By Lemma 1, $\|J^t\|$ is bounded, proving the first statement. Conversely, if $\|J^t\|$ is bounded, then by Lemma 1 again $\|A^t\|$ is bounded and by Theorem 3, D is pulse stable under all autonomous pulse processes. ▨

A necessary and sufficient condition for pulse stability

This is Theorem 5 in Roberts and Theorem 3 in "Pulse Processes..."

Theorem 5 If D is a weighted digraph and J is the canonical Jordan form of its adjacency matrix A, then the following are equivalent.

(a) D is pulse stable under all autonomous pulse processes

(b) D is pulse stable under all simple pulse processes.

(c) every eigenvalue of D has absolute value ≤ 1 and if $B_j(\lambda) \neq [\lambda]$, then $|\lambda| < 1$.

Next:

- Proof of Theorem 5
- Necessary and sufficient condition for value stability (Theorem 6)

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Theorems 2 : In a simple pulse process starting at x_i:

$$p_j(t) = \text{the } (i,j)\text{-entry of } A^t$$

$$v_j(t) = v_j(\text{start}) + (i,j)\text{-entry of } I + A + A^2 + \cdots + A^t$$

Theorem 3 : $P(t) = P(0)A^t$ (autonomous pulse process)

This is Theorem 6 in Roberts and Theorem 4 in "Pulse processes . . . "

Theorem 6 D is a weighted digraph. The following are equivalent:

(a) D is value stable under all autonomous pulse processes

(b) D is value stable under all simple pulse processes

(C) D is pulse stable under all simple pulse processes and $\lambda = 1$ is not an eigenvalue of D

a ⇒ b trivial

c ⇒ a

Lemma 1 $\|A^t\|$ is bdd $\Leftrightarrow$ $\|J^t\|$ is bounded (A any matrix

Lemma 2
- D pulse stable for simple pulse processes $\Rightarrow$ $\|J^t\|$ bounded
- $\|J^t\|$ bounded $\Rightarrow$ D is pulse stable for all autonomous pulse processes

Lemma 1′ $\left\|\sum_{t=0}^{N} A^t\right\|$ bounded $\Leftrightarrow$ $\left\|\sum_{t=0}^{N} J^t\right\|$ is bounded
$N=1,2,\ldots$ $\qquad N=1,2,\ldots$

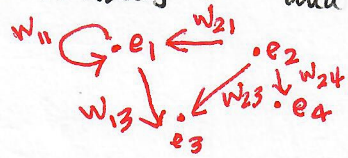
(See the top of p. 2 for the "proof")

§4 Evolution Algebras and Pulse Processes

- For (A,B) an evol. alg & natural basis $B=\{e_1,..,e_n\}$ and structure matrix $M_B(A)=[w_{ij}]$ $\quad e_j^2=\sum_i w_{ij}e_i$

let $G_B(A)$ be the graph with vertices B and adjacency matrix

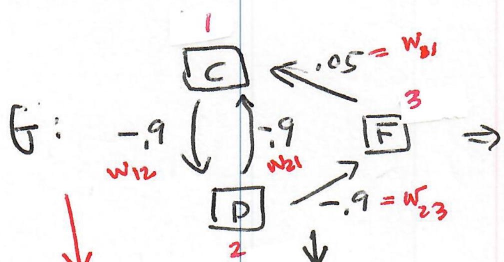
$$M_B(A)^T = \begin{matrix} w_{11} & w_{21} & w_{31} & w_{41} \\ w_{12} & w_{22} & w_{32} & w_{42} \\ w_{13} & w_{23} & w_{33} & w_{43} \\ w_{14} & w_{24} & w_{34} & w_{44} \end{matrix}$$



$$M_B(A)= \begin{matrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & & & \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{matrix}$$

- For G a weighted graph with vertices $B=\{x_1,..,x_n\}$ and adjacency matrix $M_G=[w_{ij}]$, there is uniquely an evol. alg (A,B) with structure matrix $M_B(A)=M_G^T$ $\quad e_j^2=\sum_i w_{ji}e_i \quad B=\{e_1,..,e_n\}$

Example 3



$\Rightarrow$ $B=\{e_C, e_D, e_F\}$ (1 2 3)

$$M_B(A)= \begin{bmatrix} 0_{w_{11}} & -0.9_{w_{21}} & -0.05_{w_{31}} \\ -0.9_{w_{12}} & 0_{w_{22}} & 0_{w_{32}} \\ 0_{w_{13}} & -0.9_{w_{23}} & 0_{w_{33}} \end{bmatrix}$$

$$M_G = \begin{matrix} 0_{w_{11}} & -.9_{w_{12}} & 0_{w_{13}} \\ -.9_{w_{21}} & 0_{w_{22}} & -.9_{w_{23}} \\ .05_{w_{31}} & 0_{w_{32}} & 0_{w_{33}} \end{matrix}$$

adjacency matrix

$$M_G^T = \begin{matrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{23} \\ w_{13} & w_{23} & w_{33} \end{matrix}$$

$$e_C^2 = 0\,e_C - 0.9\,e_D + 0\,e_F$$
$$e_D^2 = -.9\,e_C + 0\,e_D - .9\,e_F$$
$$e_F^2 = -.05\,e_C + 0\,e_D + 0\,e_F$$

Cor 1 of Th 7 $\quad A,B$ evol alg $e=e_1+\cdots+e_n$ $\quad G_B(A)$ associated weighted digraph with adjacency matrix M_G

double check

(i) A nontrivial evol alg $\Rightarrow \sigma_m^A(e)=\sigma(M_G)\cup\{0\}$

(ii) A trivial " " $\Rightarrow \sigma_m^A(e)=\sigma(M_G)$

Def 6 $\quad$ evolution element of evol. Alg A is $e=e_1+\cdots+e_n$ relative to B

Def 7 $\quad$ evol alg is pulse (resp. value) stable relative to B if associated graph is under all autonomous pulse processes.

since $\sigma_m^A(e) = \sigma(M_G)$ or $\sigma(M_G)\cup\{0\}$

~~5~~

Th 8 (translation of Th 3, 4, Cor 1)* of Th 7 (A, B) evol alg, $e = e_1 + \cdots + e_n$ and $M_G = M_B(A)^T$ same eigenvalues

(i) A is pulse stable rel to B under all autonom pulse processes
$\iff \sigma_m^A(e) \subseteq \overline{\mathbb{D}}$. If $\lambda \in \sigma_m^A(e)$ alg & geom multip not coincide then $\lambda \in \mathbb{D}$. i.e. $B_j(\lambda) \neq [\lambda]$ 1×1 matrix

(ii) A is value stable rel to B under all autonom. pulse processes $\iff$ A is pulse stable relative to B and $1 \notin \sigma_m^A(e)$.

These concepts are equivalent under simple pulse processes.

Cor 2 Given A, B, e. If $\sigma_m^A(e) \subseteq \overline{\mathbb{D}}$, then A is pulse and value stable under all autonomous pulse processes.

Cor 3 If $\sigma_m^A(e) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ (i.e. $|\lambda| > 1$) then A is pulse unstable under some simple pulse process, hence value unstable.

Example 4
A, M

$$e_1 \xrightarrow{w_{12}} e_2 \xrightarrow{w_{23}} e_3 \qquad M_B(A) = \begin{matrix} 0 & w_{12} & w_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$\sigma_m^A(e) \overset{?}{=} \{0\} \qquad M_G = \begin{pmatrix} 0 & 0 & 0 \\ w_{12} & 0 & 0 \\ w_3 & 0 & 0 \end{pmatrix}$$

$$\sigma(M_G) = |\lambda I - M_G| = \begin{vmatrix} \lambda & 0 & 0 \\ w_{12} & \lambda & 0 \\ w_3 & 0 & \lambda \end{vmatrix} = \lambda^3 \qquad \boxed{\lambda = 0}$$

* see 5-5-21 p. 7 for Theorem 3 (= Roberts Th 5)
see 5-9-21 p 1 for Theorem 4 (= Roberts Th 6)