

Theorem 6 $\theta: A \rightarrow \tilde{A}$ homomorphism of algebras

5-18-21

$$\theta(\alpha a) = \alpha \theta(a)$$

(i) if θ is bijective (i.e. an isomorphism) $\theta(a+b) = \theta(a) + \theta(b)$

$$\theta(ab) = \theta(a)\theta(b)$$

then $\sigma_m^{\tilde{A}}(\theta(a)) = \sigma_m^A(a)$

(ii) if θ is surjective (i.e. epimorphism)

then $\sigma_m^{\tilde{A}}(\theta(a)) \subseteq \sigma_m^A(a)$

and A is finite dimensional and commutative

this is proved of $IK = \mathbb{C}$ It holds also of $IK = \mathbb{R}$
by complexification. (Exercise) or if A is infinite dimensional
or not commutative

(i) **Case 1** A is commutative (e.g. an evolution algebra)

By Prop 2.2 (on p. 5) of a previous file)

$$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \{0\} \quad (A \text{ has no unit})$$

$$\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a) \quad (A \text{ has a unit})$$

$$\lambda \in \sigma^{\mathcal{L}(A)}(L_a) \iff L_a(b) = \lambda b \quad \text{i.e. } ab = \lambda b \quad \text{for some } b \neq 0$$

$$\iff \theta(a)\theta(b) = \lambda \theta(b) \iff \lambda \in \sigma_{\theta(a)}^{\mathcal{L}(\tilde{A})}$$

$$\iff \lambda \in \sigma_m^{\tilde{A}}(\theta(a)) \quad (A \text{ has a unit} \iff \tilde{A} \text{ has a unit})$$

Case 2 A not commutative (omitted)

(ii) see the next page

Lemma

If M is an ideal then

$$\pi: A \rightarrow A/M$$

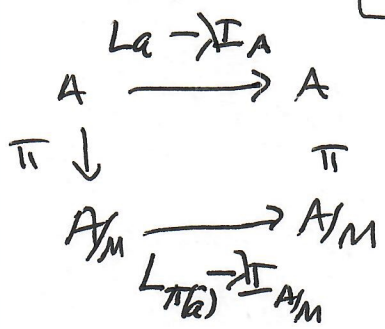
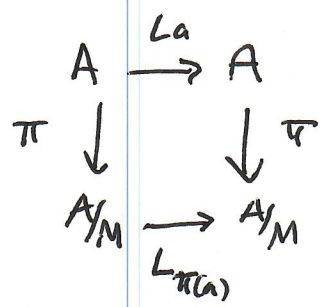
$$a \rightarrow a+M$$

$$\sigma_m^{A/M}(L_{\pi(a)}) \subseteq \sigma_m^A(La)$$

$$\text{and } \sigma_m^{A/M}(\pi(a)) \subseteq \sigma_m^{A/M}(L_{\pi(a)} \cup \{0\})$$

$$\subseteq \sigma_m^{A/M}(L_{\pi(a)}) \cup \{0\} = \sigma_m^A(a)$$

proof.



$$\pi \circ (L_a - \lambda I_A)$$

$$= (L_{\pi(a)} - \lambda I_{A/M}) \circ \pi$$

NTP

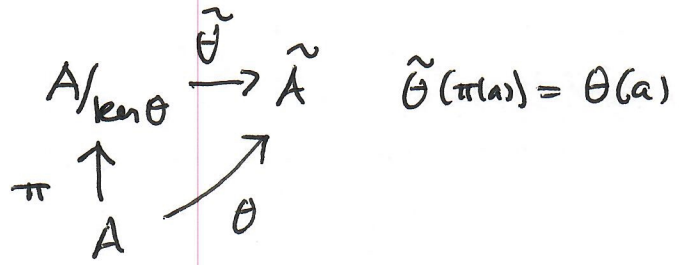
If $L_a - \lambda I_A$ invertible on A then $L_{\pi(a)} - \lambda I_{A/M}$ is invertible on A/M

Let $\pi(b) \in A/M$ NTP $\exists \pi(c) \in A/M$ $(L_{\pi(a)} - \lambda I_{A/M})\pi(c) = \pi(b)$

Since $L_a - \lambda I_A$ is invertible $\exists c \in A$ $(L_a - \lambda I_A)(c) = b$

$$\left. \begin{array}{l} \pi(L_a - \lambda I_A)(c) = (L_{\pi(a)} - \lambda I_{A/M})\pi(c) \\ \text{and } \pi(b) = \pi((L_a - \lambda I_A)(c)) = (L_{\pi(a)} - \lambda I_{A/M})\pi(c) \end{array} \right\} \Rightarrow \text{proving surjectivity of } L_{\pi(a)} - \lambda I_{A/M}$$

proof of (ii)



$\tilde{\theta}$ is an isomorphism

$$\text{so } \sigma_m^{\tilde{A}}(\theta(a)) = \sigma_m^{\tilde{A}}(\tilde{\theta}(\pi(a))) = \sigma_m^{A/\ker \theta}(\pi(a)) \subseteq \sigma_m^A(a)$$

The Jacobson radical of an evolution algebra

Proposition 5.1 A is a non-zero trivial evolution algebra

$B = \{e_i : i \in \Lambda\}$ a natural basis $M_B(A) = (w_{ij})$
 structure matrix. $a \in A$ $a = \sum \alpha_i e_i$ $\Lambda_a = \{i : \alpha_i \neq 0\}$

Then $\sigma_m^A(a) = \{d_i w_{ii} : i \in \Lambda\}$
 and $\sigma_m^A(a) \setminus \{0\} = \{d_i w_{ii} : i \in \Lambda_a\}$

(needed for Proposition 5.3 shortly)

Remark: A finite dim'd evolution algebra $B = \{e_1, \dots, e_n\}$

natural basis $M_B(A) = [w_{ij}]$ structure matrix

$(e_j^2 = \sum_{i=1}^n w_{ij} e_i)$ $a = \sum \alpha_i e_i$ $b = \sum \beta_j e_j$
 \Rightarrow $= \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ $= \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$

$ab = (\sum_i \alpha_i e_i)(\sum_j \beta_j e_j) = \sum_{i,j} \alpha_i \beta_j e_i e_j = \sum_j \alpha_j \beta_j e_j^2$
 $= \sum_j \alpha_j \beta_j (\sum_k w_{kj} e_k) = \sum_k (\sum_j \alpha_j \beta_j w_{kj}) e_k = \begin{bmatrix} \sum_j \alpha_j \beta_j w_{1j} \\ \vdots \\ \sum_j \alpha_j \beta_j w_{nj} \end{bmatrix}$

Look at

$\begin{bmatrix} w_{11} & \dots & w_{1n} \\ w_{21} & \dots & w_{2n} \\ \vdots & & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \vdots \\ \alpha_n \beta_n \end{bmatrix} = \begin{bmatrix} w_{11} \alpha_1 \beta_1 + w_{12} \alpha_2 \beta_2 + \dots + w_{1n} \alpha_n \beta_n \\ w_{21} \alpha_1 \beta_1 + w_{22} \alpha_2 \beta_2 + \dots + w_{2n} \alpha_n \beta_n \\ \dots \\ w_{n1} \alpha_1 \beta_1 + w_{n2} \alpha_2 \beta_2 + \dots + w_{nn} \alpha_n \beta_n \end{bmatrix} \therefore = ab$

$\begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \vdots \\ \alpha_n \beta_n \end{bmatrix}$

$ab = \begin{bmatrix} w_{11} & \dots & w_{1n} \\ w_{21} & \dots & w_{2n} \\ \vdots & & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$

proof of Proposition 5.1

Recall that since A has a unit, $\sigma_n^A(a) = \rho(A) = \sigma(La)$

Look at the matrix M of La with respect to e_1, \dots, e_n

$$La e_j = \sum_i d_i e_i e_j = d_j e_j^2 = d_j \sum_k w_{kj} e_k$$

$$M = \begin{bmatrix} d_1 w_{11} & d_2 w_{12} & \dots & d_n w_{1n} \\ d_1 w_{21} & d_2 w_{22} & & d_n w_{2n} \\ \vdots & & & \vdots \\ d_1 w_{n1} & d_2 w_{n2} & & d_n w_{nn} \end{bmatrix}$$

Since A is trivial non-zero $w_{ij} = 0 \quad i \neq j$

$$M = \begin{bmatrix} d_1 w_{11} & & & \\ & d_2 w_{22} & & \\ & & \circ & \\ & & & \ddots \\ \circ & & & & d_n w_{nn} \end{bmatrix}$$

$$\det(M - \lambda I) = 0 \iff d_i w_{ii} - \lambda = 0 \text{ for some } i$$



Proposition 5.3 If A is a finite dim'd non-trivial evolution algebra, $B = \{e_1, \dots, e_n\}$ natural basis

$\lambda \in \mathbb{C}$ $a = \sum \alpha_i e_i$ then

$\lambda \in \sigma_m^A(a) \iff \lambda = 0$ or λ is an eigenvalue of

$$\begin{bmatrix} w_{11} & \dots & w_{1n} \\ w_{21} & \dots & w_{2n} \\ \vdots & & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Proof. Since A does not have a unit, we know that

$$\sigma_m^A(a) = \{0\} \cup \sigma^{L(A)}(L_a)$$

look at $(L_a - \lambda I)(b) = ab - \lambda b$ ($b = \sum \beta_i e_i$)

$$= \begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} - \begin{bmatrix} \lambda \beta_1 \\ \vdots \\ \lambda \beta_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} - \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \ddots \\ 0 & & & \lambda \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \right)$$

Then $\lambda \in \sigma^{L(A)}(L_a) \iff (L_a - \lambda I)b = 0$ for some $b = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\iff \lambda$ is an eigenvalue of

$$\begin{bmatrix} w_{11} & \dots & w_{1n} \\ \vdots & & \vdots \\ w_{n1} & \dots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \square$$