source
The Jacobson radical of an evolution algebra (M. Victoria Velasco) 2018

§2 A evol. alg (arbitrary dimension)

$B = \{e_i : i \in \Lambda\}$ natural basis

$a = \sum \alpha_i e_i$ $\quad \Lambda_a^B = \{i \in \Lambda : \alpha_i \neq 0\}$ support of a

Proposition 2.2 (i) If $\dim A = \infty$, A has no unit (A = evolution algebra)

suppose e is a unit $\quad e = \sum \alpha_i e_i$ (finite sum!)

So take $i \notin \Lambda_e \therefore e e_i = 0$, but $e e_i = e_i$ supposedly $\Rightarrow\Leftarrow$

(ii) If $\dim A < \infty$ then A has a unit $\Longleftrightarrow$

$\forall$ natural basis $B = \{e_1, .., e_n\}$ $\quad e_i^2 = w_{ii} e_i \quad w_{ii} \neq 0 \quad i = 1, .., n$,

in which case $e = \frac{1}{w_{11}} e_1 + \cdots + \frac{1}{w_{nn}} e_n$ is a unit.

suppose e is a unit & $\Lambda = \{1, .., n\}$ (dimension n)

$e = \sum_{i \in \Lambda} \alpha_i e_i$ and $\alpha_i \neq 0 \ \forall i$ (if $\alpha_1 = 0$, then $e e_1$

(don't need parentheses here) $= \left(\sum_{i=2}^{n} \alpha_i e_i\right) e_1 = 0 \Rightarrow\Leftarrow$)

$\underbrace{e_i^2}$ $= e_i(e_i e) = e_i(\alpha_i e_i^2) = \alpha_i e_i \left(\sum_j w_{ji} e_j\right) = \underbrace{\alpha_i w_{ii} e_i^2}$

So either $e_i^2 = 0$ or $\alpha_i w_{ii} = 1$

$\Downarrow$ $\quad\quad \Downarrow$

$e e_i = \alpha_i e_i^2 = 0$ NO $\neq 0$ $\Rightarrow\Leftarrow$

$e = \sum \alpha_i e_i = \sum \frac{1}{w_{ii}} e_i \Rightarrow$

This proves $\boxed{\Rightarrow}$

$e_i = e_i e = \frac{1}{w_{ii}} e_i^2 = \frac{1}{w_{ii}} \left(\sum_j w_{ji} e_j\right)$

$= \sum_{j \neq i} \frac{w_{ji}}{w_{ii}} + e_i$

$\therefore w_{ji} = 0 \quad i \neq j$

Conversely suppose for every natural basis $\{e_1, \dots, e_n\} = B$
$e_i^2 = w_{ii} e_i$ with $w_{ii} \neq 0$ $\quad i = 1, \dots, n$ $\quad$ (w_{ii} depends on the basis)

Then $M_B(A) = \begin{bmatrix} w_{11} & & & \\ & w_{22} & & 0 \\ & 0 & \ddots & \\ & & & w_{nn} \end{bmatrix}$ and if $e = \sum_i \frac{1}{w_{ii}} e_i$

then $e e_j = \frac{1}{w_{jj}} e_j^2 = \frac{1}{w_{jj}} w_{jj} e_j = e_j$

so e is a unit.

$ea = e(\sum \alpha_j e_j) = \sum \alpha_j e e_j = \sum \alpha_j e_j = a$

---

unitization of an algebra A

If A is any algebra (unit or not) let $A_1 := A \oplus \mathbb{K}$ or $A \times \mathbb{K}$ $\quad (\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$

define
- $\alpha(a, \lambda) = (\alpha a, \alpha\lambda)$ (scalar multiplication) $\quad \alpha \in \mathbb{K}$, $(a, \lambda) \in A \times \mathbb{K}$
- $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$ (addition)
- $(a, \lambda)(b, \mu) = (ab, \lambda b + \mu a + \lambda\mu)$ (multiplication)

Then A_1 is an algebra, with unit $(0, 1)$ $\quad (0,1)(b,\mu) = (b, \mu)$

A is a maximal ideal in A_1

$(b, \mu)(0,1) = (b, \mu)$

complexification of a real algebra A

If A is a real algebra $(\mathbb{K} = \mathbb{R})$ let $A_{\mathbb{C}} = A \oplus A$ or $A + iA$

define
- $\underbrace{(\alpha + i\beta)}_{\in \mathbb{C}}(a, b) = (\alpha a - \beta b, \beta a + \alpha b)$ (scalar multiplication)
- $(a, b) + (a', b') = (a + a', b + b')$ (addition)
- $(a, b)(a', b') = (aa' - bb', ba' + ab')$ (multiplication)

$(a + ib)(a' + ib') = aa' - bb' + i(ba' + ab')$

Then $A_{\mathbb{C}}$ is a complex algebra $(\mathbb{K} = \mathbb{C})$, in particular a real algebra (by restricting scalars to $\mathbb{R}$)

A is a real subalgebra of $A_{\mathbb{C}}$. $\blacksquare$

Source

The multiplicative spectrum and the uniqueness of the complete normed topology. Marcos/Velasco 2014

For an associative algebra A with unit e, an element a is invertible if $\exists\, \bar{a}^{1} \in A$ with $a\bar{a}^{1} = \bar{a}^{1}a = e$

If A is a complex algebra ($\mathbb{K} = \mathbb{C}$) with unit

$$\sigma^{A}(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is NOT invertible}\}$$

is the spectrum of $a \in A$.

(not relevant but interesting)

Remark In an associative algebra B with unit

$$\sigma^{B}(b) = \underbrace{\sigma_{m}^{B}(b)}_{\text{multiplicative spectrum}} \quad \forall\, b \in B.$$

see p.5 the other file for today

Let $\lambda \notin \sigma^{B}(b)$ so $a := b - \lambda e$ is invertible if

Note: if a is invertible then $L_a^{-1} = L_{\bar{a}^{1}}$, $R_a^{-1} = R_{\bar{a}^{1}}$

$$\left.\begin{aligned} &(L_{\bar{a}^{1}} L_a x = \bar{a}^{1}(ax) = (\bar{a}^{1}a)x = x \\ &L_a L_{\bar{a}^{1}} x = a(\bar{a}^{1}x) = (a\bar{a}^{1})x = x \end{aligned}\right\} \therefore L_a^{-1} = L_{\bar{a}^{1}}$$

Then L_a, R_a are invertible (i.e. bijective)

so $\lambda \notin \sigma_{m}^{B}(b)$ i.e. $\mathbb{C} \setminus \sigma^{B}(b) \subseteq \mathbb{C} \setminus \sigma_{m}^{B}(b)$

or $\sigma_{m}^{B}(b) \subseteq \sigma^{B}(b)$.

Suppose $\lambda \notin \sigma_m^B(b)$ Then $a = b - \lambda e$ is m-invertible

so L_a and R_a are invertible (= bijective)

Look at: $L_a^{-1}(e)$ and $R_a^{-1}(e)$ $\boxed{\text{claim: Both } = a^{-1}}$

i.e. $\Big(\underbrace{L_a^{-1}(e)}_{\overset{\shortparallel}{c}}\Big)\, a \overset{?}{=} e \overset{?}{=} a.\Big(\underbrace{L_a^{-1} e}_{c}\Big)$

$c = L_a^{-1} e$ $L_a c = L_a(L_a^{-1}(e)) = e$

$L_a c = e$

$ac = e$ (c is a right inverse of a)

$R_a(R_a^{-1}(e)) = \Big(\underbrace{R_a^{-1}(e)}_{d}\Big)\, a \overset{?}{=} e \overset{?}{=} a\Big(\underbrace{R_a^{-1}(e)}_{d}\Big)$

$d = R_a^{-1} e$

$R_a(d) = e$

$da = e$ (d is a left inverse of a)

associativity!

Finally $d = de = d(ac) = (da)c = ec = c$ $\therefore\ d = c = a^{-1}$ (two sided inverse)

so $L_a^{-1}(e) = a^{-1}$, i.e. $\overset{b-\lambda e}{a}$ is invertible

so $\mathbb{C} \setminus \sigma_m^B(b) \subseteq \mathbb{C} \setminus \sigma^B(b)$

by complexification

(complex, WLOG)

Proposition 2.2 $\quad A$ an algebra without a unit $\Longrightarrow$

unitization of A

(i) $\quad \sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\} = \sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\}$

(ii) $\quad \sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\} = \sigma^{\mathcal{L}(A_1)}(R_a) \setminus \{0\}$

(iii) $\quad \boxed{\sigma^{A}_{m}(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\}}$

Note $\quad \mathcal{L}(A)$ = all linear operators on A. It is an associative algebra with a unit, namely, the identity operator on A

misprint

$\mathcal{L}(A_1)$ = all linear operators on A_1 (the unitization of A)

It is of course an associative algebra with unit, the identity operator on A_1

---

proof.

$a \in A$, $\lambda \in \mathbb{C} \setminus \{0\}$ $\quad L_a - \lambda I : A \to A$ is injective

(1) $\quad \Longleftrightarrow \quad L_{(a,0)} - \lambda I_1 : A_1 \to A_1$ is injective

$\boxed{\Longrightarrow}$

Suppose $(L_{(a,0)} - \lambda I_1)(b,\mu) = (0,0)$ for some $(b,\mu) \in A_1$

need to prove $(b,\mu) = (0,0)$

$(a,0)(b,\mu) - \lambda(b,\mu)$

$(ab + \mu a, 0) - (\lambda b, \lambda\mu)$

$(ab + \mu a - \lambda b, \lambda\mu) \Longrightarrow \mu = 0$ check since $\lambda \neq 0$

$ab - \lambda b = 0 \overset{?}{\Longrightarrow} b = 0$?

$$(L_a - \lambda I)(b) = \underbrace{ab - \lambda b}^{=(L_a - \lambda I)(b)} = 0$$

but $L_a - \lambda I$ is injective on A so $b = 0$

$\boxed{\Leftarrow}$ see page 6'

(recall $\lambda \neq 0$ is assumed)

(2) $L_a - \lambda I : A \to A$ is surjective $\Longleftrightarrow$ this is the same as (1') if A is finite dimensional (note $\dim A_1 = 1 + \dim A$)

$L_{(a,0)} - \lambda I_1 : A_1 \to A_1$ is surjective

$\boxed{\Rightarrow}$

let $(c, \mu) \in A_1$ NTP: $\exists\ (b, \sigma) \in A_1$

with $(L_{(a,0)} - \lambda I_1)(b, \sigma) = (c, \mu)$

i.e.

$$(a,0)(b,\sigma) - (\lambda b, \lambda\sigma) \stackrel{?}{=} (c, \mu)$$

$$(ab + \sigma a, 0) - (\lambda b, \lambda \sigma) \stackrel{?}{=} (c, \mu)$$

$$\left((a - \lambda) b + \sigma a, -\lambda\sigma \right) \stackrel{?}{=} (c, \mu)$$

$\therefore$ ~~take~~ $\sigma = -\mu/\lambda \Rightarrow -\lambda\sigma = \mu$ OK since $\lambda \neq 0$

Need b such that

$$(a - \lambda) b = c - \sigma a$$

Since $L_a - \lambda I$ is surjective on $A \to A$, b exists

$\boxed{\Leftarrow}$

let $c \in A$ NTP $\exists\ b \in A$ with $(L_a - \lambda I)(b) = c$

i.e. $ab - \lambda b = c$

Take $(c, 0) \in A_1$ so $\exists\ (d, \mu) \in A_1$ go to p. 7

$\boxed{\Leftarrow}$ Suppose $(L_a - \lambda I)(b) = 0$

i.e. $ab - \lambda b = 0$ NTP $b = 0$

Look at $(L_{(a,0)} - \lambda I_1)(b,0)$

$= (a,0)(b,0) - (\lambda b, 0)$

$= (ab, 0) - (\lambda b, 0)$

$= (ab - \lambda b, 0)$

$= 0$

since $L_{(a,0)} - \lambda I_1 : A_1 \to A_1$ is injective

$(b,0) = 0$ so $b = 0$

Remark We are assuming in Prop 2.2 that A has no unit. Even if A has a unit, we can still consider A_1

call it e

and $(0,1)$ is the unit of A_1.

And $(e,0)(a,\lambda) = (ea + \lambda e, 0) = (a + \lambda e, 0)$

$\neq (a,\lambda)$

so $(e,0)$ is not a unit of A_1

$$(L_{(a,0)} - \lambda I_1)(d,\mu) = (c,0)$$

$$(a,0)(d,\mu) - (\lambda d, \lambda \mu) = (c,0)$$

$$(ad + \mu a - \lambda d, \lambda \mu) = (c,0)$$

since $\lambda \neq 0$

$$\therefore \quad \mu = 0$$

$$ad - \lambda d = c$$

$$(L_a - \lambda I)(d) = c$$

so $L_a - \lambda I$ is surjective

Recall $\sigma^{\mathcal{L}(A)}(T) = \{\lambda \in \mathbb{C} : T - \lambda I : A \to A \text{ is not bijective}\}$

(3) $\quad \sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\} = \sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}$ $\quad$ ((i) in statement of Prop 2.2)

$\lambda \in \sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\} \iff$ $\lambda \neq 0$ and $L_{(a,0)} - \lambda I_1$ is not bijective $A_1 \to A_1 \iff$ either not surjective or not injective $\iff$ $\lambda \neq 0$ and $L_a - \lambda I : A \to A$ is either not surjective or not injective (so not bijective) $\iff \lambda \in \sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}$

(4) similar argument shows

$$\sigma^{\mathcal{L}(A_1)}(R_a) \setminus \{0\} = \sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\} \quad \text{((ii) in statement)}$$

If A is commutative, $L_a = R_a$ so no proof required

(otherwise the proof is essentially the same as for L_a)

(5) $a \in A$

$$\sigma_m^{A_1}(a) \overset{\mathcal{L}A_1}{=} \left(\sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \cup \sigma^{\mathcal{L}(A_1)}(R_{(a,0)})\right)$$

recall $\sigma_m^{A_1}(a) = \{\lambda \in \mathbb{C} : \cancel{a-\lambda e}$ $L_{(a,0)} - \lambda I_1 \in \mathcal{L}(A_1)$ is not m-invertible

i.e. at least one of $L_{(a,0)} - \lambda I_1$, $\cancel{L_b - \lambda I}$ $R_{(a,0)} - \lambda I_1$ $: A_1 \to A_1$ is not bijective $\}$

recall: x is m-invertible iff L_x and R_x are both bijective

$\therefore$ $\sigma_m^{A_1}(a) \setminus \{0\} \subseteq \sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \cup \sigma^{\mathcal{L}(A_1)}(R_{(a,0)})$.

so $\sigma_m^{A_1}(a) \subseteq \sigma^{\mathcal{L}(A_1)}(L_a) \cup \sigma^{\mathcal{L}(A_1)}(R_a) \cup \{0\}$

conversely

if $\lambda \in \sigma^{\mathcal{L}(A_1)}(L_{(a,0)})$ then $L_{(a,0)} - \lambda I_1 : A_1 \to A_1$ is not bijective

so $\sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \subseteq \sigma_m^{A_1}(a)$, same for $\sigma^{\mathcal{L}(A_1)}(R_{(a,0)})$.

$\therefore$ $\sigma_m^{A_1}(a) = \sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \cup \sigma^{\mathcal{L}(A_1)}(R_{(a,0)})$

(6) $$\sigma_m^{A}(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \sigma^{\mathcal{L}(A)}(R_a) \cup \{0\}$$ ((iii) in statement)

$$\sigma_m^{A}(a) \overset{\text{definition}}{:=} \sigma_m^{A_1}(a) = \sigma^{\mathcal{L}(A_1)}(L_a) \cup \sigma^{\mathcal{L}(A_1)}(R_a)$$

$L_{(a,0)}, R_{(a,0)} : A_1 \to A_1$ are not surjective since

$$(a,0)(b,\mu) = (ab + \mu a, 0)$$

i.e. $L_{(a,0)}(A_1) \subseteq A \subsetneq A_1$

so $L_{(a,0)} - 0 \cdot I_1$, $R_{(a,0)} - 0 \cdot I_1$ are not bijective: $A_1 \to A_1$

so $0 \in \sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \cap \sigma^{\mathcal{L}(A_1)}(R_{(a,0)}) = \sigma_m^{A}(a)$

Recall

(i) $\sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \setminus \{0\} = \sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}$

(ii) $\sigma^{\mathcal{L}(A_1)}(R_{(a,0)}) \setminus \{0\} = \sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\}$

By (i) and (ii) $\sigma_m^{A}(a) = \sigma^{\mathcal{L}(A_1)}(L_{(a,0)}) \cup \sigma^{\mathcal{L}(A_1)}(R_{(a,0)})$ by (5)

$$= \left(\sigma^{\mathcal{L}(A_1)}(L_a) \setminus \{0\}\right) \cup \left(\sigma^{\mathcal{L}(A_1)}(R_a) \setminus \{0\}\right) \cup \{0\}$$

$$= \left(\sigma^{\mathcal{L}(A)}(L_a) \setminus \{0\}\right) \cup \left(\sigma^{\mathcal{L}(A)}(R_a) \setminus \{0\}\right) \cup \{0\}$$