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Theorem 6 $\quad \theta : A \longrightarrow \tilde{A}$ homomorphism of algebras $\quad$ 5-18-21

$\theta(\alpha a) = \alpha\, \theta(a)$

$\theta(a+b) = \theta(a) + \theta(b)$

$\theta(ab) = \theta(a)\theta(b)$

(i) if θ is bijective (i.e. an isomorphism)

then $\sigma_m^{\tilde{A}}(\theta(a)) = \sigma_m^{A}(a)$

(ii) if θ is surjective (i.e. epimorphism)

then $\sigma_m^{\tilde{A}}(\theta(a)) \subseteq \sigma_m^{A}(a)$

this is proved if $\mathbb{K} = \mathbb{C}$ and A is finite dimensional and commutative. It holds also if $\mathbb{K} = \mathbb{R}$ by complexification. (Exercise) or if A is infinite dimensional or not commutative)

(i) $\boxed{\text{case 1}}$ A is commutative (e.g. an evolution algebra)

By Prop 2.2 (on p.⑤ of a previous file)

$\sigma_m^{A}(a) = \sigma^{\mathcal{L}(A)}(L_a) \cup \{0\}$ (if A has no unit)

$\sigma_m^{A}(a) = \sigma^{\mathcal{L}(A)}(L_a)$ (if A has a unit)

$\lambda \in \sigma^{\mathcal{L}(A)}(L_a) \iff L_a(b) = \lambda b$ i.e. $ab = \lambda b$ for some $b \neq 0$

$\iff \theta(a)\theta(b) = \lambda\theta(b) \iff \lambda \in \sigma^{\mathcal{L}(\tilde{A})}_{\theta(a)}$

$\iff \lambda \in \sigma_m^{\tilde{A}}(\theta(a))$ (A has a unit $\iff$ $\tilde{A}$ has a unit)

$\boxed{\text{Case 2}}$ A not commutative (omitted)

(ii) see the next page

Lemma

if M is an ideal then $\sigma^{\mathcal{L}(A/M)}(L_{\pi(a)}) \subseteq \sigma^{\mathcal{L}(A)}(L_a)$

$\pi : A \to A/M$

$a \to a+M$

$\boxed{\text{and } \sigma_m^{A/M}(\pi(a)) = \sigma^{\mathcal{L}(A/M)}(L_{\pi(a)}) \cup \{0\} \subseteq \sigma^{\mathcal{L}(A)}(L_a) \cup \{0\} = \sigma_m^{A}(a)}$

proof.

$$\begin{tikzcd} A \arrow[r, "L_a"] \arrow[d, "\pi"'] & A \arrow[d, "\pi"] \\ A/M \arrow[r, "L_{\pi(a)}"'] & A/M \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "L_a - \lambda I_A"] \arrow[d, "\pi"'] & A \arrow[d, "\pi"] \\ A/M \arrow[r, "L_{\pi(a)} - \lambda I_{A/M}"'] & A/M \end{tikzcd}$$

$\pi \circ (L_a - \lambda I_A)$

$= (L_{\pi(a)} - \lambda I_{A/M}) \circ \pi$

NTP

if $L_a - \lambda I_A$ invertible on A then $L_{\pi(a)} - \lambda I_{A/M}$ is invertible on A/M

let $\pi(b) \in A/M$ NTP $\exists\, \pi(c) \in A/M$ $(L_{\pi(a)} - \lambda I_{A/M})\pi(c) = \pi(b)$

since $L_a - \lambda I_A$ is invertible $\exists\, c \in A$ $(L_a - \lambda I_A)(c) = b$

$\underbrace{\pi\,(L_a - \lambda I_A)}(c) = \underbrace{(L_{\pi(a)} - \lambda I_{A/M})}\pi(c)$

and $\pi(b) = \underbrace{\pi((L_a - \lambda I_A)}(c)) = \underbrace{(L_{\pi(a)} - \lambda I_{A/M})\pi(c)}$

$\Big\}\Rightarrow$ proving surjectivity of $L_{\pi(a)} - \lambda I_{A/M}$ ∎

proof of (ii)

$$\begin{tikzcd} A/\ker\theta \arrow[r, "\tilde{\theta}"] & \tilde{A} \\ A \arrow[u, "\pi"] \arrow[ur, "\theta"'] & \end{tikzcd}$$

$\tilde{\theta}(\pi(a)) = \theta(a)$

$\tilde{\theta}$ is an isomorphism

so $\sigma_m^{\tilde{A}}(\theta(a)) = \sigma_m^{\tilde{A}}(\tilde{\theta}\pi(a)) = \sigma_m^{A/\ker\theta}(\pi(a)) \subseteq \sigma_m^{A}(a)$ ∎

The Jacobson radical of an evolution algebra

<u>Proposition 5.1</u> A is a non-zero trivial evolution algebra

$B = \{e_i : i \in \Lambda\}$ a natural basis $M_B(A) = (\omega_{ij})$

structure matrix. $a \in A$ $a = \sum \alpha_i e_i$ $\Lambda_a = \{i : \alpha_i \neq 0\}$

Then $\sigma_m^A(a) = \{\alpha_i w_{ii} : i \in \Lambda\}$

and $\sigma_m^A(a) \setminus \{0\} = \{\alpha_i w_{ii} : i \in \Lambda_a\}$

(needed for Proposition 5.3 shortly)

<u>Remark</u>: A finite dim'l evolution algebra $B = \{e_1, .., e_n\}$

natural basis $M_B(A) = [w_{ij}]$ structure matrix

$$\left(e_j^2 = \sum_{i=1}^n w_{ij} e_i \right) \qquad a = \sum \alpha_i e_i = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad b = \sum \beta_j e_j = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$\Rightarrow$

$$ab = \Big(\sum_i \alpha_i e_i\Big)\Big(\sum_j \beta_j e_j\Big) = \sum_{i,j} \alpha_i \beta_j e_i e_j = \sum_j \alpha_j \beta_j e_j^2$$

$$= \sum_j \alpha_j \beta_j \Big(\sum_k w_{kj} e_k\Big) = \sum_k \Big(\sum_j \alpha_j \beta_j w_{kj}\Big) e_k = \begin{bmatrix} \sum_j \alpha_j \beta_j w_{1j} \\ \vdots \\ \sum_j \alpha_j \beta_j w_{nj} \end{bmatrix}$$

Look at

$$\begin{bmatrix} w_{11} & \cdots & w_{1n} \\ w_{21} & \cdots & w_{2n} \\ & \cdots & \\ w_{n1} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \vdots \\ \alpha_n \beta_n \end{bmatrix} = \begin{bmatrix} w_{11}\alpha_1\beta_1 + w_{12}\alpha_2\beta_2 + \cdots + w_{1n}\alpha_n\beta_n \\ w_{21}\alpha_1\beta_1 + w_{22}\alpha_2\beta_2 + \cdots + w_{2n}\alpha_n\beta_n \\ \cdots \\ w_{n1}\alpha_1\beta_1 + w_{n2}\alpha_2\beta_2 + \cdots + w_{nn}\alpha_n\beta_n \end{bmatrix} \therefore = ab$$

$$\begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ & & & \\ 0 & \cdots & & \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \beta_1 \\ \alpha_2 \beta_2 \\ \vdots \\ \alpha_n \beta_n \end{bmatrix}$$

$$\boxed{\therefore\ ab = \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ w_{21} & \cdots & w_{2n} \\ \vdots & & \\ w_{n1} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ & & \vdots & \\ 0 & \cdots & & \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}}$$

proof of Proposition 5.1

Recall that since A has a unit, $\sigma_m^A(a) = \sigma^{\mathcal{L}(A)}(L_a)$

Look at the matrix M of L_a with respect to $e_1, \dots, e_n$

$$L_a e_j = \sum_i \alpha_i e_i e_j = \alpha_j e_j^2 = \alpha_j \sum_k w_{kj} e_k$$

$$M = \begin{bmatrix} \alpha_1 w_{11} & \alpha_2 w_{12} & \cdots & \alpha_n w_{1n} \\ \alpha_1 w_{21} & \alpha_2 w_{22} & & \alpha_n w_{2n} \\ \vdots & & & \vdots \\ \alpha_1 w_{n1} & \alpha_2 w_{n2} & & \alpha_n w_{nn} \end{bmatrix}$$

Since A is trivial non-zero $w_{ij} = 0$ $i \neq j$

$$M = \begin{bmatrix} \alpha_1 w_{11} & & 0 \\ & \alpha_2 w_{22} & \\ 0 & & \ddots & \\ & & & \alpha_n w_{nn} \end{bmatrix}$$

$\det(M - \lambda I) = 0 \iff \alpha_i w_{ii} - \lambda = 0$ for some i

$\blacksquare$

<u>Proposition 5.3</u> If A is a finite dim'l non-trivial evolution algebra, $B = \{e_1, .., e_n\}$ natural bases $\lambda \in \mathbb{C}$ $a = \sum \alpha_i e_i$ then

$$\lambda \in \sigma_m^A(a) \iff \lambda = 0 \text{ or } \lambda \text{ is an eigenvalue of}$$

$$\begin{bmatrix} \omega_{11} & \cdots & w_{1n} \\ w_{21} & \cdots & w_{2n} \\ \vdots & & \\ w_{n1} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix}$$

Proof. Since A does not have a unit, we know that

$$\sigma_m^A(a) = \{0\} \cup \sigma^{\mathcal{L}(A)}(L_a)$$

Look at $(L_a - \lambda I)(b) = ab - \lambda b \qquad (b = \sum \beta_i \alpha_i)$

$$= \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ & & \\ w_{n1} & \cdots & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} - \begin{bmatrix} \lambda\beta_1 \\ \vdots \\ \lambda\beta_n \end{bmatrix}$$

$$= \left(\begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \\ w_{n1} & & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix} - \begin{bmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \lambda & \\ 0 & & & \ddots \lambda \end{bmatrix} \right) \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

Then $\lambda \in \sigma^{\mathcal{L}(A)}(L_a) \iff (L_a - \lambda I) b = 0$ for some $b = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \neq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\iff \lambda$ is an eigenvalue of

$$\begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \\ w_{n1} & & w_{nn} \end{bmatrix} \begin{bmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_n \end{bmatrix}$$

∎