Theorem 2 : In a simple pulse process starting at x_i:

$p_j(t)$ = the (i,j)-entry of A^t

$v_j(t) = v_j(\text{start}) +$ (i,j)-entry of $I + A + A^2 + \cdots + A^t$

Theorem 3 : $P(t) = P(0)A^t$ (autonomous pulse process)

Theorem 6 D is a weighted digraph. The following are equivalent:

(a) D is value stable under all autonomous pulse processes

(b) D is value stable under all simple pulse processes

(c) D is pulse stable under all simple pulse processes and $\lambda = 1$ is not an eigenvalue of D

a $\Rightarrow$ b trivial

c $\Rightarrow$ a

Lemma 1 $\|A^t\|$ is bdd $\iff$ $\|J^t\|$ is bounded (A any matrix)

Lemma 2
- D pulse stable for simple pulse processes $\Rightarrow$ $\|J^t\|$ bounded
- $\|J^t\|$ bounded $\Rightarrow$ D is pulse stable for all autonomous pulse processes

Lemma 1' $\|\sum_{t=0}^{N} A^t\|$ bounded $N=1,2,\ldots$ $\iff$ $\|\sum_{t=0}^{N} J^t\|$ is bounded $N=1,2,\ldots$

(see the top of p. (2) for the "proof")

$$A = S^{-1} J S$$

$$A^t = S^{-1} J^t S$$

$$I + A + \cdots + A^t = S^{-1}(I + J + J^2 + \cdots + J^t) S$$

$$\left\| \sum_{t=0}^{N} A^t \right\| \le \|S^{-1}\| \|S\| \left\| \sum_{t=0}^{N} J^t \right\| \le \underset{\|S^{-1}\|^2 \|S\|^2}{\underset{\|}{\text{constant}}} \left\| \sum_{t=0}^{N} A^t \right\|$$

$\boxed{(c) \Rightarrow (a)}$

If D is pulse stable for all simple pulse processes, then we saw in the proof of lemma 2 that $\|A^t\|$ is bounded

(see p.(6) of the notes for May 5)

We need to show $\underbrace{\left\| \sum_{k=0}^{t} A^k \right\|}_{B(t)}$ is bounded

since

$V_j(t) =$ (i,j) entry of $I + A + A^2 + \cdots + A^N = B(N)$

$V(t) = (V_1(t), \ldots, V_n(t))$

$$= (0, 0, \ldots, i, \ldots 0) \begin{bmatrix} b_{11}(t) & \cdots & b_{1n}(t) \\ \vdots & & \\ b_{n1}(t) & \cdots & b_{nn}(t) \end{bmatrix}$$

By Lemma 1' it is sufficient to show $\left\| \sum_{k=0}^{t} J^k \right\|$ is bounded.

in fact $\left\| \sum_{k=0}^{t} B_j^t \right\|$ is bounded

$\boxed{\text{case 1} \quad \text{if } B_j = [\lambda]}$, then $\sum_{t=0}^{N} B_j^t = \sum_{t=0}^{N} \lambda^t$

$|\lambda| \le 1$ by pulse stability and $\lambda \ne 1$ by hypotheses

So $\sum_{t=0}^{N} \lambda^t$ converges! NO! $1 + \lambda + \cdots + \lambda^N = \dfrac{1 - \lambda^{N+1}}{1 - \lambda}$ $\quad \lambda^N$ doesn't converge

But it is bounded since $\left| \sum_{t=0}^{N} \lambda^t \right| = \left| \dfrac{1 - \lambda^{N+1}}{1 - \lambda} \right| \le \dfrac{2}{|1 - \lambda|}$ $\quad (\lambda \ne 1)$

Case 2

if $B_j = b^j_{k,l}$ is $n \times n$ $n \ge 2$ then by pulse stability $|\lambda| < 1$

If $k > l$ $\sum_{t=0}^{N} b^{j,t}_{k,l} = 0$. If $k \le l$ as in the proof of Th 5

(See p.(3) of the notes for May 8)

$$\left| b^{J,t}_{k,\ell} \right| \le \frac{t^{n-1}}{(n-1)!} |\lambda|^{t-n+1}$$ for sufficiently large t

and $$\left| \sum_{t=0}^{N} b^{J,t}_{k,\ell} \right| \le \sum_{t=0}^{N} \left| b^{J,t}_{k,\ell} \right| \le \sum_{t=0}^{N} \frac{1}{(n-1)!} t^{n-1} |\lambda|^{t-n+1}$$

Ratio Test $$\frac{(t+1)^{n-1} |\lambda|^{t+1-n+1}}{t^{n-1} |\lambda|^{t-n+1}} = \left(\frac{t+1}{t}\right)^{n-1} |\lambda| \to |\lambda| < 1$$ as $t \to \infty$.

So $\sum_{t=0}^{N} b^{J,t}_{k,\ell}$ converges $\forall (k,\ell)$ so is bounded, say

$$\left| \sum_{t=0}^{N} b^{J,t}_{k,\ell} \right| \le M < \infty$$

$$\left\| \sum_{t=0}^{N} B_J^t \right\|^2 = \sum_{k,\ell} \left| \sum_{t=0}^{N} b^{J,t}_{k,\ell} \right|^2$$

$$\le \sum_{k,\ell} M^2 = n^2 M^2 < \infty$$

This completes the proof of (c) $\Rightarrow$ (a)

$\boxed{b \Rightarrow c}$ Since value stability implies pulse stability, we only need to prove that $\lambda = 1$ is not an eigenvalue.

Suppose it was. We shall show that $\left\| \sum_{t=0}^{N} J^t \right\|$ is not bounded

By pulse stability $B_J(\lambda) = [\lambda]$ ($\lambda = 1$!) 1×1 matrix

So $$\sum_{t=0}^{N} B_J^t = \sum_{t=0}^{N} \lambda^t = \sum_{t=0}^{N} 1 = N+1$$ and

$\left\| \sum_{t=0}^{N} B_J^t \right\| = N+1$ is unbounded, hence so is $\left\| \sum_{t=0}^{N} J^t \right\|$