k,\ell}^{j,t}| = \left|\binom{t}{\ell-k}\lambda^{t-\ell+k}\right| \leq \underbrace{\frac{t^{n-1}}{(n-1)!}}_{\to\infty}\,\underbrace{|\lambda|^{t-(n-1)}}_{\to 0}$$

$\lim\limits_{t\to\infty} |\lambda|^{t-(n-1)} = 0$, since $|\lambda| < 1$. $\lim\limits_{t\to\infty} \dfrac{t^{n-1}}{(n-1)!} = +\infty$

By l'Hopital's rule $\lim\limits_{t\to\infty} |b_{k,\ell}^{j,t}| = 0$ so $|b_{k,\ell}^{j,t}(\lambda)|$ is bounded $t = 0, 1, 2, \ldots$

This proves Theorem 5 $\blacksquare$

<u>Review</u>

Theorem 4 — $\exists$ eigenvalue λ, $|\lambda| > 1 \Longrightarrow$ pulse unstable for some simple pulse process

Theorem 5 — every eigenvalue λ, $|\lambda| \leq 1$ and $|\lambda| < 1$ if $B_j(\lambda) \neq [\lambda]$ $\Longleftrightarrow$ pulse stable under all autonomous pulse processes

Theorem 6 — pulse stable under all simple pulse processes and 1 is not an eigenvalue $\Longleftrightarrow$ value stable under all autonomous pulse processes

(value stable $\Rightarrow$ pulse stable)