199c 051221_8PP

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Homework 2—Math199C-due April 30, 2021

1. Let f_1, f_2, f_3 be a basis for the zygotic algebra:

$$f_1f_1 = f_1, \quad f_1f_2 = f_2f_1 = \frac{1}{2}(f_1 + f_2), \quad f_1f_3 = f_3f_1 = f_2$$

$$f_2f_2 = \frac{1}{2}f_1 + \frac{1}{4}f_3 + \frac{1}{2}f_2, \quad f_2f_3 = f_3f_2 = \frac{1}{2}(f_3 + f_2), \quad f_3f_3 = f_3$$

Define, for $i = 1, 2, 3$, $e_i = \alpha_{1i}f_1 + \alpha_{2i}f_2 + \alpha_{3i}f_3$, and assume that e_1, e_2, e_3 is a natural basis, that is, assume that the zygotic algebra is an evolution algebra.

(a) If $e_ie_j = Af_1 + Bf_2 + Cf_3$, then calculate A, B and C for $i \neq j$.

(b) Let M be the matrix with respect to f_1, f_2, f_3 of the linear transformation T defined by $Tf_i = e_i$. Why is M a non-singular matrix?

2. Let A be any algebra

(a) Show that A is power associative, that is, $(a^ia^j)a^k = a^i(a^ja^k)$ for all $i, j, k \geq 1$ and all $a \in A$, if and only if $a^{k+\ell} = a^ka^\ell$ for all $k, \ell \geq 1$ and all $a \in A$.

(b) (Revised) For an evolution algebra A, define $A^2 = \{\sum_{i=1}^n a_ib_i : a_i, b_i \in A\}$

Prove

(a) A^2 is the linear span of e_i^2, where $e_1, e_2, \ldots, e_m$ is any natural basis.

(b) $A^2 = A$ if and only if $\det M_B(A) \neq 0$

(c) If $\det M_B(A) \neq 0$ for some natural basis B, then $\det M'_B(A) \neq 0$ for any other natural basis

(d) If A is power associative and $\det M_B(A) \neq 0$ for some natural basis B, then $M_B(A)$ is diagonal

3. Let A be an evolution algebra with natural basis e_1, e_2, e_3, with $e_1^2 = e_1 + e_2$, $e_2^2 = -e_1 - e_2$, $e_3^2 = -e_2 + e_3$. Let A' be the linear span of the two vectors $u_1 = e_1 + e_2$ and $u_2 = e_1 + e_3$. Show that A' is a two dimensional subalgebra of A but it is not an evolution algebra, that is, it does not have a natural basis. Hence it is not an evolution subalgebra.

4. Let A be an evolution algebra with natural basis e_1, e_2, e_3 with $e_1^2 = e_2 + e_3$, $e_2^2 = e_1 + e_2$, $e_3^2 = -e_1 - e_2$. Let I be the linear span of the two vectors $u_1 = e_1^2 = e_2 + e_3$ and $u_2 = e_2^2 = e_1 + e_2$. Show that I is a two dimensional ideal of A, but it is not an evolution algebra, that is, it does not have a natural basis. Hence it is not an evolution ideal.

5. Let A be an evolution algebra with natural basis e_1, e_2, e_3, with $e_1^2 = e_3$, $e_2^2 = e_1 + e_2$, $e_3^2 = e_3$. Let I be the linear span of the two vectors $e_1 + e_2$ and e_3. Show that I is an evolution ideal but that it does not have the extension property, that is, no natural basis of I can be extended to a natural basis of A. Explicitly, if $u = \alpha(e_1+e_2)+\beta e_3$ and $v = \gamma(e_1 + e_2) + \delta e_3$ is a natural basis for I, then $\{u, v, w\}$, where $w = \lambda e_1 + \mu e_2 + \rho e_3$ cannot be a natural basis for A.

Exercise 2ⓐ solution (2b solution p②-③) 4/16/21 ①

A is power assoc.

$(a^i a^j)a^k = a^i(a^j a^k) \iff a^{k+l} = a^k a^l$ continued 5/1/21

$\forall\, i,j,k \geq 1$ $\forall\, k,l \geq 1$

on p. ②

def $a^1 = a$

$a^{k+1} = a a^k = a^k a$

$\boxed{\Rightarrow}$ induction $k=1$ $a^{l+1} = a a^l$ (defn)

assume $a^{k+l} = a^k a^l$ for some k $\forall\, l$

then $a^{k+1} a^l = \cancel{a(a^k a)}$

$(a a^k) a^l = a(a^k a^l) = a(a^{k+l})$

$= a^{k+l+1}$

$\boxed{\Leftarrow}$ $i=1$ $(a a^j) a^k \stackrel{?}{=} a(a^j a^k) = a(a^{j+k}) = a^{j+k+1}$

$\cancel{a(a^j a^k)}$

$\|$

$a^{j+1} a^k$

$\|$

a^{j+k+1}

<u>induction</u> assume $(a^i a^j) a^k = a^i(a^j a^k)$ for some i $\forall\, j\, k$

$(a^{i+1} a^j) a^k = a^{i+j+1} a^k = a^{i+j+1+k}$ $\cancel{= a^{i+j+1} a^k}$

$\stackrel{?}{=} a^{i+1}(a^j a^k)$ $\|$ $\cancel{= (a\, a^{i+j}) a^k}$

~~it works~~ WORKS

$(a^{i+1})(a^{j+k})$

$= a^{i+1}(a^j a^k)$ ∎

2b A is an evolution algebra

(a) $A^2 \overset{?}{=} \operatorname{span}\{e_i^2 : e_1, \ldots, e_m \text{ nat. basis}\}$

$$A^2 = \sum_{i=1}^{n} a_i b_i \quad : a_i, b_i \in A, \quad i = 1, 2, \ldots$$

$a = \sum \alpha_i e_i \qquad b = \sum \beta_j e_j$

$$ab = \sum_i \sum_j \alpha_i \beta_j e_i e_j = \sum_i \alpha_i \beta_i e_i^2$$

This proves $A^2 \subset \operatorname{span}\{e_i^2\}$

conversely $e_i^2 \in A^2$ so equality holds. $\boxed{\,}$

(b) $A^2 = A \overset{?}{\Longleftrightarrow} \det M_B(A) \neq 0$

Suppose $A^2 = A$ Then $A = \operatorname{span}\{e_i^2\}$

so $e_1^2, e_2^2, \ldots e_m^2$ is a basis

But $e_i^2 = \sum a_{ji} e_j$ so $[a_{ij}]$ is non-singular

conversely if $[a_{ij}]$ is non singular, then $e_1^2, \ldots, e_m^2$ is a basis, so $A^2 = \operatorname{span}\{e_i^2\} = A$. $\boxed{\,}$

(c) $\det M_B(A) \neq 0 \overset{?}{\Longleftrightarrow} \det M_{B'}(A) \neq 0 \quad \forall$ nat basis B, B'

$B = \{e_1, \dots, e_n\} \qquad e_i = \sum p_{ji} f_j$

$B' = \{f_1, \dots, f_n\} \qquad f_i = \sum q_{ji} e_j$

$\det M_B(A) \neq 0 \Leftrightarrow A^2 = A \Leftrightarrow \det M_{B'}(A) \neq 0$ $\boxed{}$

(d) A power assoc, $\det M_B(A) \neq 0 \Rightarrow M_B(A)$ diagonal

$$e_i^2(e_i e_i) = \sum_l \left(\sum_k w_{ki}^2 w_{lk} \right) e_l \qquad M_B(A) = [w_{ij}]$$

$$(e_i^2 e_i) e_i = \sum_l \left(w_{ii}^2 w_{li} \right) e_l$$

$$\therefore \sum_k w_{ki}^2 w_{lk} = w_{ii}^2 w_{li} \quad \forall i, \forall l$$

$$M_B(A) \begin{bmatrix} w_{11}^2 & w_{12}^2 & & w_{1n}^2 \\ w_{21}^2 & w_{22}^2 & \cdots & w_{2n}^2 \\ \vdots & & \ddots & \\ w_{n1}^2 & \cdots & \cdots & w_{nn}^2 \end{bmatrix} = M_B(A) \begin{bmatrix} w_{11}^2 & 0 & \cdots & 0 \\ 0 & w_{22}^2 & & \\ \vdots & & \ddots & \\ 0 & & & w_{nn}^2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} w_{11}^2 & \cdots & w_{1n}^2 \\ \vdots & & \\ \vdots & & \\ w_{n1}^2 & \cdots & w_{nn}^2 \end{bmatrix} = \begin{bmatrix} w_{11}^2 & 0 & \cdots & 0 \\ 0 & w_{22}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & w_{nn}^2 \end{bmatrix}$$ $\boxed{}$

#3

$e_1^2 = e_1 + e_2$

$e_2^2 = -e_1 - e_2$

$e_3^2 = -e_2 + e_3$

$$M_B(A) = \begin{matrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}$$

$u_1 = e_1 + e_2 \qquad u_2 = e_1 + e_3 \qquad u_1 u_2 = e_1^2 = e_1 + e_2 = u_1$

$u_1^2 = e_1^2 + e_2^2 = 0 \qquad u_2^2 = e_1^2 + e_3^2 = e_1 + e_2 - e_2 + e_3 = e_1 + e_3 = u_2$

A' = Subalg gen by u_1, u_2 is ~~not~~ an evolution algebra

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suppose $v_1 = \alpha u_1 + \beta u_2 \quad v_2 = \gamma u_1 + \delta u_2$ is a natural basis

$$0 = v_1 v_2 = \alpha\gamma \underbrace{u_1^2}_{0} + \beta\gamma \underbrace{u_2 u_1}_{u_1} + \alpha\delta \underbrace{u_2 u_1}_{u_1} + \beta\delta \underbrace{u_2^2}_{u_2}$$

$A' = \{ \sum u_1^k u_2^l$

$\alpha u_1 + \beta u_2 = 0$

$\alpha e_1 + \alpha e_2 + \beta e_1 + \beta e_3 = 0$

$(\alpha + \beta) e_1 + \alpha e_2 + \beta e_3 = 0$

$\alpha = \beta = 0$

$\beta = 0 \quad \alpha\delta = 0 \Rightarrow \delta = 0 \Rightarrow ↯$

$v_1 = \alpha u_1 \quad v_2 = \gamma u_1 \Rightarrow ↯$ u_1, u_2 indep.

$\beta\gamma = 0 \Rightarrow \beta = 0$

$\beta\gamma + \alpha\delta = 0$

$\beta\delta = 0$

$\delta = 0$

let $A' = \text{span } u_1, u_2 \qquad A'$ has dim 2

$u_1 u_2 = \qquad a = \alpha u_1 + \beta u_2 \qquad b = \gamma u_1 + \delta u_2$

$$ab = \alpha\gamma u_1^2 + \beta\gamma u_2 u_1 + \alpha\delta u_1 u_2 + \beta\delta u_2^2$$

$$= \alpha\gamma \cdot 0 + (\beta\gamma + \alpha\delta) u_1 + \beta\delta u_2$$

$\boxed{A' \text{ is an algebra}}$

#4

$$M_B(A) = \begin{matrix} 0 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{matrix}$$

$u_1 = e_1^2 = e_2 + e_3 \qquad u_2 = e_2^2 = e_1 + e_2$

u_1, u_2 lin indep

$\alpha u_1 + \beta u_2 = 0$

$\alpha(e_2+e_3) + \beta(e_1+e_2) = 0$

$\beta e_1 + (\alpha+\beta)e_2 + \alpha e_3 = 0$

$\alpha = 0$
$\beta = 0$

$$I = sp\ u_1, u_2 = \{ \overbrace{\alpha e_1 + (\alpha+\beta)e_2 + \beta e_3}^{x} : \alpha, \beta \in \mathbb{K} \}$$

I is an ideal of dimension 2

$e_1 x = \alpha(e_2+e_3) = \alpha u_1$

$e_2 x = (\alpha+\beta)(e_1+e_2) = (\alpha+\beta) u_2$

$e_3 x = \beta(-e_1 - e_2) = -\beta u_2$

$v_1 = \alpha e_1 + (\alpha+\beta)e_2 + \beta e_3$

$v_2 = \lambda e_1 + (\lambda+\mu)e_2 + \mu e_3$

$$v_1 v_2 = \alpha\lambda \underbrace{(e_2+e_3)}_{u_1} + (\alpha+\beta)(\lambda+\mu)\underbrace{(e_1+e_2)}_{u_2} + \beta\mu(-e_1-e_2)$$

$v_1 v_2 = 0 \Rightarrow \alpha\lambda = 0 \qquad (\alpha+\beta)(\lambda+\mu) = \beta\mu$

- $\alpha = 0$
  - $\lambda = 0$
  - $\lambda \neq 0$
- $\lambda = 0$
  - $\alpha = 0$
  - $\alpha \neq 0$

3 cases

$\alpha=0 \quad \lambda\neq 0$	$\alpha\neq 0 \quad \lambda=0$	$\alpha=0, \lambda=0$
$\beta\lambda+\cancel{\beta\mu}=\cancel{\beta\mu}$	$\alpha\mu+\cancel{\beta\mu}=\cancel{\beta\mu}$	$\beta\mu=\beta\mu$
$\beta\lambda=0$	$\mu=0$	
$\beta=0$		$v_1=\beta(e_2+e_3)$
$\Downarrow$	$\Downarrow$	$v_2=\mu(e_2+e_3)$
$v_1=0$	$v_2=0$	
$\rightrightarrows\leftleftarrows$	$\rightrightarrows\leftleftarrows$	v_1, v_2 not lin. indep.

#5

$$\begin{matrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

$x=\alpha(e_1+e_2)+\beta e_3$

$e_1x=\alpha e_3$

$e_2x=\alpha(e_1+e_2)$ $\quad \therefore$ ideal

$e_3x=\beta e_3$

e_1+e_2, e_3 lin. indep

$\alpha(e_1+e_2)+\beta e_3=0$ ✓

$\Rightarrow \alpha=\beta=0$

$(e_1+e_2)e_3=0$

OK

$I=sp\{e_1+e_2, e_3\}$ evolution ideal

nat bais e_1+e_2, e_3

not the extension property

Let $u=\alpha(e_1+e_2)+\beta e_3$
$v=\gamma(e_1+e_2)+\delta e_3$ be a natural basis for I

which extends to a nat. basis $\{u,v,w\}$ of A

$w=\lambda e_1+\mu e_2+\rho e_3$

$uv = 0$

$\alpha\gamma(e_1^2 + e_2^2) + \beta\delta e_3^2 = 0$

$\alpha\gamma(e_3 + e_1 + e_2) + \beta\delta e_3 = 0$

$\boxed{\alpha\gamma = 0}$

$\alpha\gamma + \beta\delta = 0$

$\boxed{\beta\delta = 0}$

$uw = 0$

$u = \alpha e_1 + \alpha e_2 + \beta e_3$

$w = \lambda e_1 + \mu e_2 + \rho e_3$

$\alpha\lambda e_3 + \alpha\mu(e_1 + e_2) + \beta\rho e_3 = 0$

$\boxed{\cancel{\alpha\mu = 0} \quad \cancel{\alpha\lambda = 0}}$

$\boxed{\alpha\mu = 0 \quad \alpha\lambda + \beta\rho = 0}$

$vw = 0$

$v = \gamma e_1 + \gamma e_2 + \delta e_3$

$w = \lambda e_1 + \mu e_2 + \rho e_3$

$\gamma\lambda e_3 + \gamma\mu(e_1 + e_2) + \delta\rho e_3 = 0$

$\boxed{\gamma\mu = 0 \quad \gamma\lambda + \delta\rho = 0}$

3 cases

$\alpha = 0 \quad \gamma \neq 0$

$\beta\delta = 0$

$\beta\rho = 0$

$\mu = 0$

$\beta = 0$

$u = 0$

NO GO

$\beta \neq 0$

~~$u = \beta e_3$~~

~~$v = \gamma(e_1 + e_2)$~~

~~$w = \lambda e_1$~~

$\delta = 0$

$\rho = 0$

$u = \beta e_3$

$v = \gamma(e_1 + e_2)$

$w = \lambda e_1$

$vw = \gamma\lambda e_3$

$\lambda = 0 \quad w = 0$ NO GO

$\alpha \neq 0 \quad \gamma = 0$

$\beta\delta = 0$

$\mu = 0$

$\alpha\lambda + \beta\rho = 0$

$\delta\rho = 0$

$\delta = 0$

$u = \alpha e_1 + \alpha e_2 + \beta e_3$

$v = 0$

NO GO

$\delta \neq 0$

$\beta = 0$

$\rho = 0$

$u = \alpha e_1 + \alpha e_2$

$v = \delta e_3$

$w = \lambda e_1 + \mu e_2$

$uw = \alpha\lambda e_3 + \alpha\mu(e_1 + e_2)$

$\alpha\lambda = 0 \quad \lambda = 0$

$\alpha\mu = 0 \quad \mu = 0$

$w = 0$ NO GO

$\alpha = 0 \quad \gamma = 0$

$u = \beta e_3$

$v = \delta e_3$

u, v not lin. indep

NO GO