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The Jacobson radical of an evolution algebra (M. Victoria Velasco) 2018

§2 A evol. alg (arbitrary dimension)

$B = \{e_i : i \in \Lambda\}$ natural basis

$a = \sum \alpha_i e_i \qquad \Lambda_a^B = \{i \in \Lambda : \alpha_i \neq 0\}$ support of a

Proposition 2.2 (i) If $\dim A = \infty$, A has no unit (A = evolution algebra)

Suppose e is a unit $\quad e = \sum \alpha_i e_i$ (finite sum!)

So take $i \notin \Lambda_e \therefore ee_i = 0$, but $ee_i = e_i$ supposedly $\Rightarrow\Leftarrow$

(ii) If $\dim A < \infty$ then A has a unit $\Longleftrightarrow$

$\forall$ natural basis $B = \{e_1, \ldots, e_n\} \quad e_i^2 = w_{ii} e_i \quad w_{ii} \neq 0 \quad i = 1, \ldots, n$,

in which case $e = \frac{1}{w_{11}} e_1 + \cdots + \frac{1}{w_{nn}} e_n$ is a unit.

Suppose e is a unit & $\Lambda = \{1, \ldots, n\}$ (dimension n)

$e = \sum_{i \in \Lambda} \alpha_i e_i$ and $\alpha_i \neq 0 \ \forall i$ (if $\alpha_1 = 0$, then ee_1

(don't need parentheses here) $= \left(\sum_{i=2}^{n} \alpha_i e_i\right) e_1 = 0 \Rightarrow\Leftarrow$)

$e_i^2 = e_i(e_i e) = e_i(\alpha_i e_i^2) = \alpha_i e_i \left(\sum_j w_{ji} e_j\right) = \alpha_i w_{ii} e_i^2$

So either $e_i^2 = 0$ or $\alpha_i w_{ii} = 1$

$\Downarrow \qquad \Downarrow$

$ee_i = \alpha_i e_i^2 = 0$ NO GO $\Rightarrow\Leftarrow$

$e = \sum \alpha_i e_i = \sum \frac{1}{w_{ii}} e_i \Rightarrow$

This proves $\boxed{\Rightarrow}$

$e_i = e_i e = \frac{1}{w_{ii}} e_i^2 = \frac{1}{w_{ii}}\left(\sum_j w_{ji} e_j\right)$

$= \sum_{j \neq i} \frac{w_{ji}}{w_{ii}} + e_i$

$\therefore w_{ji} = 0 \quad i \neq j$

Conversely suppose for every natural basis $\{e_1, .., e_n\} = B$
$e_i^2 = w_{ii} e_i$ with $w_{ii} \neq 0$ $i = 1, .., n$ (w_{ii} depends on the basis)

Then $M_B(A) = \begin{bmatrix} w_{11} & & & \\ & w_{22} & & 0 \\ & 0 & \ddots & \\ & & & w_{nn} \end{bmatrix}$ and if $e = \sum_i \frac{1}{w_{ii}} e_i$

then $e e_j = \frac{1}{w_{jj}} e_j^2 = \frac{1}{w_{jj}} w_{jj} e_j = e_j$

so e is a unit.

$ea = e(\sum \alpha_j e_j) = \sum \alpha_j e e_j = \sum \alpha_j e_j = a$

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unitization of an algebra A

If A is any algebra (unit or not) let $A_1 := A \oplus \mathbb{K}$ or $A \times \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

define
- $\alpha(a, \lambda) = (\alpha a, \alpha\lambda)$ (scalar multiplication) $\alpha \in \mathbb{K}$, $(a,\lambda) \in A \times \mathbb{K}$
- $(a, \lambda) + (b, \mu) = (a+b, \lambda+\mu)$ (addition)
- $(a,\lambda)(b,\mu) = (ab, \lambda b + \mu a + \lambda\mu)$ (multiplication)

Then A_1 is an algebra, with unit $(0, 1)$ $(0,1)(b,\mu) = (b,\mu)$

A is a maximal ideal in A_1

$(b,\mu)(0,1) = (b,\mu)$

complexification of a real algebra A

If A is a real algebra ($\mathbb{K} = \mathbb{R}$) let $A_{\mathbb{C}} = A \oplus A$ or $A + iA$

define
- $(\underbrace{\alpha + i\beta}_{\in \mathbb{C}})(a, b) = (\alpha a - \beta b, \beta a + \alpha b)$ (scalar multiplication)
- $(a,b) + (a',b') = (a+a', b+b')$ (addition)
- $(a,b)(a',b') = (aa' - bb', ba' + ab')$ (multiplication)

$(a+ib)(a'+ib') = aa' - bb' + i(ba' + ab')$

Then $A_{\mathbb{C}}$ is a complex algebra ($\mathbb{K} = \mathbb{C}$), in particular a real algebra (by restricting scalars to $\mathbb{R}$)

A is a <u>real</u> subalgebra of $A_{\mathbb{C}}$.

Source

The multiplicative spectrum and the uniqueness of the complete normed topology. Marcos/Velasco 2014

For an associative algebra A with unit e, an element a is invertible if $\exists\, a^{-1} \in A$ with $a a^{-1} = a^{-1} a = e$

If A is a complex algebra ($\mathbb{K} = \mathbb{C}$) with unit

$$\sigma^A(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is NOT invertible}\}$$

is the spectrum of $a \in A$.

(not relevant but interesting)

Remark In an associative algebra B with unit

$$\sigma^B(b) = \underbrace{\sigma_m^B(b)}_{\text{multiplicative spectrum}} \quad \forall\, b \in B.$$

see p. 5 the other file for today

Let $\lambda \notin \sigma^B(b)$ so $a := b - \lambda e$ is invertible

Note: if a is invertible then $L_a^{-1} = L_{a^{-1}}$, $R_a^{-1} = R_{a^{-1}}$

$$\left.\begin{array}{l} L_{a^{-1}} L_a x = a^{-1}(ax) = (a^{-1}a)x = x \\ L_a L_{a^{-1}} x = a(a^{-1}x) = (aa^{-1})x = x \end{array}\right\} \therefore L_a^{-1} = L_{a^{-1}}$$

Then L_a, R_a are invertible (i.e. bijective)

so $\lambda \notin \sigma_m^B(b)$ i.e. $\mathbb{C} \setminus \sigma^B(b) \subseteq \mathbb{C} \setminus \sigma_m^B(b)$

or $\sigma_m^B(b) \subseteq \sigma^B(b)$.

Suppose $\lambda \notin \sigma_m^B(b)$ Then $a = b - \lambda e$ is m-invertible

so L_a and R_a are invertible (= bijective)

Look at: $L_a^{-1}(e)$ and $R_a^{-1}(e)$ $\boxed{\text{claim: Both } = a^{-1}}$

i.e. $\underbrace{\left(L_a^{-1}(e)\right)}_{c} a \overset{?}{=} e \overset{?}{=} a\,(\underbrace{L_a^{-1} e}_{c})$

$c = L_a^{-1} e$ $\quad L_a c = L_a(L_a^{-1}(e)) = e$

$L_a c = e$

$ac = e$ (c is a right inverse of a)

$R_a(R_a^{-1}(e)) = \left(\underbrace{R_a^{-1}(e)}_{d}\right) a \overset{?}{=} e \overset{?}{=} a(\underbrace{R_a^{-1}(e)}_{d})$

$d = R_a^{-1} e$

$R_a(d) = e$

$da = e$ (d is a left inverse of a)

Finally $d = de = d(ac) \overset{\text{associativity!}}{=} (da)c = ec = c$ $\quad \therefore\ d = c = a^{-1}$ (two sided inverse)

so $L_a^{-1}(e) = a^{-1}$, i.e. $a = b - \lambda e$ is invertible

so $\mathbb{C} \setminus \sigma_m^B(b) \subseteq \mathbb{C} \setminus \sigma^B(b)$ $\blacksquare$

by complexification

Proposition 2.2 A an (complex, WLOG) algebra without a unit $\Rightarrow$

unitization of A

(i) $\sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\} = \sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\}$

(ii) $\sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\} = \sigma^{\mathcal{L}(A_1)}(R_a) \setminus \{0\}$

(iii) $\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\}$

Note $\mathcal{L}(A)$ = all linear operators on A. It is an associative algebra with a unit, namely, the identity operator. on A

$\mathcal{L}(A)$ = all linear operators on A_1 (the unitization of A)

It is of course an associative algebra with unit, the identity operator on A_1

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proof.

$a \in A$, $\lambda \in \mathbb{C} \setminus \{0\}$ $\quad L_a - \lambda I : A \to A$ is injective

(1) $\iff L_a - \lambda I : A_1 \to A_1$ is injective

$\boxed{\Rightarrow}$

Suppos $(L_a - \lambda I)(b, \mu) = (0, 0)$ for some $(b, \mu) \in A_1$

need to prove $(b, \mu) = (0, 0)$

$(a, 0)(b, \mu) - \lambda(b, \mu)$

$(ab + \mu a, 0) - (\lambda b, \lambda \mu)$

$(ab + \mu a - \lambda b, \lambda \mu) \Rightarrow \mu = 0$ check

$ab - \lambda b = 0 \overset{?}{\Longrightarrow} b = 0$?

$(L_a - \lambda I)(b) = ab - \lambda b = 0$

but $L_a - \lambda I$ is injective on A so $b = 0$

(2) $L_a - \lambda I : A \to A$ is surjective $\Longleftrightarrow$

$L_a - \lambda I : A_1 \to A_1$ is surjective

$\boxed{\Rightarrow}$

let $(c, \mu) \in A_1$ NTP: $\exists\ (b, \sigma) \in A_1$

with $(L_a - \lambda I)(b, \sigma) = (c, \mu)$

i.e. $(a, 0)(b, \sigma) - (\lambda b, \lambda \sigma) \overset{?}{=} (c, \mu)$

$(ab + \sigma a, 0) - (\lambda b, \lambda \sigma) \overset{?}{=} (c, \mu)$

$((a - \lambda) b + \sigma a, -\lambda \sigma) \overset{?}{=} (c, \mu)$

$\therefore$ take $\sigma = -\mu/\lambda$

Need b such that

$(a - \lambda) b = c - \sigma a$

Since $L_a - \lambda I$ is surjective on $A \to A$, b exists

$\boxed{\Leftarrow}$

let $c \in A$ NTP $\exists\ b \in A$ with $(L_a - \lambda I)(b) = c$

i.e. $ab - \lambda b = c$

$(c, 0) \in A_1$ so $\exists\ (d, \mu) \in A_1$

$$(L_a - \lambda I)(d, \mu) = (c, 0)$$

$$(a, 0)(d, \mu) - (\lambda d, \lambda \mu) = (c, 0)$$

$$(ad + \mu a - \lambda d, \lambda \mu) = (c, 0)$$

$$\therefore \quad \mu = 0$$

$$ad - \lambda d = c$$

$$(L_a - \lambda I)(d) = c$$

Recall $\sigma^{\mathcal{L}(A)}(T) = \{\lambda \in \mathbb{C} : T - \lambda I : A \to A \text{ is not bijective}\}$

A_1 $\qquad$ A

(3) $\quad \sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\} = \sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}$ $\quad$ ((i) in statement)

$\lambda \in \sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\} \iff L_a - \lambda I$ is not bijective $A_1 \to A_1 \iff$ either not surjective or not injective $\iff L_a - \lambda I : A \to A$ is either not surjective or not injective $\iff \lambda \in \sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}$

(4) Similar argument shows

$$\sigma^{\mathcal{L}(A_1)}(R_a) \setminus \{0\} = \sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\}$$ ((ii) in statement)

(5) $$\sigma_m^{A_1}(a) = \sigma^{\mathcal{L}(A_1)}(L_a) \cup \sigma^{\mathcal{L}(A_1)}(R_a)$$

A_1

recall $\sigma_m^A(a) = \{\lambda \in \mathbb{C} :$ $a - \lambda e$ is not m-invertible

i.e. at least one of $L_a - \lambda I, L_b - \lambda I$ $: A_1 \to A_1$ is not bijective $\}$

$\therefore$ $\sigma_m^A(a) \subseteq \sigma^{\mathcal{L}(A_1)}(L_a) \cup \sigma^{\mathcal{L}(A_1)}(R_a)$.

if $\lambda \in \sigma^{\mathcal{L}(A_1)}(L_a)$ then $L_a - \lambda I$ $: A_1 \to A_1$ is not bijective

so $\sigma^{\mathcal{L}(A_1)}(L_a) \subseteq \sigma_m^A(a)$, same for $\sigma^{\mathcal{L}(A_1)}(R_a)$.

(6) $$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\}$$ ((iii) in statement)

$\sigma_m^A(a) \overset{\text{definition}}{:=} \sigma_m^{A_1}(a) = \sigma^{\mathcal{L}(A_1)}(L_a) \cup \sigma^{\mathcal{L}(A_1)}(R_a)$ by (5)

$L_a, R_a : A_1 \to A_1$ are not surjective

$(a, 0)(b, \mu) = (ab + \mu a, 0)$ i.e.

$L_a(A_1) \subseteq A$

so $L_a - 0 \cdot I$, $R_a - 0 \cdot I$ are not bijective $: A_1 \to A_1$

so $0 \in \sigma^{\mathcal{L}(A_1)}(L_a) \cap \sigma^{\mathcal{L}(A_1)}(R_a) = \sigma_m^A(a)$

proving $\supseteq$ in (6)

By (i) and (ii) $\sigma_m^A(a) = \sigma^{\mathcal{L}(A_1)}(L_a) \cup \sigma^{\mathcal{L}(A_1)}(R_a)$

$= (\sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\}) \cup (\sigma^{\mathcal{L}(A_1)}(R_a) \setminus \{0\}) \cup \{0\}$

$= (\sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}) \cup (\sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\}) \cup \{0\}$