Time-Frequency Analysis
Math 211A—Spring 2006

Bernard Russo

June 20, 2006

Contents

1 April 3—Inversion Theorem

Besides the "text" [10], we shall refer to (for example) the books [8] and [2].
The class will meet as follows

MW 11:00-11:50 ET204
Th 11:00-12:30 MSTB 256 (or 254)

1.1 Fourier Integrals

For $f \in L^1 = L^1(\mathbf{R})$, the Fourier transform is defined by

$$\hat{f}(\omega) = \int f(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \qquad (\omega \in \mathbf{R}).$$

The map $f \mapsto \hat{f}$ is a bounded operator from L^1 to L^∞.

Exercise 1 *If $g \in L^1$ and is positive with integral 1, then for all continuous functions on $\mathbf{R}$ with compact support,*

$$\lim_{s\to 0} \int g_s(t)\phi(t)\,dt = \phi(0),$$

where $g_s(t) = s^{-1}g(s^{-1}t)$.

In the proof of Theorem 1.1, we shall use the fact that if $f(t) = e^{-t^2}$, then $\hat{f}(\omega) = \pi^{1/2}e^{-\omega^2/4}$.

Theorem 1.1 (Theorem 2.1,p.23) *If $f \in L^1$ and if $\hat{f} \in L^1$, then*

$$f(t) = \frac{1}{2\pi}\int \hat{f}(\omega)e^{i\omega t}\,d\omega \qquad (t \in \mathbf{R}).$$

Proof: Define

$$I_\epsilon(t) = \frac{1}{2\pi}\int_\omega \int_u f(u)e^{-\epsilon^2\omega^2/4}e^{i\omega(t-u)}\,du\,d\omega.$$

By Lebesgue Dominated Convergence,

$$I_\epsilon(t) = \frac{1}{2\pi}\int_\omega \hat{f}(\omega)e^{-\epsilon^2\omega^2/4}e^{i\omega t}\,d\omega \to \frac{1}{2\pi}\int \hat{f}(\omega)e^{i\omega t}\,d\omega.$$

By Fubini's Theorem

$$I_\epsilon(t) = \frac{1}{2\pi}\int_u \int_\omega f(u)e^{-\epsilon^2\omega^2/4}e^{i\omega(t-u)}\,d\omega\,du.$$

Therefore

$$I_\epsilon(t) = \int_u \left(\frac{1}{2\pi}\int_\omega e^{-\epsilon^2\omega^2/4+i\omega(t-u)}\,d\omega\right) f(u)\,du = \int_u g_\epsilon(t-u)f(u)\,du,$$

where $g_\epsilon(x) = \epsilon^{-1}g_1(\epsilon^{-1}x)$ and $g_1(x) = \pi^{-1/2}e^{-x^2}$.

Exercise 2 *Complete (if possible) the proof of Theorem 1.1 by showing that $I_\epsilon(t) \to f(t)$.*

2 Wednesday April 5—Plancherel Theorem and Uncertainty Principle

2.1 Elementary Properties of the Fourier Transform

The space $L^1 = L^1(\mathbf{R})$ is a Banach *-algebra with convolution as product and $f^*(t) = \overline{f(-t)}$ as involution. The convolution of two functions $f, g \in L^1$ is defined by

$$f * g(x) = \int f(y)g(x-y)\,dy.$$

Theorem 2.1 (Theorem 2.2,p.24) *If $f, h \in L^1$, then $g := h * f \in L^1$ and $\hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega)$.*

Proof:

$$\begin{aligned}\hat{g}(\omega) &= \int e^{-it\omega} h * f(t)\,dt = \int_t (\int_u f(t-u)h(u)\,du)e^{-i\omega t}\,dt \\ &= \int_u (\int_t f(t-u)e^{-i\omega t}\,dt)h(u)\,du = \int_u (\int_t f(t)e^{-i\omega(t+u)}\,dt)h(u)\,du \\ &= \int \hat{f}(\omega)h(u)e^{-i\omega u}\,du = \hat{f}(\omega)\hat{h}(\omega).\end{aligned}$$

□

Elementary properties of the Fourier transform (see [10, p. 25])

2.2 Plancherel Theorem

Theorem 2.2 (Theorem 2.3,p.26) *If $f, h \in L^1 \cap L^2$, then*

$$\int f(t)\overline{h(t)}\,dt = \frac{1}{2\pi}\int \hat{f}(\omega)\overline{\hat{h}(\omega)}\,d\omega.$$

Proof: With h^* defined by $h^*(t) = \overline{h(-t)}$, we have

$$\hat{h^*}(\omega) = \int \overline{h(-t)}e^{-i\omega t}\,dt = \int \overline{h(t)}e^{i\omega t}\,dt = \overline{\int h(t)e^{-i\omega t}\,dt} = \overline{\hat{h}(\omega)}.$$

So with $g := f * h^*$, $\hat{g}(\omega) = \hat{f}(\omega)\hat{h^*}(\omega) = \hat{f}(\omega)\overline{\hat{h}(\omega)}$. Hence, by Theorem 1.1,

$$\int f(u)\overline{h(u)}\,du = g(0) = \frac{1}{2\pi}\int \hat{g}(\omega)\,d\omega = \frac{1}{2\pi}\int \hat{f}(\omega)\overline{\hat{h}(\omega)}\,d\omega.$$

Exercise 3 *Correct the proof of Theorem 2.2 by showing independently (if possible) that if $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$.*

Examples of Fourier transforms (see [10, p. 27])

2.3 Uncertainty Principle

Uncertainty Principle (see [10, p. 30]) The state of a one-dimensional particle is described by a wave function $f \in L^2$.

- The probability density that this particle is located at t is $\frac{1}{\|f\|^2}|f(t)|^2$
- The probability density that its momentum is equal to ω is $\frac{1}{2\pi\|f\|^2}|\hat{f}(\omega)|^2$
- The average location of this particle is $u = \frac{1}{\|f\|^2}\int f|f(t)|^2\,dt$
- The average momentum of this particle is $\xi = \frac{1}{2\pi\|f\|^2}\int \omega|\hat{f}(\omega)|^2\,d\omega$
- The temporal variance around the average location is

$$\sigma_t^2 = \frac{1}{\|f\|^2}\int (t-u)^2|f(t)|^2\,dt$$

- The frequency variance around the average momentum is

$$\sigma_\omega^2 = \frac{1}{2\pi\|f\|^2}\int (\omega-\xi)^2|\hat{f}(\omega)|^2\,d\omega$$

Theorem 2.3 (Theorem 2.5,p.31) $\sigma_t^2\sigma_\omega^2 \geq 1/4$, *with equality holding if and only if* $f(t) = a\exp(i\xi t - b(t-u)^2)$ *for* $a, b \in \mathbf{C},\ u, \xi \in \mathbf{R}$.

Proof:[1]

Step 1 We may assume that $u = \xi = 0$. The general case reduces to this by considering the function $g(t) = e^{-i\xi t}f(t+u)$.

Step 2 Since $i\omega\hat{f}(\omega)$ is the Fourier transform of $f'(t)$, by Plancherel's theorem and the Schwarz inequality

$$\begin{aligned}
\sigma_t^2\sigma_\omega^2 &= \frac{1}{2\pi\|f\|^4}\int |tf(t)|^2\,dt \int |\omega\hat{f}(\omega)|^2\,d\omega \\
&= \frac{1}{\|f\|^4}\int |tf(t)|^2\,dt \int |f'(t)|^2\,dt \\
&\geq \frac{1}{\|f\|^4}\left(\int |tf(t)f'(t)|\,dt\right)^2 \\
&\geq \frac{1}{\|f\|^4}\left|\int tf(t)f'(t)\,dt\right|^2
\end{aligned}$$

[1] The proof of the inequality $\sigma_t^2\sigma_\omega^2 \geq 1/4$ given here assumes that f is differentiable and that $\sqrt{|t|}f(t) \to 0$ as $|t| \to \infty$. This suffices in general since such functions are dense in L^2

Step 3 $(|f'(t)|^2)' = f(t)\overline{f'(t)} + f'(t)\overline{f(t)}$

Step 4 $\left|\int t\overline{f(t)}f'(t)\,dt\right| = \left|\int t\overline{f'(t)}f(t)\,dt\right|$

Step 5 By integrating by parts, and using $\sqrt{|t|}f(t) \to 0$,

$$\int t(|f(t)|^2)'\,dt = \Big|t|f(t)|^2\Big|_{-\infty}^{\infty} - \int |f(t)|^2\,dt = -\|f\|^2$$

Step 6 $|\int t(|f(t)|^2)'\,dt| = \left|\int t(f(t)\overline{f'(t)} + f'(t)\overline{f(t)})\,dt\right| \leq 2\int |t\overline{f(t)}f'(t)|\,dt$

Step 7 $\|f\|^4 = |\int t(|f(t)|^2)'\,dt|^2 \leq \left(2\int |t\overline{f(t)}f'(t)|\,dt\right)^2 \leq 4\sigma_t^2\sigma_\omega^2\|f\|^4.$

This proves the inequality. The proof of equality will be given later.

Theorem 2.4 (Theorem 2.6,p.32) *If $f \neq 0$ has compact support, then $\hat{f}$ cannot be zero on a whole interval. Similarly, if $\hat{f} \neq 0$ has compact support, then f cannot be zero on a whole interval.*

3 Thursday April 6—Time-frequency proof of Plancherel's theorem; the uncertainty principle revisited

Since we are following [8], we shall use the definition that is used there, namely,

$$\hat{f}(\omega) = \int_{\mathbf{R}} f(x)e^{-2\pi i x\omega}\,dx \text{ for } f \in L^1(\mathbf{R}) \text{ and } \omega \in \mathbf{R}.$$

3.1 The fundamental operators of time-frequency analysis

The fundamental operators of time-frequency analysis are

- Translation by x: $T_x f(t) = f(t-x) \quad (t, x \in \mathbf{R})$
- Modulation by ω: $M_\omega f(t) = e^{2\pi i\omega t}f(t) \quad (t, \omega \in \mathbf{R})$

The products $T_x M_\omega$ and $M_\omega T_x$ are called "time-frequency shifts."

Some properties of these operators are

- $T_x M_\omega = e^{-2\pi i\omega x}M_\omega T_x$
- $\|T_x M_\omega f\|_p = \|f\|_p$ for $1 \leq p \leq \infty$
- $(T_x f)\widehat{} = M_{-x}\hat{f}$

- $(M_\omega f)\widehat{\ } = T_\omega \hat{f}$
- $(T_x M_\omega f)\widehat{\ } = e^{-2\pi i\omega x} T_\omega M_{-x} \hat{f}$

We'll use the following two properties of derivatives and Fourier transforms for suitably smooth f and $k \geq 1$.

- $\widehat{f^{(k)}}(\omega) = (2\pi i\omega)^k \hat{f}(\omega)$
- $((-2\pi i x)^k f(x))\widehat{\ }(\omega) = \hat{f}^{(k)}(\omega)$

3.2 Plancherel Theorem

Lemma 3.1 ([8, Lemma 1.5.1. p. 17]) *If $\varphi_a(x) = e^{-\pi x^2/a}$, then $\hat{\varphi}_a(\omega) = a^{1/2}\varphi_{1/a}(\omega)$.*

Proof: $\varphi_a'(x) = (-2\pi x/a)\varphi_a(x)$ which can be written as $X\varphi_a(x) = (-a/2\pi)\varphi_a'(x)$ where X is the operator of multiplication by x: $Xg(x) = xg(x)$. Then

$$\begin{aligned} \hat{\varphi}_a'(\omega) &= (-2\pi i X\varphi_a)\hat{\ }(\omega) = (-2\pi i)(-a/2\pi)\varphi_a'(\omega) \\ &= (ia\varphi_a')\hat{\ }(\omega) = ia(2\pi i)\hat{\varphi}_a(\omega). \end{aligned}$$

Thus, $\hat{\varphi}_a$ is the solution of the differential equation $y' = -2a\pi xy$, so $\hat{\varphi}_a(\omega) = Ce^{-\pi a\omega^2}$, with $C = \hat{\varphi}_a(0) = a^{1/2}$. □

Lemma 3.2 ([8, Lemma 1.5.2. p. 18]) *For $a > 0$ and $x, \omega, u, \eta \in \mathbf{R}$,*

$$(T_x M_\omega \varphi_a | T_u M_\eta \varphi_a) = (a/2)^{1/2} e^{\pi i(u-x)(\eta+\omega)} \varphi_{2a}(u-x)\varphi_{2/a}(\eta-\omega)$$

Proof:

Step 1 $(\varphi_a | M_\omega T_x \varphi_a) = e^{-\pi i x\omega}(a/2)^{1/2}\varphi_{2a}(x)\varphi_{2/a}(\omega)$

$$\begin{aligned} \text{proof: } (\varphi_a | M_\omega T_x \varphi_a) &= \int e^{-\pi t^2/a} e^{-\pi(t-x)^2/a} e^{-2\pi i\omega t}\, dt \\ &= e^{-\pi x^2/2a} \int e^{-2\pi(t-x/2)^2/a} e^{-2\pi i\omega t}\, dt \\ &= \varphi_{2a}(x)(T_{x/2}\varphi_{a/2})\hat{\ }(\omega) = e^{-\pi i x\omega}(a/2)^{1/2}\varphi_{2a}(x)\varphi_{2/a}(\omega) \end{aligned}$$

Step 2 $M_{-\omega} T_{u-x} M_\eta = e^{-2\pi i\eta(u-x)} M_{\eta-\omega} T_{u-x}$

proof: In the left hand side, substitute $T_{u-x}M_\eta = e^{-2\pi i(u-x)\eta} M_\eta T_{u-x}$

Step 3

$$\begin{aligned} (T_x M_\omega \varphi_a | T_u M_\eta \varphi_a) &= (\varphi_a | M_{-\omega} T_{u-x} M_\eta \varphi_a) \\ &= e^{2\pi i\eta(u-x)}(\varphi_a | M_{\eta-\omega} T_{u-x}\varphi_a) \quad \text{(by Step 2)} \\ &= e^{2\pi i\eta(u-x)} e^{-\pi i(u-x)(\eta-\omega)}(a/2)^{1/2}\varphi_{2a}(u-x)\varphi_{2/a}(\eta-\omega) \quad \text{by Step 1)} \\ &= (a/2)^{1/2} e^{\pi i(u-x)(\eta+\omega)}\varphi_{2a}(u-x)\varphi_{2/a}(\eta-\omega) \square \end{aligned}$$

Lemma 3.3 **([8, Lemma 1.5.3. p. 18])** *For any $a > 0$, the set*

$$\{T_x M_\omega \varphi_a : x, \omega \in \mathbf{R}\}$$

is total in L^2.

Proof: Will be given later.

Theorem 3.4 **([8, Theorem 1.1.2. p. 5])** *The Fourier transform extends from $L^1 \cap L^2$ to a unitary operator on L^2.*

Proof: By Lemma 3.3, the set $X = \text{span}\{T_x M_\omega \varphi_a : x, \omega \in \mathbf{R}\}$ is dense in L^2. Let F denote the restriction of the Fourier transform to X. Since $F(T_x M_\omega \varphi_1) = M_{-x} T_\omega \hat{\varphi}_1 = e^{-2\pi i x\omega} T_\omega M_{-x} \hat{\varphi}_1 = e^{-2\pi i x\omega} T_\omega M_{-x} \varphi_1$, $F(X) = X$ is also dense in L^2.

Claim: $(T_x M_\omega \varphi_1 | T_u M_\eta \varphi_1) = (F(T_x M_\omega \varphi_1) | F(T_u M_\eta \varphi_1))$

Assuming this claim for the moment, if we write $f = \sum_1^n c_k T_{x_k} M_{\omega_k} \varphi_1 \in X$, then $\hat{f} = \sum_1^n c_k F T_{x_k} M_{\omega_k} \varphi_1$ and $\|f\|_2^2 = (f|f) = \sum_{k,l} c_k \bar{c}_l (T_{x_k} M_{\omega_k} \varphi_1 | T_{x_l} M_{\omega_l} \varphi_1) = \sum_{k,l} c_k \bar{c}_l (F T_{x_k} M_{\omega_k} \varphi_1 | F T_{x_l} M_{\omega_l} \varphi_1) = (\hat{f}|\hat{f}) = \|\hat{f}\|_2^2$, completing the proof.

The claim follows by two applications of Lemma 3.2. In the first place, $F T_x M_\omega \varphi_1 = M_{-x} F M_\omega \varphi_1 = M_{-x} T_\omega F \varphi_1 = M_{-x} T_\omega \varphi_1$, and similarly $F T_u M_\eta = M_{-u} T_\eta \varphi_1$, so that

$$\begin{aligned}
(F(T_x M_\omega \varphi_1) | F(T_u M_\eta \varphi_1)) &= (M_{-x} T_\omega \varphi_1 | M_{-u} T_\eta \varphi_1) \\
&= (e^{-2\pi i x\omega} T_\omega M_{-x} \varphi_1 | e^{-2\pi i u\eta} T_\eta M_{-u} \varphi_1) \\
&= e^{-2\pi i (x\omega - u\eta)} (T_\omega M_{-x} \varphi_1 | T_\eta M_{-u} \varphi_1) \\
&= e^{-2\pi i (x\omega - u\eta)} (1/2)^{1/2} e^{\pi i (\eta - \omega)(-u - x)} \varphi_2(\eta - \omega) \varphi_2(-u + x) \\
&= (1/2)^{1/2} e^{\pi i (\eta u + \omega u - \eta x - \omega x)} \varphi_2(\eta - \omega) \varphi_2(-u + x) \\
&= (T_x M_\omega \varphi_1 | T_u M_\eta \varphi_1) \square
\end{aligned}$$

3.3 The uncertainty principle of Donoho and Stark

The results in this subsection might be proved later.

Theorem 3.5 (**[8, Theorem 2.3.1,page 30]**) *Suppose that $f \in L^2(\mathbf{R}^d)$ is ϵ_T-concentrated on $T \subset \mathbf{R}^d$ and$\hat{f}$ is ϵ_Ω-concentrated on $\Omega \subset \mathbf{R}^d$, that is*

$$\left(\int_{\mathbf{R}^d \setminus T} |f|^2 \right)^{1/2} \leq \epsilon_T \|f\|_2 \text{ and } \left(\int_{\mathbf{R}^d \setminus \Omega} |\hat{f}|^2 \right)^{1/2} \leq \epsilon_\Omega \|\hat{f}\|_2.$$

Then $|T||\Omega| \geq (1 - \epsilon_T - \epsilon_\Omega)^2$.

Corollary 3.6 *If $f \in L^2(\mathbf{R}^d)$ is supported on T and $\hat{f}$ is supported on Ω, then $|T||\Omega| \geq 1$.*

One can use the Poisson summation formula to prove the following qualitative uncertainty principle.

Theorem 3.7 (**[8, Theorem 2.3.3,page 32]**) *If $f \in L^1(\mathbf{R}^d)$ is supported on T, $\hat{f}$ is supported on Ω, and $|T||\Omega| < \infty$, then $f = 0$.*

4 Monday April 10, 2006

4.1 Completion of the proof of Theorem 2.3

Theorem 2.3 was proved under the assumption that f is differentiable and that $\sqrt{|t|}f(t) \to 0$ as $|t| \to \infty$. There does not seem to be a simple density argument to complete the proof. Moreover, the assumption is actually redundant, given that you can assume without loss of generality that $tf(t)$ and $\omega\hat{f}(\omega)$ both belong to L^2. Insight to this is from [7, p. 210], as follows.

If f is a distribution, then $\widehat{(f')}$ is the distributional Fourier transform of its distributional derivative, and it will follow from the assumption $\omega\hat{f}(\omega) \in L^2$ that $\widehat{(f')} \in L^2$ and therefore $f' \in L^2$ as well. More precise details follow.

First let's recall some basic definitions for distributions. If f is a distribution, that is, a linear functional satisfying a continuity assumption on the space of test functions (= C^∞-functions with compact support), then its distributional derivative is the distribution f' defined by $\langle f', \varphi\rangle = -\langle f, \varphi'\rangle$. The distributional Fourier transform of f is the distribution $\hat{f}$ defined by $\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$.

Thus

$$\begin{aligned}
\langle (f')\hat{}, \varphi\rangle &= \langle f', \hat{\varphi}\rangle = -\langle f, (\hat{\varphi})'\rangle \\
&= 2\pi i \int f(\omega)\omega\hat{\varphi}(\omega)\, d\omega = 2\pi i \int \omega f(\omega)\overline{\psi(\omega)}\, d\omega \\
&= i \int (\omega f(\omega))\hat{}\, \overline{\hat{\psi}(\omega)}\, d\omega = i \int (\omega f(\omega))\hat{}\, \varphi(\omega)\, d\omega,
\end{aligned}$$

where $\psi(\omega) := \overline{\hat{\varphi}(\omega)}$ from which it follows that $\overline{\hat{\psi}} = \varphi$. This shows that $(f')\hat{} = i(\omega f(\omega))\hat{}$ so that $(f')\hat{} \in L^2$ and therefore $f' \in L^2$.

What about the use of the assumption $\sqrt{|t|}f(t) \to 0$ in the integration by parts in Step 5? Not to worry. Since $tf(t) \in L^2$ and $f' \in L^2$, we have $t|f(t)|^2)' = 2t\,(\operatorname{Re} f(t))\overline{f'(t)} \in L^1$ and since f is absolutely continuous (see Exercise 4), by [14, p. 239], for each $c < d$,

$$\int_c^d t(|f(t)|^2)'\, dt = \Big[t|f(t)|^2\Big]_c^d - \int_c^d |f(t)|^2\, dt.$$

Both limits $\lim_{d\to\infty} d|f(d)|^2$ and $\lim_{c\to-\infty} c|f(c)|^2$ exist, and they must both be zero, otherwise $|f|^2$ would be asymptotic to $1/|x|$, which is not integrable. □

Exercise 4 *Prove that f in the proof of Theorem 2.3 is absolutely continuous.*

4.2 Absolute Convergence of Fourier series and the proof of Lemma 3.3

The proof of the following lemma was distilled from the proof in several variables in [12, Cor. 1.9,p. 249].

Lemma 4.1 *A periodic function on* $[0,2\pi]$ *with a continuous first derivative is the uniform limit of its Fourier series.*

Proof: Since f' is in $L^2[0,2\pi]$ and $(f')\hat{}(m) = 2\pi i m \hat{f}(m)$, we have $\sum_{m\in\mathbf{Z}}(2\pi m)^2|\hat{f}(m)|^2 < \infty$. Then by Schwarz's inequality,

$$\sum_{m\neq 0} |\hat{f}(m)| \leq \left(\sum |\hat{f}(m)|^2 |2\pi m|^2\right)^{1/2} \left(\sum |2\pi m|^{-2}\right)^{1/2} < \infty,$$

so the Fourier series for f, namely, $\sum \hat{f}(m)e^{2\pi i m x}$ converges uniformly. Since this series converges to f in $L^2[0,2\pi]$, it converges to f uniformly. □

Proof of Lemma 3.3: We leave it as a simple exercise, using translation operators, that it suffices to prove the lemma in the case that $a=1$. So let $X := \text{span}\,\{T_x M_\omega \varphi_1 : x,\omega \in \mathbf{R}\}$. Let k be a C^∞ function with compact support on $\mathbf{R}$, let $K := \max_{x\in\mathbf{R}} |k(x)|e^{\pi x^2}$ and let $\epsilon > 0$. Choose $c > 0$ such that the support of k lies in $[-c,c]$ and such that $\int_{|x|\geq c} e^{-2\pi x^2}\,dx < \frac{1}{2}(\frac{\epsilon}{K+1})^2$.

Expand $k(x)e^{\pi x^2}$ as a Fourier series $\sum_{-\infty}^{\infty} b_n e^{\pi i n x/c}$ in $L^2[-c,c]$. By Lemma 4.1, the series converges uniformly to $k(x)e^{\pi x^2}$. Pick a partial sum $s(x) = \sum_{|n|\leq M} b_n e^{\pi i n x/c}$ so that $\sup_{x\in[-c,c]} |k(x)e^{\pi x^2} - s(x)| < \epsilon/\sqrt{2}$ and $\|s\|_\infty \leq K+1$.

Consider $h(x) := s(x)\varphi_1(x) = \sum_{|n|\leq M} b_n M_{n/2c} e^{-2\pi x^2}$ which belongs to X. Then

$$|k(x)-h(x)|^2 = |(k(x)e^{\pi x^2} - h(x)e^{\pi x^2})e^{-\pi x^2}|^2 = |k(x)e^{\pi x^2} - s(x)|^2 e^{-2\pi x^2},$$

so that

$$\begin{aligned}
\|k-h\|_2^2 &= \int_{|x|\leq c} |k(x)e^{\pi x^2} - s(x)|^2 e^{-2\pi x^2}\,dx + \int_{|x|\geq c} |s(x)|^2 e^{-2\pi x^2}\,dx \\
&\leq \frac{\epsilon^2}{2}\int_{|x|\leq c} e^{-2\pi x^2}\,dx + (K+1)^2 \int_{|x|\geq c} e^{-2\pi x^2}\,dx \\
&\leq \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.
\end{aligned}$$

Since the C^∞-functions with compact support are dense in L^2, this shows that X is dense in L^2. □

4.3 Does the Plancherel Theorem imply the L^1-Inversion Theorem?

Exercise 5 *Show (if possible) by using integral operators, that Theorem 1.1 follows from Theorem 3.4*

4.4 Unfinished business for chapter 2

- A proof of Theorem 2.3 based on commutators of self-adjoint operators

- The proof of Theorem 3.7
- Proof for the case of equality in the uncertainty principle
- Proof of Theorem 2.4
- Problems, chapter 2

5 Wednesday April 19, 2006—Sampling analog signals I

5.1 Poisson summation formula

We shall prove the following theorem in the case $T = 1$ following [13, pp. 117-119]. Let us first recall that if $\delta = \delta(t)$ denotes the Dirac distribution: $\langle \delta, \varphi \rangle = \varphi(0)$, then $\langle \hat{\delta}, \varphi \rangle = \langle \delta, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int \varphi(t)\,dt = \langle 1, \varphi \rangle$ so that $\hat{\delta} = 1$, or $\hat{\delta}(\xi) = 1$ for all $\xi \in \mathbf{R}$. Next, letting $\mathcal{F}_x$ denote the Fourier transform in the variable x, we have[2] $\langle \mathcal{F}_x(\delta(x-y)), \varphi \rangle = \langle \delta(x-y), \hat{\varphi} \rangle = \hat{\varphi}(y) = \int \varphi(\xi) e^{iy\xi}\,d\xi = \langle e^{iy\,\cdot}, \varphi \rangle$ so that $\mathcal{F}_x(\delta(x-y))(\xi) = e^{iy\xi}$.

Theorem 5.1 (Poisson Formula [10, Theorem 2.4, p. 28]) *In the sense of distributions,* $\sum_{-\infty}^{\infty} e^{-inT\omega} = \frac{2\pi}{T} \sum_{-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$.

Proof: Given a function $\varphi : \mathbf{R} \to \mathbf{C}$, let $P\varphi(x) = \sum_{k\in\mathbf{Z}} \varphi(x - 2k\pi)$. $P\varphi$ exists as a 2π-periodic function if φ has compact support or decays rapidly enough at infinity. Expand $P\varphi$ in a Fourier series $P\varphi(x) \sim \sum c_k e^{ikx}$ where[3] $c_k = \frac{1}{2\pi}\int_0^{2\pi} P\varphi(x) e^{-ikx}\,dx$. We note that

$$\begin{aligned} c_k &= \frac{1}{2\pi}\sum_{j\in\mathbf{Z}} \int_0^{2\pi} \varphi(x - 2\pi j) e^{-ikx}\,dx = \frac{1}{2\pi}\sum_{j\in\mathbf{Z}} \int_{-2\pi j}^{-2\pi(j-1)} \varphi(y) e^{-iky}\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(y) e^{-iky}\,dy = \frac{1}{2\pi}\hat{\varphi}(-k). \end{aligned}$$

Now compute $P\varphi(0)$ in two ways. First, from the definition of $P\varphi$, $P\varphi(0) = \sum_{k\in\mathbf{Z}} \varphi(2\pi k) = \langle \sum_{k\in\mathbf{Z}} \delta(x - 2\pi k), \varphi \rangle$. Second, from inversion of Fourier series,

$$\begin{aligned} P\varphi(0) &= \sum_{k\in\mathbf{Z}} \widehat{P\varphi}(k) = \sum_{k\in\mathbf{Z}} c_k = \frac{1}{2\pi}\sum_{k\in\mathbf{Z}} \hat{\varphi}(k) \\ &= \frac{1}{2\pi}\langle \sum_{k\in\mathbf{Z}} \delta(x-k), \hat{\varphi} \rangle = \frac{1}{2\pi}\langle \mathcal{F}(\sum_{k\in\mathbf{Z}} \delta(x-k)), \varphi \rangle. \end{aligned}$$

Equating these two values proves the theorem. □

[2] The Fourier transform in [13] uses a plus sign

[3] Inconsistent with the previous footnote, the Fourier coefficient in [13] uses a minus sign

5.2 Sampling analog signals I

A uniform sampling of an analog signal f (f is in L^2 but will usually be continuous so that $f(nT)$ makes sense) corresponds to a weighted Dirac sum $f_d(t) = \sum_{-\infty}^{\infty} f(nT)\delta(t-nT)$, a distribution, with Fourier transform $(f_d)\hat{}(\omega) = \sum_{-\infty}^{\infty} f(nT)e^{-inT\omega}$.

In the following proposition we use the following two definitions. If f is a distribution and ψ is a function of an appropriate regularity, then the distribution ψf is defined by $\langle \psi f, \varphi\rangle = \langle f, \psi\varphi\rangle$, and the distribution $\psi * f$ is defined by $\langle \psi * f, \varphi\rangle = \langle f, \tilde{\psi} * \varphi\rangle$, where $\tilde{\psi}(t) := \psi(-t)$.

Proposition 5.2 ([10, Prop. 3.1, p. 43]) $(f_d)\hat{}(\omega) = \frac{1}{T}\sum_{-\infty}^{\infty} \hat{f}(\omega - \frac{2k\pi}{T})$.

Proof: Since $f(nT)\delta(\cdot - nT) = f\delta(\cdot - nT)$, we have$f_d(t) = \sum_{n\in\mathbf{Z}} f(t)\delta(t - nT) = f(t)c(t)$, where $c(t) = \sum_{n\in\mathbf{Z}}\delta(t-nT)$ is a "Dirac comb." From $f_d = fc$ we shall show that $\hat{f}_d = \frac{1}{2\pi}\hat{f} * \hat{c}$. We shall also show that $\hat{f} * \delta = \hat{f}$. Assuming these two facts for the moment, we have from Proposition 5.1, $\hat{f}_d(\omega) = \frac{1}{2\pi}\hat{f} * \hat{c}(\omega) = \frac{1}{2\pi}\hat{f} * \left[\frac{2\pi}{T}\sum_{k\in\mathbf{Z}}\delta(\omega - \frac{2\pi k}{T})\right] = \frac{1}{T}\left[\sum_{k\in\mathbf{Z}}\hat{f} * \delta(\omega - \frac{2\pi k}{T})\right] = \frac{1}{T}\left[\sum_{k\in\mathbf{Z}}\hat{f}(\omega - \frac{2\pi k}{T})\right]$, as required.

It remains to show that if ψ is a function and f is a distribution, then $(\psi f)\hat{} = \frac{1}{2\pi}\hat{\psi} * \hat{f}$ and $\psi * \delta = \psi$. In the first place, $\langle(\psi f)\hat{}, \varphi\rangle = \langle \psi f, \hat{\varphi}\rangle = \langle f, \hat{\check{\psi}}\hat{\varphi}\rangle = \langle f, (\check{\psi} * \varphi)\hat{}\rangle = \langle \hat{f}, \check{\psi} * \varphi\rangle$. On the other hand, $\frac{1}{2\pi}\langle\hat{\psi} * \hat{f}, \varphi\rangle = \frac{1}{2\pi}\langle \hat{f}, \tilde{\hat{\psi}} * \varphi\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{\tilde{\psi}} * \varphi\rangle$. But, $\check{\psi} = \frac{1}{2\pi}\hat{\tilde{\psi}}$ so $(\psi f)\hat{} = \frac{1}{2\pi}\hat{\psi} * \hat{f}$ is proved. As to $\psi * \delta = \psi$, $\langle \psi * \delta, \varphi\rangle = \langle \delta, \tilde{\psi} * \varphi\rangle = \tilde{\psi} * \varphi(0) = \int \tilde{\psi}(y)\varphi(-y)\,dy = \int \psi(-y)\varphi(-y)\,dy = \langle \psi, \varphi\rangle$. □

6 Thursday April 20, 2006—Sampling analog signals II; Proof of Theorem 3.7

6.1 Sampling analog signals II

Theorem 6.1 ([10, Theorem 3.1, p. 44]) *If the support of* $\hat{f}$ *lies in* $[-\pi/T, \pi/T]$*, then* $f(t) = \sum_{-\infty}^{\infty} f(nT)h_T(t-nT)$*, where* $h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}$.

Proof: We start by noting that $\hat{h}_T(\omega) = T1_{[\pi/T,\pi/T]}(\omega)$, which is the first example on [10, p. 27]. Next, by Proposition 5.2 and the assumption about the support of $\hat{f}$, $\hat{f}_d(\omega) = \frac{1}{T}\sum_{n\in\mathbf{Z}}\hat{f}(\omega - \frac{2\pi n}{T}) = \hat{f}(\omega)/T$ for $|\omega| \le \pi/T$. Putting these two facts together results in $\hat{f} = \hat{h}_T\hat{f}_d = \widehat{h_T * f_d}$ so that $f = h_T * f_d = h_T * [\sum_n f(nT)\delta(t-nT)] = \sum_n f(nT)\,h_T(t-nT)$, since $\langle h_T * [\delta(t-nT)], \varphi\rangle = \langle \delta(t-nT), \tilde{h}_T * \varphi\rangle = \tilde{h}_T * \varphi(nT) = \int \tilde{h}_T(nT-y)\varphi(y)\,dy = \int h_T(y-nT)\varphi(y)\,dy = \langle h_T(\cdot - nT), \varphi\rangle$. □

Exercise 6 *Show that* $f(t)=\sum_{-\infty}^{\infty} f(nT)h_T(t-nT)$ *converges in* L^2*.*

An example where the support of $\hat f$ **is not included in** $[-\pi/T,\pi/T]$: $f(t)=\cos\omega_0 t$**.** This is referred to as a high-frequency oscillation. The Fourier transform $\hat f(\omega)=\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$ has support equal to $\{\omega_0,-\omega_0\}$. Now take $\omega_0\in(\pi/T,2\pi/T)$. Then

$$\begin{aligned}\hat h_T(\omega)\hat f_d(\omega) &= [\frac{1}{T}\sum_k \hat f(\omega-\frac{2\pi k}{T})][T1_{[-\pi/T,\pi/T]}(\omega)]\\ &= \pi 1_{[-\pi/T,\pi/T]}(\omega)\sum_k[\delta(\omega-\omega_0-\frac{2\pi k}{T})+\delta(\omega+\omega_0-\frac{2\pi k}{T})]\\ &= \pi[\delta(\omega+\omega_0-\frac{2\pi}{T})+\delta(\omega-\omega_0-\frac{2\pi}{T})].\end{aligned}$$

Thus $f_d*h_T(t)=\cos(\frac{2\pi}{T}-\omega_0)$ which is an oscillation with a lower frequency.

Note that, of course, $f_d*h_T\neq f$. In such a case, one looks for $\tilde f$ such that the support of $\hat{\tilde f}$ lies in $[-\pi/T,\pi/T]$ and $\|f-\tilde f\|_2\le\|f-g\|_2$ for all g with the support of $\hat g$ in $[-\pi/T,\pi/T]$. Since

$$\|f-\tilde f\|^2=\frac{1}{2\pi}\int|\hat f-\hat{\tilde f}|^2\,d\omega=\frac{1}{2\pi}\int_{|\omega|>\pi/T}|\hat f|^2\,d\omega+\frac{1}{2\pi}\int_{|\omega|\le\pi/T}|\hat f-\hat{\tilde f}|^2\,d\omega,$$

we choose $\tilde f$ such that the second integral is zero, that is, such that $\hat{\tilde f}=\hat f 1_{[-\pi/T,\pi/T]}=\frac{1}{T}\hat h_T\hat f$, corresponding to $\tilde f=\frac{1}{T}f*h_T$.

Conclusion: The function $\tilde f$ is called the filtering of f and $\tilde f$ can be recovered from the samples $\tilde f(nT)$ since the support of $\hat{\tilde f}$ lies in $[-\pi/T,\pi/T]$.[4]

Proposition 6.2 ([10, Prop. 3.2, p. 47]) *With* $h_T(t)=\frac{\sin(\pi t/T)}{\pi t/T}$, $\{h_T(t-nT)\}_{n\in\mathbf{Z}}$ *is an orthogonal basis of the subspace* $U_T\subset L^2$ *of functions* f *with Fourier transform* $\hat f$ *supported in* $[-\pi/T,\pi/T]$*. If* $f\in U_T$*, then* $f(nT)=(1/T)(f(\cdot)|h_T(\cdot-nT))$*.*

Proof: Since $\mathcal{F}h_T=\hat h_T=T1_{[-\pi/T,\pi/T]}$, $h_T\in U_T$ and we can calculate the inner product by means of Parseval's formula:

$$\begin{aligned}(h_T(\cdot-nT)|h_T(\cdot-pT)) &= \frac{1}{2\pi}(\mathcal{F}h_T(\cdot-nT)|\mathcal{F}h_T(\cdot-pT))\\ &= \frac{1}{2\pi}\int T^2 1_{[-\pi/T,\pi/T]}(\omega)e^{-i(n-p)T\omega}\,d\omega=T\delta_{p,n}.\end{aligned}$$

Thus $\{h_T(\cdot-nT)/\sqrt{T}\}_{n\in\mathbf{Z}}$ is an orthonormal set in U_T. It is a basis by Exercise 6. Since $f=\sum\sqrt{T}f(nT)\,h_T(\cdot-nT)/\sqrt{T}$, by properties of a basis $\sqrt{T}f(nT)=(f|H_T(\cdot-nT)/\sqrt{T})$. □

[4] I'll explain the meaing of the term "aliasing" in this context later

The following well-known result is included here for completeness. Since we already proved that a continuously differentiable function has an absolutely and uniformly convergent Fourier series, and such functions are dense in L^2, it follows that the trigonometric polynomials are dense in L^2, proving the completeness of the orthonormal system $\{e^{ik\omega}\}_{k\in\mathbf{Z}}$.

Theorem 6.3 ([10, Theorem 3.2, p. 51]) *$\{e^{ik\omega}\}_{k\in\mathbf{Z}}$ is an orthonormal basis of $L^2[-\pi,\pi]$.*

6.2 Poisson formula revisited; proof of Theorem 3.7

Lemma 6.4 ([8, Lemma 1.4.1, p. 14]) *If $f \in L^1$ and $\alpha > 0$, then*

$$\int_{\mathbf{R}} f(x)\,dx = \int_0^\alpha \left(\sum_{k\in\mathbf{Z}} f(x+\alpha k)\right) dx.$$

Proof: Let $F(k,x) = f(x+\alpha k)$, for $(k,x) \in \mathbf{Z}\times[0,\alpha]$. Then $\sum_{\mathbf{Z}} \int_0^\alpha |F(k,x)|\,dx = \sum_{\mathbf{Z}} \int_0^\alpha |f(x+k\alpha)|\,dx = \sum_{\mathbf{Z}} \int_{\alpha k}^{\alpha(k+1)} |f(x)|\,dx = \int_{\mathbf{R}} |f(x)|\,dx < \infty$ so by Tonelli's theorem the function F is integrable over the product space $\mathbf{Z}\times[0,\alpha]$. Then, by Fubini's theorem,

$$\int_{\mathbf{R}} f(x)\,dx = \sum_{\mathbf{Z}} \int_{\alpha k}^{\alpha(k+1} f(x)\,dx = \sum_{\mathbf{Z}} \int_0^\alpha f(x+k\alpha)\,dx = \int_0^\alpha \sum_{\mathbf{Z}} f(x+\alpha k)\,dx.\Box$$

Proposition 6.5 ([8, Proposition 1.4.2, p. 15]—Poisson summation) *If $|f(x)| \le C(1+|x|)^{-1-\epsilon}$ and $|\hat{f}(\omega)| \le C(1+|\omega|)^{-1-\epsilon}$, then*

$$\sum_{n\in\mathbf{Z}} f(x+n) = \sum_{n\in\mathbf{Z}} \hat{f}(n)e^{2\pi inx} \tag{1}$$

with both sums converging absolutely for every $x \in \mathbf{R}$.

Proof: By the decay assumptions, both series converge absolutely. Let $\varphi(x)$ denote the left side of (1). The φ is one-periodic and integrable on $[0,1]$ since $\int_0^1 |\varphi(x)|\,dx \le \sum_{\mathbf{Z}} \int_0^1 |f(x+n)|\,dx = \int_{\mathbf{R}} |f(y)|\,dy < \infty$.

Compute the Fourier coefficients of $\varphi|_{[0,1]}$ as follows, using Lemma 6.4 at one point: $\hat{\varphi}(n) = \int_0^1 \varphi(x)e^{-2\pi inx}\,dx = \int_0^1 \sum_{\mathbf{Z}} f(x+k)e^{-2\pi in(x+k)}\,dx = \int_{\mathbf{R}} f(x)e^{-2\pi inx}\,dx = \hat{f}(n)$. Since $\sum_{\mathbf{Z}} |\hat{f}(n)| < \infty$, φ is the absolutely convergent limit of its Fourier series, proving (1). $\Box$

Remark 6.6 *If f is as in Proposition 6.5, then for every $x, \omega \in \mathbf{R}$,*

$$\sum_{n\in\mathbf{Z}} f(n-x)e^{2\pi i\omega(n-x)} = \sum_{n\in\mathbf{Z}} \hat{f}(n-\omega)e^{-2\pi inx}.$$

Proof: In (1), rewritten with the variable changed to t, that is,

$$\sum_{n\in\mathbf{Z}} f(t+n) = \sum_{n\in\mathbf{Z}} \hat{f}(n)e^{2\pi i n t}$$

replace f by $T_x M_\omega f$ and set $t=0$. You will get

$$\sum_{\mathbf{Z}} f(n-x)e^{2\pi i\omega(n-x)} = \sum_{\mathbf{Z}} T_x M_\omega f(n) = \sum_{\mathbf{Z}} (T_x M_\omega f)\hat{}(n) = \sum_{\mathbf{Z}} \hat{f}(n-\omega)e^{-2\pi i x n},$$

as required. □

Proof of Theorem 3.7: We may assume without loss of generality that $|T| < 1$; otherwise, you can replace $f(x)$ by $f(ax)$ for a suitable $a > 0$. From Lemma 6.4,

$$\int_0^1 \sum_{\mathbf{Z}} 1_T(x+n)\,dx = \int_{\mathbf{R}} 1_T(x)\,dx = |T| < 1 \tag{2}$$

and

$$\int_0^1 \sum_{\mathbf{Z}} 1_\Omega(\omega+n)\,d\omega = |\Omega| < \infty. \tag{3}$$

From (3) it follows that $\sum_{\mathbf{Z}} 1_\Omega(\omega+n)$ is finite for almost every $\omega \in [0,1]$ so that

$$\text{The cardinality of } \{n \in \mathbf{Z} : \hat{f}(\omega+n) \neq 0\} \text{ is finite for almost every } \omega \in [0,1]. \tag{4}$$

From (2) it follows that

$$\text{There is a set } \tilde{T} \subset [0,1] \text{ such that } |\tilde{T}| > 0,\ f(x+n) = 0,\ \forall n \in \mathbf{Z},\ \forall x \in \tilde{T}. \tag{5}$$

(Otherwise, the integral would be equal to 1.)

For each $b \in [0,1]$, let $\varphi_b(x)$ be the "periodization" of $M_b f$, that is, $\varphi_b(x) = \sum_{\mathbf{Z}} f(x+n)e^{2\pi i b(x+n)}$. By Remark 6.6, $\varphi_b(x) = \sum_{\mathbf{Z}} \hat{f}(n-b)e^{2\pi i n x}$. Then by (4), φ_b is a trigonometric polynomial for a. e. $b \in [0,1]$ and by (5), $\varphi_b(x) = 0$ for all $x \in \tilde{T}$ and a. e. $b \in [0,1]$.

However, a trigonometric polynomial cannot vanish on a set of positive measure unless it vanishes identically. Thus $\varphi_b(x) = 0$ for a. e. $x \in \mathbf{R}$ and for a. e. $b \in [0,1]$, so that $\hat{f}(n-b) = 0$ for a. e. $b \in [0,1]$ and all $n \in \mathbf{Z}$. Hence, $\hat{f} = 0$, and $f = 0$. □

7 Monday April 24, 2006—Discrete and Fast Fourier transform

7.1 Discrete Fourier Transform

All proofs in this subsection are straightforward. However, I am including them for the convenience of the lazy reader.

For a function g on $\mathbf{R}$ of period P, the Fourier coefficients are given by $c_k = \frac{1}{P}\int_0^P g(x)e^{i2\pi kx/P}\,dx$. Let's approximate c_k by a Riemann sum: $c_k \sim \frac{1}{P}\sum_0^{N-1} g(x_j)e^{-i2\pi kx_j/P}(P/N)$, where $x_j = jP/N$, $j = 0, \ldots, N-1$. We can thus write

$$c_k \sim \frac{1}{N}\sum_0^{N-1} g(jP/N)e^{-2\pi ijk/N}.$$

This motivates the following definition. Give $\{h_j\}_{j=0}^{N-1} \subset \mathbf{C}$, the discrete Fourier transform of $\{h_j\}_{j=0}^{N-1}$ is the sequence $\{H_k\}_{k\in\mathbf{Z}}$ defined by

$$H_k = \sum_0^{N-1} h_j e^{-i2\pi kj/N}.$$

The discrete Fourier transform satisfies the following three properties: Let H_k be the DFT of h_j and G_k the DFT of g_j.

linearity For scalars a, b, $aH_k + bG_k$ is the DFT of $ah_j + bg_j$.

Proof: Let $k_j = ah_j + bg_j$. Then $K_k = \sum_0^{N-1} k_j e^{-i2\pi kj/N} = \sum_0^{N-1}(ah_j + bg_j)e^{-i2\pi kj/N} = aH_k + bG_k$.

periodicity $H_{k+N} = H_k$ for $k \in \mathbf{Z}$

Proof: Let $W = e^{-i2\pi jk/N}$. Then $H_{k+N} = \sum_0^{N-1} h_j W^{j(k+N)} = \sum_0^{N-1} h_j W^{jk}(W^N)^j = \sum_0^{N-1} h_j W^{jk} = H_k$.

inversion $h_j = \frac{1}{N}\sum_0^{N-1} H_k e^{i2\pi jk/N}$

Proof: With $W = e^{-i2\pi jk/N}$, we have

$$\frac{1}{N}\sum_0^{N-1} H_k e^{i2\pi jk/N} = \frac{1}{N}\sum_0^{N-1} H_k W^{-jk} = \frac{1}{N}\sum_0^{N-1}[\sum_{m=0}^{N-1} h_m W^{mk}]W^{-jk} = \frac{1}{N}\sum_{m=0}^{N-1} h_m[\sum_{k=0}^{N-1} W^{(m-j)k}] = h_j.$$

Given a finite sequence $\{G_k\}_0^{N-1} \subset \mathbf{C}$, the inverse DFT is defined by $g_j = \sum_0^{N-1} G_k e^{i2\pi jk/N}$, for $j \in \mathbf{Z}$. Linearity, periodicity, and inversion also hold for the inverse DFT. The inversion formula is $G_k = \frac{1}{N}\sum_0^{N-1} g_j e^{-i2\pi kj/N}$.

Two other properties of the DFT are

periodicity $H_{N-k} = H_{-k}$ for $k \in \mathbf{Z}$

Proof: This is trivial.

Parseval equality $\sum_0^{N-1} |h_j|^2 = \frac{1}{N}\sum_o^{N-1} |H_k|^2$

Proof:

$$\sum_{j=0}^{N-1} |h_j|^2 = \sum_{j=0}^{N-1}\left|\frac{1}{N}\sum_{k=0}^{N-1} H_k e^{i2\pi jk/N}\right|^2 = \frac{1}{N^2}\left(\sum_{j=0}^{N-1} H_k e^{i2\pi jk/N}\right)\left(\sum_{l=0}^{N-1} \overline{H}_l e^{-i2\pi jl/N}\right)$$

$$= \frac{1}{N^2}\sum_{j=0}^{N-1}\sum_{k,l=0}^{N-1} H_k\overline{H}_l e^{i2\pi j(k-l)/N} = \frac{1}{N^2}\sum_{k,l=0}^{N-1}\left(\sum_{j=0}^{N-1} e^{i2\pi j(k-l)/N}\right) H_k\overline{H}_l$$

$$= \frac{1}{N^2}\sum_{k=0}^{N-1} N|H_k|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|H_k|^2.$$

7.2 Fast Fourier Transform

Let $\{X_k\}$ be the DFT of $\{x_m\}$ so that

$$X_k = \sum_{0}^{N-1} x_m W^{mk} \tag{6}$$

where $W = e^{-i2\pi/N}$.

We now suppose that $N = N_1 N_2$ and note that $0 \le m \le N-1$ if and only if $m = N_1 m_2 + m_1$ where $0 \le m_1 \le N_1 - 1$ and $0 \le m_2 \le N_2 - 1$. Similarly, $0 \le k \le N-1$ if and only if $k = N_2 k_1 + k_2$ where $0 \le k_1 \le N_1 - 1$ and $0 \le k_2 \le N_2 - 1$.

Now we can rewrite (6) as

$$\begin{aligned} X_{N_2k_1+k_2} &= \sum_{m_1=0}^{N-1}\sum_{m_2=0}^{N-1} x_{N_1m_2+m_1} W^{(N_1m_2+m_1)(N_2k_1+k_2)} \\ &= \sum_{m_1=0}^{N-1} W^{N_2m_1k_1} W^{m_1k_2} \sum_{m_2=0}^{N-1} x_{N_1m_2+m_1} W^{N_1m_2k_2} \text{ (since } W^{N_1N_2m_2k_1} = 1) \\ &= \sum_{m_1=0}^{N-1} W_1^{m_1k_1} W^{m_1k_2} \sum_{m_2=0}^{N-1} x_{N_1m_2+m_1} W_2^{m_2k_2}, \end{aligned}$$

where $W_1 = e^{-i2\pi/N_1} = W^{N_2}$ and $W_2 = e^{-i2\pi/N_2} = W^{N_1}$.

Thus, the computation of X_k, where $k = N_2k_1 + k_2$, is accomplished in three steps.

Step 1 For fixed m_1 and k_2, let $Y_{m_1,k_2} = \sum_{m_2=0}^{N-1} x_{N_1m_2+m_1} W_2^{m_2k_2}$. Then

$$\{Y_{m_1,k_2}\}_{k_2=0}^{N_2-1} \text{ is the DFT of } \{x_{N_1m_2+m_1}\}_{m_2=0}^{N_2-1}.$$

Step 2 Multiply Y_{m_1,k_2} by the "twiddle factors" W^{m_1,k_2}.

Step 3 Compute the DFT $\{Z_{k_1}\}_{k_1=0}^{N_1-1}$ of $\{z_{m_1} := W^{m_1k_2} Y_{m_1,k_2}\}_{m_1=0}^{N_1-1}$; of course, $Z_{k_1} = X_{N_2k_1+k_2}$.

We shall now discuss the "complexity issues" associated with the direct DFT and the "3 step"-DFT. The direct computation using (6) involves $(N-1)^2$ multiplications and N^2 additions. In what follows, we shall not bother with the additions[5]. Also, for simplicity, let's call this (at most) N^2 multiplications.

Compare this with the complexity of the "3-step"-DFT. Step 1 consists of taking N_1 DFTs of N_2 terms, hence there are $N_1N_2^2$ multiplications to perform. Step 2 consists of taking N_1N_2 multiplications. Step 3 consists of N_2 DFTs of N_1 terms, hence requires $N_2N_1^2$ multiplications. Thus the direct DFT method requires $N_1^2N_2^2 = (N_1N_2)(N_1N_2)$ multiplications and the "3-step"-DFT requires (only) $N_1N_2(N_1 + N_2 + 1)$ multiplications. For example, for $N_1 = N_2 = 100$, the direct method requires about 50 times the number of multiplications.

Let us now look at the so-called Radix-2 FFT-algorithm. Consider a DFT X_k of an N-point sequence x_m with $N = 2^t$ and let $N_1 = 2$ and $N_2 = 2^{t-1}$. Then, since $W^{N/2} = -1$,

$$X_k = \sum_{m=0}^{N/2-1} x_{2m}W^{2mk} + W^k \sum_{m=0}^{N/2-1} x_{2m+1}W^{2mk} \text{ for } 0 \le k \le N/2 - 1,$$

and

$$X_{k+N/2} = \sum_{m=0}^{N/2-1} x_{2m}W^{2mk} - W^k \sum_{m=0}^{N/2-1} x_{2m+1}W^{2mk} \text{ for } 0 \le k \le N/2 - 1.$$

Here, the computation of an N-point DFT is computed by two DFTs of length $N/2$, namely $\sum_{m=0}^{N/2-1} x_{2m}(W^2)^{mk}$ and $\sum_{m=0}^{N/2-1} x_{2m+1}(W^2)^{mk}$, plus $N/2$ multiplications by W^k. (Again, we are ignoring the additions.)

Now each of these DFTs can be split into two DFTs of length $N/4$ together with $N/4$ multiplications by W^{2k}. Thus, at this stage the N^2 multiplications required by the direct DFT have been replaced by $2(N/2)^2 + 2(N/4)$ multiplications. Repeating this bisection procedure for t steps, and observing that computing a 1-point DFT involves no multiplications at all, you will have in the end replaced N^2 multiplications by $tN/2 = (N/2)\log_2 N$ multiplications. This is the well-known reduction in complexity for the FFT.

The discussion of the fast Fourier transform given here is from [11, chap. 4] (which is reference [51] in [10]). The original paper that introduced the fast Fourier transform (FFT) technique is [3], which is reference 4.6 in [11]. A more recent paper that surveys FFT algorithms is [6] (which is reference [177] in [10]). The papers [3] and [6] can be downloaded from my website: https://math.uci.edu/∼brusso/211as06.html.

[5] Addition is easier than multiplication, even for a computer.

8 Wednesday April 26, 2006—Time-frequency atoms I: Windowed Fourier Transform

8.1 Uncertainty revisited in the context of time-frequency

Let us revisit the uncertainty principle and compare the "Fourier Kingdom" (FK) terminology with a "Time meets Frequency" (TF) version.

(1)
FK: the state of a 1-dimensional particle is a wave function $f \in L^2$
TF: a signal $f \in L^2$ is correlated with a family of waveforms $\{\phi_\gamma\} \subset L^2$ with $\|\phi_\gamma\| = 1$ called *time-frequency atoms*. The *linear time-frequency transform* of $f \in L^2$ is

$$Tf(\gamma) = \int_{\mathbf{R}} f(t)\overline{\phi_\gamma(t)}\, dt$$

We shall study two instances of this. The *windowed Fourier transform* is the case $\gamma = (\xi, u) \in \mathbf{R}^2$ and $\phi_\gamma(t) = g_{\xi,u}(t) = e^{i\xi t} g(t-u)$ for a fixed "window" $g \in L^2$. The *wavelet transform* uses $\gamma = (s, u) \in (0, \infty) \times \mathbf{R}$ and $\phi_\gamma(t) = \psi_{s,u}(t) = \psi((t-u)/s)/\sqrt{s}$ for a fixed "wavelet" $\psi \in L^2$. The corresponding transforms are then[6]

$$Sf(u, \xi) = (f|g)u, \xi) = \int f(t) g(t-u) e^{-i\xi t}\, dt$$

and

$$Wf(u, s) = (f|\psi_{u,s}) = \int f(t) \frac{1}{\sqrt{s}} \overline{\psi\left(\frac{t-u}{s}\right)}\, dt.$$

(2)
FK: the probability density that the particle is located at t is $\|f\|^{-2}|f(t)|^2$ and the average location is $u = \|f\|^{-2} \int t|f(t)|^2\, dt$
TF: $|\phi_\gamma(t)|^2$ is a probability density centered at $u_\gamma = \int t|\phi_\gamma(t)|^2\, dt$

(3)
FK: the probability density that the particle's momentum is equal to ω is $\frac{1}{2\pi\|f\|^2}|\hat{f}(\omega)|^2$ with average momentum $\xi = \frac{1}{2\pi\|f\|^2}| \int \omega|\hat{f}(\omega)|^2\, d\omega$
TF: $\frac{1}{2\pi}|\hat{\phi}_\gamma(\omega)|^2$ is a probability density and the center frequency of $\hat{\phi}_\gamma$ is

$$\xi_\gamma = \frac{1}{2\pi} \int \omega|\hat{\phi}_\gamma(\omega)|^2\, d\omega$$

(4)
FK: the temporal variance around the average location is $\sigma_t^2 = \frac{1}{\|f\|^2} \int (t - u)^2|f(t)|^2\, dt$ and the frequency variance around the average momentum is

$$\sigma_\omega^2 = \frac{1}{2\pi\|f\|^2} \int (\omega - \xi)^2 |\hat{f}(\omega)|^2\, d\omega$$

[6]S is for "short-time Fourier transform

TF: the (time-) spread around u_γ is measured by the variance $\sigma_t^2(\gamma) = \int (t-u_\gamma)^2|\phi_\gamma(t)|^2\,dt$ and the (frequency-) spread around ξ_γ is $\sigma_\omega^2 = \frac{1}{2\pi}\int(\omega - \xi_\gamma)^2|\hat{\phi}_\gamma(\omega)|^2\,d\omega$

(5)

FK: uncertainty principle is $\sigma_t\sigma_\omega \geq 1/2$

TF: uncertainty principle is $\sigma_t(\gamma)\sigma_\omega(\gamma) \geq 1/2$

The time-frequency resolution of ϕ_γ is represented by rectangular *Heisenberg boxes* of area at least 1/2:

$$[u_\gamma - \frac{\sigma_t(\gamma)}{2}, u_\gamma + \frac{\sigma_t(\gamma)}{2}] \times [\xi_\gamma - \frac{\sigma_\omega(\gamma)}{2}, \xi_\gamma + \frac{\sigma_\omega(\gamma)}{2}]$$

8.2 Windowed Fourier transform

We shall assume that the window $g \in L^2$ is real valued and "symmetric" (=even) and is normalized to have norm 1. We observe the following

- The center of the probability density $|\phi_\gamma(t)|^2$ is independent of ξ: $u_\gamma = u$

  proof: $u_\gamma = \int t|g_{u,\xi}(t)|^2\,dt = \int f|g(t-u|^2\,dt = \int t|g(t)|^2\,dt + u\|g\| = 0 + u = u$

- The time spread around u is independent of u and ξ (that is, of γ)

  proof: $\sigma_t^2(\gamma) = \int(t-u)^2|g_{u,\xi}(t)|^2\,dt = \int(t-u)^2|g(t-u)|^2\,dt = \int t^2|g(t)|^2\,dt$.

- The center frequency ξ_γ is independent of u: $\xi_\gamma = \xi$

  proof: $\xi_\gamma = \frac{1}{2\pi}\int\omega|\hat{g}_{u,\xi}(\omega)|^2\,d\omega = \frac{1}{2\pi}\int\omega|\hat{g}(\omega-\xi)|^2\,d\omega = \frac{1}{2\pi}\int(\omega+\xi)|\hat{g}(\omega)|^2\,d\omega = 0 + \xi$ since $\hat{g}$ is also real valued and symmetric.

- The frequency spread around ξ is independent of u and ξ

  proof: $\sigma_\omega^2(\gamma) = \frac{1}{2\pi}\int(\omega-\xi)^2|\hat{g}_{u,\xi}(\omega)|^2\,d\omega = \frac{1}{2\pi}\int(\omega-\xi)^2|\hat{g}(\omega-\xi)|^2\,d\omega = \frac{1}{2\pi}\int\omega^2|\hat{g}(\omega)|^2\,d\omega$

Two examples:

- $f(t) = e^{i\xi_0 t}$.

  Although $f \notin L^2$ we still have $Sf(u,\xi) = e^{-i(\xi-\xi_0)u}\hat{g}(\xi - \xi_0)$, since

  $Sf(u,\xi) = \int f(t)g(t-u)e^{-i\xi t}\,dt = \int g(t-u)e^{-i(\xi-\xi_0)t}\,dt = \int g(t)e^{-i(\xi-\xi_0)(t+u)}\,dt$.

- $f(t) = \delta(t - u_0)$.

  Although f is not a function, we have $Sf(u,\xi) = g(u_0 - u)e^{-i\xi u_0}$, since

  $Sf(u,\xi) = \langle f, g(\cdot - u)e^{-i\xi\cdot}\rangle = \langle \delta(t-u_0), g(t-u)e^{-i\xi t}\rangle = g(u_0 - u)e^{-i\xi u_0}$.

Theorem 8.1 ([10, Theorem 4.1, p. 73]) *For $f \in L^2$,*

$$f(t) = \tfrac{1}{2\pi}\int\int Sf(u,\xi)g(t-u)e^{i\xi t}\,d\xi\,du \quad \text{(inversion)}$$

and

$$\int |f(t)|^2\,dt = \tfrac{1}{2\pi}\int\int |Sf(u,\xi)|^2\,d\xi\,du \quad \text{(conservation of energy).}$$

Proof: We first note that

$$Sf(u,\xi) = e^{-iu\xi}\int f(t)g(t-u)e^{i\xi(u-t)} = e^{-iu\xi}\, f * g_\xi(u)$$

where $g_\xi(t) := g(t)e^{i\xi t}$ (recall that $g(-t) = g(t)$). Since $\hat{g}_\xi(\omega 0 = \int g(t)e^{i\xi t - it(\omega+\xi)}\,dt = \hat{g}(\omega)$, we have $Sf(\cdot,\xi)\hat{}(\omega) = \hat{f}(\omega+\xi)\hat{g}_\xi(\omega+\xi) = \hat{f}(\omega+\xi)\hat{g}(\omega)$, which together with $g(t-\cdot)\hat{}(\omega) = \hat{g}(\omega)e^{-it\omega}$ and Parseval gives

$$\begin{aligned}
\tfrac{1}{2\pi}\int\int Sf(u,\xi)g(t-u)e^{i\xi t}\,d\xi\,du &= \tfrac{1}{2\pi}\int\left(\int Sf(u,\xi)g(t-u)\,du\right)e^{i\xi t}\,d\xi \\
&= \tfrac{1}{2\pi}\int\left(\tfrac{1}{2\pi}\int \hat{f}(\omega+\xi)\hat{g}(\omega)\overline{\hat{g}(\omega)}e^{it\omega+it\xi}\,d\omega\right)d\xi \\
&= \tfrac{1}{2\pi}\int |\hat{g}(\omega)^2\left(\tfrac{1}{2\pi}\int \hat{f}(\omega+\xi)e^{i(\xi+\omega)t}\,d\xi\right)d\omega = f(t)
\end{aligned}$$

where we have made the assumption that $\hat{f} \in L^1$ to use Fubini and Fourier inversion.

Exercise 7 *Complete the proof of the inversion formula by showing that the set of functions in L^2 for which the inversion formula holds almost everywhere is a closed set.*

The proof of energy conservation is straightforward: $\frac{1}{2\pi}\int\int |Sf(u,\xi)|^2\,d\xi\,du = \frac{1}{2\pi}\int\frac{1}{2\pi}\int|\hat{f}(\omega+\xi)\hat{g}(\omega)|^2\,d\omega\,d\xi = \frac{1}{2\pi}\int\frac{1}{2\pi}\int|\hat{f}(\omega+\xi)|^2\,d\xi)|\hat{g}(\omega)|^2\,d\omega = \|f\|^2\|g\|^2 = \|f\|^2$. □

Proposition 8.2 ([10, Proposition 4.1, p. 74]) *Let $\Phi \in L^2(\mathbf{R}^2)$. There exists $f \in L^2$ such that $Sf = \Phi$ if and only if for a. e. $(u_0,\xi_0) \in \mathbf{R}^2$,*

$$\Phi(u_0,\xi_0) = \tfrac{1}{2\pi}\int\int \Phi(u,\xi)(g_{u,\xi}|g_{u_0,\xi_0})\,du\,d\xi.$$

Proof: Suppose first that such f exists. Then

$$\begin{aligned}\Phi(u_0,\xi_0) &= Sf(u_0,\xi_0) = \int f(t)g(t-u_0)e^{-i\xi_0 t}\,dt \\ &= \tfrac{1}{2\pi}\int\left(\int\int Sf(u,\xi)g_{u,\xi}(t)\,du\,d\xi\right)\overline{g_{u_0,\xi_0}(t)}\,dt \\ &= \tfrac{1}{2\pi}\int\int \Phi(u,\xi)\left(\int g_{u,\xi}(t)\overline{g_{u_0,\xi_0}(t)}\,dt\right)du\,d\xi.\end{aligned}$$

Conversely, suppose the condition on Φ holds. Then define f by

$$f(t) = \tfrac{1}{2\pi}\int\int \Phi(u,\xi)g(t-u)e^{i\xi t}\,d\xi$$

Exercise 8 *Show that f is well defined and belongs to L^2.*

Assuming the result of Exercise 8, we have $Sf(u_0,\xi_0) = \int f(t)\overline{g_{u_0,\xi_0}(t)}\,dt = \int\left(\frac{1}{2\pi}\int\int\Phi(u,\xi)g_{u,\xi}(t)\,d\xi\,du\right)\overline{g_{u_0,\xi_0}(t)}\,dt = \Phi(u_0,\xi_0)$. □

9 Thursday April 27, 2006—Time-frequency atoms II: Real and Analytic Wavelets

9.1 Real Wavelets

A *wavelet* is defined to be a function $\psi \in L^2$ with $\|\psi\| = 1$ and $\int \psi(t)\,dt = 0$. It is also supposed to be "centered in a neighborhood of zero" but I haven't yet figured out what exactly that means.[7] The *wavelet transform* of a signal $f \in L^2$ is defined to be

$$Wf(u,s) = (f|\psi_{u,s}) = \int f(t)\overline{\psi_{u,s}(t)}\,dt = \int f(t)\frac{1}{\sqrt{s}}\overline{\psi\left(\frac{t-u}{s}\right)}\,dt,$$

where $\psi_{u,s}(t) = \psi((t-u)/s)/\sqrt{s}$ for $s > 0$ so that $\|\psi_{u,s}\| = 1$.

We first note these two facts.

- $Wf(u,s) = f * \psi_s^*(u)$, where ψ_s denotes $\psi_{0,s}$, since

$$f * \psi_s^*(u) = \int f(t)\psi_s^*(u-t)\,dt = \int f(t)\frac{1}{\sqrt{s}}\overline{\psi((-u+t)/s)}\,dt.$$

- $\widehat{\psi_s^*}(\omega) = \sqrt{s}\,\overline{\widehat{\psi_s}}(s\omega)$, since

$$\widehat{\psi_s^*}(\omega) = \int \psi_s^*(t)e^{-it\omega}\,dt = \int \frac{1}{\sqrt{s}}\overline{\psi(-t/s)}e^{-it\omega}\,dt = \sqrt{s}\int\overline{\psi(t)}e^{-it\omega}\,dt.$$

[7]Stay tuned!

Theorem 9.1 ([10, Theorem 4.3, p. 81]) *Let ψ be a real-valued wavelet and suppose that $C_\psi := \int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega < \infty$. Then for all $f \in L^2$*

$$f(t) = \frac{1}{C_\psi} \int_0^\infty \int Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \, \frac{ds}{s^2} \tag{7}$$

and

$$\int |f(t)|^2 \, dt = \frac{1}{C_\psi} \int_0^\infty \int |Wf(u,s)|^2 \, du \, \frac{ds}{s^2}.$$

Proof: We first prove that energy is conserved.

$$\begin{aligned}
\int_0^\infty \int |Wf(u,s)|^2 \, du \frac{ds}{s^2} &= \int_0^\infty \int |f * \psi_s^*(u)|^2 \, du \frac{ds}{s^2} \\
&= \int_0^\infty \tfrac{1}{2\pi} \int |(f * \psi_s^*)\hat{}(\omega)|^2 \, d\omega \frac{ds}{s^2} \\
&= \int_0^\infty \tfrac{1}{2\pi} \int |\hat{f}(\omega)\widehat{\psi_s^*}(\omega)|^2 \, d\omega \frac{ds}{s^2} \\
&= \int_0^\infty \tfrac{1}{2\pi} \int |\hat{f}(\omega)\sqrt{s}\hat{\psi}(s\omega)|^2 \, d\omega \frac{ds}{s^2} \\
&= \tfrac{1}{2\pi} \int |\hat{f}(\omega)|^2 \, d\omega \int_0^\infty \frac{|\hat{\psi}(s\omega)|^2}{s} \, ds = \|f\|^2 C_\psi.
\end{aligned}$$

Let $b(t)$ denote the right side of (7), so that $b(t) = C_\psi^{-1} \int_0^\infty f * \psi_s^* * \psi_s(t) \, \frac{ds}{s^2}$

Exercise 9 *Show (rigorously) that b exists and that $\hat{b}(\omega) = C_\psi^{-1} \int_0^\infty \hat{f}(\omega)\widehat{\psi_s^*}(\omega)\widehat{\psi_s}(\omega) \, \frac{ds}{s^2}$.*

Assuming Exercise 9, we now have

$$\hat{b}(\omega) = C_\psi^{-1} \int_0^\infty \hat{f}(\omega)\sqrt{s}\hat{\psi}(s\omega)\overline{\sqrt{s}\hat{\psi}(\omega)} \, \frac{ds}{s^2} = C_\psi^{-1} \hat{f}(\omega) \int_0^\infty |\hat{\psi}(s\omega)|^2 \, \frac{ds}{s} = \hat{f}(\omega).$$

In the last step we have used the assumption that ψ is real so that $\hat{\psi}$ is even.
□

9.2 Analytic Wavelets

A signal $f \in L^2$ is said to be *analytic* if its Fourier transform vanishes at negative frequencies: $\hat{f}(\omega) = 0$ for $\omega < 0$.

To motivate the following definition of analytic part of an arbitrary signal, note that for any signal f,

$$\widehat{\mathrm{Re}\, f}(\omega) = \begin{cases} \hat{f}(\omega)/2 & \omega \geq 0 \\ \overline{\hat{f}(-\omega)}/2 & \omega < 0. \end{cases} \tag{8}$$

Hence if f is analytic, then $\hat{f}(\omega) = 2(\mathrm{Re}\, f)\hat{}$ if $\omega \geq 0$ (and $\hat{f}(\omega) = 0$, by definition, if $\omega < 0$).

We thus define the *analytic part* f_a of an arbitrary signal $f \in L^2$ to be the inverse Fourier transform of $2\hat{f}1_{[0,\infty)}$, that is, $\hat{f}_a = 2\hat{f}1_{[0,\infty)}$.

Example: if $f(t) = a\cos(\omega_0 t + \phi)$, with $\omega_0 > 0$, then $f_a(t) = ae^{i(\omega_0 t+\phi)}$.

Proof: (In this example, the signal is not in L^2 so we treat it as a distribution.) We know that $\hat{f}(\omega) = \pi a[e^{i\phi}\delta(\omega - \omega_0) + e^{-i\phi}\delta(\omega + \omega_0)]$. We calculate

$$\begin{aligned}
\langle \hat{f}_a, \varphi\rangle &= \langle 2\hat{f}1_{[0,\infty)}, \varphi\rangle = \langle \hat{f}, 2\,1_{[0,\infty)}\varphi\rangle \\
&= \langle \pi a[e^{i\phi}\delta(\omega - \omega_0) + e^{-i\phi}\delta(\omega + \omega_0)], 2\,1_{[0,\infty)}\varphi\rangle \\
&= 2\pi a e^{i\phi}\,1_{[0,\infty)}(\omega_0)\varphi(\omega_0) + 2\pi a e^{-i\phi}\,1_{[0,\infty)}(-\omega_0)\varphi(-\omega_0) \\
&= 2\pi a e^{i\phi}\varphi(\omega_0) = 2\pi a e^{i\phi}\langle \delta(\omega - \omega_0), \varphi\rangle
\end{aligned}$$

so that $\hat{f}_a(\omega) = 2\pi a e^{i\phi}\delta(\omega - \omega_0)$ and $f_a(t) = ae^{i(\omega_0 t+\phi)}$. □

Theorem 9.2 ([10, Theorem 4.4, p. 86]) *If ψ is an analytic wavelet, then for all $f \in L^2$,*

(a) $Wf(u,s) = \frac{1}{2}Wf_a(u,s)$

(b) *If f is real-valued, then*

$$f(t) = \frac{2}{C_\psi} Re \int_0^\infty \int Wf(u,s)\psi_s(t-u)\,du\,\frac{ds}{s^2}$$

and

$$\|f\|^2 = \frac{2}{C_\psi} Re \int_0^\infty \int |Wf(u,s)|^2\,du\,\frac{ds}{s^2}.$$

Proof: We have $Wf(\cdot,s)\hat{}(\omega) = (f * \psi_s^*)\hat{}(\omega) = \hat{f}(\omega)\widehat{\psi_s^*}(\omega) = \hat{f}(\omega)\overline{\sqrt{s}\hat{\psi}(\omega)}$ and $Wf_a(\cdot,s)\hat{}(\omega) = (f_a * \psi_s^*)\hat{}(\omega) = 2\hat{f}(\omega)1_{[0,\infty)}(\omega)\overline{\sqrt{s}\hat{\psi}(\omega)}$. Since ψ is analytic, $Wf(\cdot,s)\hat{}(\omega) = Wf_a(\cdot,s)\hat{}(\omega)/2$ and (a) follows.

As in the proof of Theorem 9.1, the inverse wavelet formula reconstructs[8] the analytic part of f. Thus

$$f_a(t) = \frac{1}{C_\psi}\int_0^\infty \int Wf_a(u,s)\psi_s(t-u)\,du\,\frac{ds}{s^2},$$

and since $f = \mathrm{Re}\, f_a$ the reconstruction formula is proved. To see that $f = \mathrm{Re}\, f_a$, note that by (8),

$$\widehat{\mathrm{Re}\, f_a}(\omega) = \begin{cases} \hat{f}_a(\omega)/2 & \omega \geq 0 \\ \overline{\hat{f}_a(-\omega)}/2 & \omega < 0 \end{cases} = \begin{cases} \hat{f}(\omega) & \omega \geq 0 \\ \overline{\hat{f}(-\omega)} & \omega < 0. \end{cases}$$

[8] We don't need to have a real wavelet here since $\hat{f}_a$ vanishes on negative frequencies. (See the last remark in the proof of Theorem 9.1)

However, since f is real-valued $\overline{\hat{f}(-\omega)} = \hat{f}(\omega)$ for all ω and so indeed we do have $\widehat{\operatorname{Re} f_a} = \hat{f}$.

We can now prove the conservation of energy. We note first that energy conservation holds[9] for f_a, that is,

$$\|f_a\|^2 = \int |f_a(t)|^2\,dt = \frac{1}{C_\psi}\int_0^\infty \int |Wf_a(u,s)|^2\,du\,\frac{ds}{s^2} = \frac{4}{C_\psi}\int_0^\infty \int |Wf(u,s)|^2\,du\,\frac{ds}{s^2}$$

It remains to show that $\|f_a\|^2 = 2\|f\|^2$. For this, we use again the fact that f is real so that $\overline{\hat{f}(-\omega)} = \hat{f}(\omega)$ for all ω and therefore $\|\hat{f}1_{[0,\infty)}\| = \|\hat{f}\|/2$ to get $\|f_a\|^2 = \frac{1}{2\pi}\|\hat{f}_a\|^2 = \frac{1}{2\pi}\|2\hat{f}1_{[0,\infty)}\|^2 = \frac{1}{\pi}\|\hat{f}\|^2 = 2\|f\|^2$. □

10 Monday May 1, 2006—Frames

The discrete windowed Fourier transform and the discrete wavelet transform are best studied through the frame formalism. Recall from Proposition 6.2 that if the support of $\hat{f}$ is contained in the interval $[-\pi/T, \pi/T]$, then the signal f can be recovered at regularly sampled points by taking inner product with a family of vectors, in this case $\{h_T(t-nT)\}_{n\in\mathbf{Z}}$, where $h_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}$. This motivated the discovery of conditions under which one can recover a vector f in Hilbert space from its inner products with a family of vectors $\{\phi_n\}_{n\in\Gamma}$.

A sequence $\{\phi_n\}_{n\in\Gamma}$ of vectors in a Hilbert space H is a *frame* if there exist positive constants A and B such that for all $f \in H$,

$$A\|f\|^2 \le \sum_{n\in\Gamma} |(f|\phi_n)|^2 \le B\|f\|^2. \tag{9}$$

The frame is *tight* if $A = B$, and the *frame operator* U is the linear map from H to $\ell^2(\Gamma)$ defined by $Uf = \{(f|\phi_n)\}_{n\in\Gamma}$.

Examples (elementary)

- If e_1, e_2 is an orthonormal basis of a two-dimensional Hilbert space, then $\{\phi_1, \phi_2, \phi_3\}$ is a tight frame with $A = B = 3/2$, where

  $$\phi_1 = e_1,\ \phi_2 = (-1/2)e_1 + (\sqrt{3}/2)e_2,\ \phi_3 = (-1/2)e_1 - (\sqrt{3}/2)e_2.$$

  Proof: Let $w = (f|e_1)$ and $z = (f|e_2)$. Then $\sum_1^3 |(f|\phi_n)|^2 = |w|^2 + |-(1/2)w + (\sqrt{3}/2)z|^2 + |-(1/2)w - (\sqrt{3}/2)z|^2 = |w|^2 + 2|(1/2)w|^2 + 2|(\sqrt{3}/2)z|^2$ (parallelogram law) $= |(f|e_1)|^2 + (1/2)|(f|e_1)|^2 + (3/2)|(f|e_2)|^2 = (3/2)\|f\|^2$.

[9] Again, instead of ψ being real-valued, which it isn't, we rely on f_a being analytic—see the previous footnote

- For each $k = 0, 1, \dots K-1$, let $\{e_{k,n}\}_{n\in\mathbf{Z}}$ be an orthonormal basis of a Hilbert space H. Then $\cup_{k=0}^{K-1}\{e_{k,n}\}_{n\in\mathbf{Z}}$ is a tight frame with $A = B = K$.

  Proof: For each k and $f \in H$, $\|f\|^2 = \sum_{n\in\mathbf{Z}} |(f|e_{k,n})|^2$. Hence $K\|f\|^2 = \sum_{k=0}^{K-1}\sum_{n\in\mathbf{Z}} |(f|e_{k,n})|^2$.

- Any finite set of vectors $\phi, \dots, \phi_n$ in a Hilbert space H is a frame for the subspace V they generate.

  Proof: Let us just note that you can take $B = \sum_m \sum_j |(\phi_m|\psi_j)|^2$ where $\psi_1, \dots, \psi_k$ is an orthonormal basis for V and $A^{-1} = \sum_j \sum_m |a_{jm}|^2$ where the numbers a_{jm} are defined by $\psi_j = \sum_m a_{jm}\phi_m$. (See [10, Problem 5.8] which claims that there is no control on the constants A and B.)

We shall use the notion of pseudo-inverse to reconstruct f from its frame coefficients. We shall use the following (unusual) definition of linear independence. A sequence of vectors in a Hilbert space is *linearly independent* if whenever $\sum \alpha_n e_n = 0$ and $\sum |\alpha_n|^2 < \infty$, then $\alpha_n = 0$ for every n.

Proposition 10.1 ([10, Prop. 5.1, p. 128]) *If $\{\phi_n\}_{n\in\Gamma}$ is a frame which is linearly dependent, then $U(H) \neq \ell^2(\Gamma)$ and U admits infinitely many left inverses.*

Proof: There is a vector $x \in \ell^2(\Gamma)$, $x \neq 0$ with $\sum x[n]\phi_n = 0$. Then for $f \in H$, $(Uf|x) = \sum Uf[n]\overline{x[n]} = \sum(f|\phi_n)\overline{x[n]} = (f|\sum x[n]\phi_n) = 0$. This shows that $U(H)$ is not dense in $\ell^2(\Gamma)$.

By the frame inequality (9), U is one-to-one. Let $U_1 : U(H) \to H$ be the inverse of U and let $U_2 : [U(H)]^\perp \to H$ be any operator. Then $\overline{U}^{-1} : \ell^2(\Gamma)$ defined by $x = x_1 + x_2 \mapsto U_1x_1 + U_2x_2$ is a left inverse for U, where $x_1 \in U(H)$ and $x_2 \in [U(H)]^\perp$. □

The *pseudo-inverse* of U, denoted by $\tilde{U}^{-1}$, is the left inverse in Proposition 10.1 corresponding to $U_2 = 0$.

Theorem 10.2 ([10, Theorem 5.1, p. 128]) *If U is a frame operator with frame bounds A and B, then U^*U is invertible, and the pseudo-inverse $\tilde{U}^{-1}$ of U satisfies*

(a) $\tilde{U}^{-1} = (U^*U)^{-1}U^*$.

(b) $\|\tilde{U}^{-1}\| \leq \|\overline{U}^{-1}\|$

(c) $\|\tilde{U}^{-1}\| \leq A^{-1/2}$

Proof: If $f \in H$ and $U^*Uf = 0$, then $\|Uf\|^2 = (U^*Uf|f) = 0$ and so $f = 0$ and U^*U is one-to-one. If $g \in [(U^*U)H]^\perp$, then $(g|U^*Ug) = 0$ and again $g = 0$, so U^*U has dense range. Since $\|(U^*U)f\| = \sup_{\|g\|\leq 1} |(U^*Uf|g)| \geq (U^*Uf|f/\|f\|) = \|Uf\|^2/\|f\| \geq A\|f\|$, U^*U has closed range and is therefore invertible. We now prove (a)-(c).

(a) Since U^*U is invertible, it suffices to prove that $U^*U\tilde{U}^{-1} = U^*$. So let $x \in \ell^2(\Gamma)$. If $x \in [U(H)]^\perp$, then $\tilde{U}^{-1}x = 0$ and for all $f \in H$, $0 = (Uf|x) = (f|U^*x)$ so $U^*x = 0$ and $U^*U\tilde{U}^{-1}x = 0$ as well. On the other hand, if $x \in U(H)$, say $x = Uf$, then $U\tilde{U}^{-1}x = U(\tilde{U}^{-1}Uf) = Uf = x$ and so $U^*U\tilde{U}^{-1}x = U^*x$, as required.

(b) If $x \in \ell^2(\Gamma)$, with $x = x_1 + x_2 \neq 0$ and $x_1 \in U(H)$ and $x_2 \in [U(H)]^\perp$, then

$$\frac{\|\tilde{U}^{-1}x\|}{\|x\|} = \frac{\|\tilde{U}^{-1}x_1\|}{\|x\|} = \frac{\|\overline{U}^{-1}x_1\|}{\|x\|} \leq \frac{\|\overline{U}^{-1}x_1\|}{\|x_1\|} \leq \|\overline{U}^{-1}|_{U(H)}\| \leq \|\overline{U}^{-1}\|.$$

Thus $\|\tilde{U}^{-1}\| = \sup_{x\neq 0} \frac{\|\tilde{U}^{-1}x\|}{\|x\|} \leq \|\overline{U}^{-1}\|$.

(c) For $x_1 \in U(H)$, pick $f \in H$ with $Uf = x_1$. Then for any $x \in \ell^2(\Gamma)$, with $x = x_1 + x_2$, $\tilde{U}^{-1}x = \tilde{U}^{-1}x_1 = \tilde{U}^{-1}Uf = f$, and by the frame inequality (9), $\|\tilde{U}^{-1}x\| = \|f\| \leq A^{-1/2}\|Uf\| = A^{-1/2}\|x_1\| \leq A^{-1/2}\|x\|$.

□

11 Thursday May 4, 2006—Riesz bases

11.1 Dual frame

Theorem 11.1 ([10, Theorem 5.2, p. 129]) *If $\{\phi_n\}_{n\in\Gamma}$ is a frame for H with frame bounds A, B, then $\{\tilde{\phi}_n\}$ defined by $\tilde{\phi}_n = (U^*U)^{-1}\phi_n$ is also a frame, called the dual frame, with frame bounds B^{-1}, A^{-1}. Moreover for every $f \in H$,*

$$f = \sum_n (f|\phi_n)\tilde{\phi}_n = \sum_n (f|\tilde{\phi}_n)\phi_n.$$

If $A = B$, then $\tilde{\phi}_n = A^{-1}\phi_n$.

Proof:

- If $x \in \ell^2(\Gamma)$ and $f \in H$, then $(U^*x|f) = (x|Uf) = \sum_n x[n]\overline{(f|\phi_n)} = (\sum_n x[n]\phi_n|f)$, so[10] that $U^*x = \sum_n x[n]\phi_n$.
- $\tilde{U}^{-1}x = (U^*U)^{-1}U^*x = (U^*U)^{-1}\sum_n x[n]\phi_n = \sum_n x[n]\tilde{\phi}_n$ so that for every $f \in H$, $f = \tilde{U}^{-1}Uf = \sum_n Uf[n]\tilde{\phi}_n = \sum_n (f|\phi_n)\tilde{\phi}_n$ which proves one of the assertions of the theorem.
- For every $f, g \in H$, by the preceding bullet, $(f|g) = \sum_n (f|\phi_n)(\tilde{\phi}_n|g) = (f|\sum_n (g|\tilde{\phi}_n)\phi_n)$, so that $g = \sum_n (g|\tilde{\phi}_n)\phi_n$ which proves another assertion of the theorem.

[10] For the last step we need to know that the series $\sum_n x[n]\phi_n$ converges. This will certainly be so if $\sup_n \|\phi_n\| < \infty$

- By the first bullet, $U^*Uf = \sum_n (f|\phi_n)\phi_n$ so that $(U^*Uf|f) = \sum_n |(f|\phi_n)|^2$ and by the frame inequality $A\|f\|^2 \le (U^*Uf|f) \le B\|f\|^2$. This says that $A \cdot I \le U^*U \le B \cdot I$. Let us also note for use in the next bullet that by the functional calculus for the positive operator U^*U, we have $B^{-1} \cdot I \le (U^*U)^{-1} \le A^{-1} \cdot I$. If $A = B$, then $U^*U = A \cdot I$ so that $\tilde{\phi}_n = A^{-1}\phi$, which proves another assertion.

- Finally, $(U^*U)^{-1}f = (U^*U)^{-1}\sum_n (f|\tilde{\phi}_n)\phi_n = \sum_n (f|\tilde{\phi}_n)\tilde{\phi}_n$, so that $((U^*U)^{-1}f|f) = (\sum_n (f|\tilde{\phi}_n)\tilde{\phi}_n|f) = \sum_n |(f|\tilde{\phi}_n)|^2$. From $B^{-1}\|f\|^2 \le ((U^*U)^{-1}f|f) \le A^{-1}\|f\|^2$ we see that $\{\tilde{\phi}_n\}$ is a frame with frame bounds B^{-1}, A^{-1}. □

11.2 Riesz bases

A frame $\{\phi_n\}_{n\in\Gamma}$ which is linearly independent is a *Riesz basis*. Linear independence is interpreted here in the following sense: if $\{a[n]\} \in \ell^2(\Gamma)$ and $\sum_n a[n]\phi_n = 0$, then $a[n] = 0$ for every $n \in \Gamma$.

Some facts about Riesz bases.

- $U(H) = \ell^2(\Gamma)$ (**Proof:** If $x \in [U(H)]^\perp$, then for all $f \in H$, $0 = (Uf|x) = \sum_n (f|\phi_n)\overline{x[n]} = (f|\sum_n x[n]\phi_n)$, so by linear indepdence, $x = 0$.)

- $(\phi_p|\tilde{\phi}_n) = \delta_{n,p}$, where $\{\tilde{\phi}_n\}$ is the dual frame. (**Proof:** By Theorem 11.1, $\phi_p = \sum_n (\phi_p|\tilde{\phi}_n)$; now use the linear independence.)

- The dual frame of a Riesz basis is itself a Riesz basis. (**Proof:** If $\sum_n c_n\tilde{\phi}_n = 0$ with $\{c_n\} \in \ell^2(\Gamma)$, then $0 = (\sum_n c_n\tilde{\phi}_n|\phi_p) = \sum_n c_n(\tilde{\phi}_n|\phi_p) = c_p$.)

- If $\|\phi_n\| = 1$ for every n, then $A \le 1 \le B$. (**Proof:** By Theorem 11.1, $B^{-1}\|\phi_p\|^2 \le \sum_n |(\phi_p|\tilde{\phi}_n)|^2 \le A^{-1}\|\phi_p\|^2$, so by the second bullet, $B^{-1} \le 1 \le A^{-1}$.)

Partial Reconstruction: If ϕ_n is a frame for a subspace V of the signal space H, then $f \in H$ cannot be recovered from the data $\{(f|\phi_n)\}$. However, its projection P_Vf can, since if U denotes the frame operator for the frame ϕ_n in V, then $P_Vf = \tilde{U}^{-1}UP_Vf = \sum_n (P_Vf|\phi_n) = \sum_n (f|\phi_n)$.

Proposition 11.2 ([10, Prop. 7.1, p. 222]) *If $\theta \in L^2$, then $\{\theta(\cdot - n)\}_{n\in\mathbf{Z}}$ is a Riesz basis of the space V_0 that it generates if and only if there exist positive constants A, B, such that for a. e. $\omega \in [-\pi, \pi]$,*

$$\frac{1}{B} \le \sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega - 2k\pi)|^2 \le \frac{1}{A} \tag{10}$$

Proof: By definition, $V_0 = \{\sum_n a[n]\theta(\cdot - n) : \sum |a[n]|^2 < \infty\}$. The proof in both directions is based on the following formula for $f \in V_0$. If $f = \sum_n a[n]\theta(\cdot - n)$ and $g \in L^2[0, 2\pi]$ is such that $\hat{g}(n) = a[n]$, then

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |g(\omega)|^2 \left(\sum_{k=-\infty}^{\infty} \hat{\theta}(\omega + 2k\pi)|^2 \right) d\omega \tag{11}$$

Assume first that the constants A, B exist such that (10) holds. Then from (11),

$$\frac{1}{2\pi B} \int_0^{2\pi} |g(\omega)|^2 \, d\omega \leq \|f\|^2 \leq \frac{1}{2\pi A} \int_0^{2\pi} |g(\omega)|^2 \, d\omega.$$

Since $\frac{1}{2\pi} \int_0^{2\pi} |g(\omega)|^2 \, d\omega = \sum_n |a[n]|^2$, this shows that

$$A\|f\|^2 \leq \sum_n |a[n]|^2 \leq B\|f\|^2, \tag{12}$$

so that $\{\theta(\cdot - n)\}_{n \in \mathbf{Z}}$ is a frame for V_0. The linear independence follows from (12), since if $f = 0$ then $a[n] = 0$ for every n.

Conversely, suppose that $\{\theta(\cdot - n)\}_{n \in \mathbf{Z}}$ is a Riesz basis for V_0 and suppose that (10) is false. Let

$$S_\epsilon = \{\omega \in [0, 2\pi] : \frac{1}{B} - \epsilon \geq \sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega - 2k\pi)|^2\}$$

and

$$T_\epsilon = \{\omega \in [0, 2\pi] : \frac{1}{A} + \epsilon \leq \sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega - 2k\pi)|^2\}$$

One of these sets has positive measure. Suppose it is S_ϵ, let $g_0 = 1_{S_\epsilon}$ and let $a[n] = \hat{g}_0(n)$ so that $f_0 = \sum_n a[n]\theta(\cdot - n) \in V_0$. From (11),

$$\|f_0\|^2 \leq \frac{1}{2\pi} \int_{S_\epsilon} |g_0(\omega)|^2 \, d\omega (\frac{1}{B} - \epsilon) = \sum_n |a[n]|^2 (\frac{1}{B} - \epsilon),$$

which contradicts the frame inequality.

It remains to prove (11). To be honest, I can only prove the slightly weaker statement

$$\|f\|^2 = \lim_{|N| \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\sum_{-N}^{N} a[n] e^{-in\omega}|^2 \left(\sum_{k=-\infty}^{\infty} \hat{\theta}(\omega + 2k\pi)|^2 \right) d\omega. \tag{13}$$

However, this suffices in the above argument. Before continuing, let us put the proof of (11) as a formal exercise.

Exercise 10 *Prove (11) if it is correct.*

We now prove (13), which as noted above, suffices to complete the proof of the proposition.

Let $f_N = \sum_{-N}^{N} a[n]\theta(\cdot - n)$, so that $f_N \to f$ in L^2-norm and $\widehat{f_N} \to \hat{f}$ in L^2-norm as $|N| \to \infty$. Since $f_N \in L^2$ (and not necessarily in L^1) we compute its Fourier transform by approximation, namely, letting $f_N^{(M)} := 1_{[-M,M]} f_N \in L^1 \cap L^2$ so that $f_N^{(M)}$ converges to f_N in L^2 as $|M| \to \infty$, and $\hat{f} = L^2\text{-}\lim_{|N|\to\infty}(L^2\text{-}\lim_{|M|\to\infty} \widehat{f_N^{(M)}})$. Now

$$\widehat{f_N^{(M)}}(\omega) = \int_{-M}^{M} \sum_{-N}^{N} a[n]\theta(t-n)e^{-it\omega}\,dt = \sum_{-N}^{N} a[n]e^{-in\omega} \int_{-M-n}^{M-n} \theta(t)e^{-it\omega}\,dt$$

so that in addition to

$$\hat{\theta}(\omega) = L^2\text{-}\lim_{|M|\to\infty} \int_{-M-n}^{M-n} \theta(t)e^{-it\omega}\,dt \tag{14}$$

we have

$$\widehat{f_N}(\omega) = L^2\text{-}\lim_{|M|\to\infty} \left(\sum_{-N}^{N} a[n]e^{-in\omega} \int_{-M-n}^{M-n} \theta(t)e^{-it\omega}\,dt \right). \tag{15}$$

By considering subsequences that coverge in both (14) and (15) almost everywhere, we have for a. e. ω,

$$\widehat{f_N}(\omega) = \left(\sum_{-N}^{N} a[n]e^{-in\omega} \right) \hat{\theta}(\omega). \tag{16}$$

Finally,

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi}\|\hat{f}\|^2 = \lim_{|N|\to\infty} \frac{1}{2\pi}\|\widehat{f_N}\|^2 = \lim_{|N|\to\infty} \frac{1}{2\pi} \int |\sum_{-N}^{N} a[n]e^{-in\omega}|^2 |\hat{\theta}(\omega)|^2\,d\omega \\ &= \lim_{|N|\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} |\sum_{-N}^{N} a[n]e^{-in\omega}|^2 |\hat{\theta}(\omega + 2k\pi)|^2\,d\omega \\ &= \lim_{|N|\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |\sum_{-N}^{N} a[n]e^{-in\omega}|^2 \left(\sum_{k=-\infty}^{\infty} |\hat{\theta}(\omega+2k\pi)|^2 \right) d\omega, \text{ proving (13).} \square \end{aligned}$$

12 Monday May 8, 2006—Multiresolutional Approximation; scaling function

12.1 Multiresolution Approximation

A *multiresolution approximation* is a sequence $\{V_j : j \in \mathbf{Z}\}$ of subspaces $V_j \subset L^2$ with the following six properties.

1. $\forall (j,k) \in \mathbf{Z}^2,\ f(t) \in V_j \Leftrightarrow f(t-2^j n) \in V_j$

   ("V_j is invariant by any translation proportional to the scale 2^j")

2. $\forall j \in \mathbf{Z},\ V_{j+1} \subset V_j$

   ("causality property; V_{j+1} gives a coarser resolution that V_j")

3. $\forall j \in \mathbf{Z},\ f(t) \in V_j \Leftrightarrow f(t/2) \in V_{j+1}$

   ("dilating by 2 enlarges details and defines an approximation at a coarser resolution")

4. $\cap_{j=-\infty}^{\infty} V_j = \{0\}$

   ("we lose all details of f when the resolution $2^{-j} \to 0$; $\lim_{j\to\infty} P_{V_j} f = 0$")

5. $\cup_{j=-\infty}^{\infty} V_j$ is dense in L^2

   ("when the resolution $2^{-j} \to \infty$ the signal approximation converges to the original signal; $\lim_{j\to -\infty} \|f - P_{V_j} f\| = 0$")

6. $\exists\ \theta \in L^2$ such that $\{\theta(t-n) : n \in \mathbf{Z}\}$ is a Riesz basis of V_0.

   ("provides a discretization theorem; θ is interpreted as a 'unit resolution cell' "—By property 3, $\{2^{-j/2}\theta(2^{-j}t - n) : n \in \mathbf{Z}\}$ is a Riesz basis for V_j with the same frame bounds)

12.2 Remark on Exercise 10

In (16) we know that there is a subsequence $\{N_k\}$ such that $\widehat{f_{N_k}} \to \hat{f}$ almost everywhere and $\sum_{-N_k}^{N_k} a[n]e^{-in\omega} \to g$ almost everywhere. [11] Hence passing to the limit in (16) results in $\hat{f}(\omega) = g(\omega)\hat{\theta}(\omega)$ almost everywhere. From this, (11) follows easily.

12.3 Three examples

Example 1 (piecewise constant MRA): Let

$$V_j = \{g \in L^2 : g \text{ is constant on } [n2^j, (n+1)2^j)\ \forall n \in \mathbf{Z}\}$$

Properties 1,2,3,4 follow immediately by inspection; $\cap_{j\in\mathbf{Z}} V_j$ is the set of constants, hence equal to $\{0\}$; $\cup_{j\in\mathbf{Z}} V_j$ is the set of "dyadic" step functions, hence dense; $\theta = 1_{[0,1)}$ and the $\theta(\cdot - n) = 1_{[n,n+1)}$ constitute an orthonormal basis in V_0.

Example 2 (Shannon MRA): Let

$$V_j = \{g \in L^2 :\ \text{the support of } \hat{g} \subset [-2^{-j}\pi, 2^{-j}\pi]\}$$

[11] By Carleson's theorem, we don't really need to use a subsequence for one of these limits, but we do on the other one; see [10, p. 53]

Property 1 follows from $f(\cdot - 2^j k)\hat{}(\omega) = e^{2^j i\pi\omega}\hat{(}\omega)$.

Property 2 follows from $[-2^{-(j+1)}\pi, 2^{-(j+1)}\pi] \subset [-2^{-j}\pi, 2^{-j}\pi]$

Property 3 follows from $f(t/2)\hat{}(\omega) = 2\hat{f}(2\omega)$

If $f \in V_j$ for all j, then $\hat{f}$ is supported on $\{0\}$, so $\hat{f} = 0$ and $f = 0$, which is property 4.

If $f \in L^2$, then $\hat{f} = L^2\text{-}\lim_{j\to-\infty} 1_{[-\pi/2^j,\pi/2^j]}\hat{f}$. Then letting $f_j \in L^2$ be defined by $\hat{f}_j = 1_{[-\pi/2^j,\pi/2^j]}\hat{f}$, we have $f_j \to f$ in L^2 and the support of $\hat{f}_j \subset [-\pi/2^j, \pi/2^j]$, which is property 5.

By Proposition 6.2, with $\theta(t) = \sin \pi t/\pi t$, $\{\theta(\cdot - n) : n \in \mathbf{Z}\}$ is an orthonormal basis[12] of V_0.

Example 3 (spline MRA): We won't discuss this now as there are not many details in [10]. However, this is an important example which will illustrate some theorems that follow. So we shall probably return to it later.

12.4 Scaling function

Theorem 12.1 ([10, Theorem 7.1, p. 225]) *Let $\{V_j\}$ be a multiresolution approximation with associated "unit resolution cell" θ. Let ϕ be a scaling function for this MRA, that is*[13] *,*

$$\hat{\phi}(\omega) = \hat{\theta}(\omega)/(\sum_{-\infty}^{\infty} |\hat{\theta}(\omega + 2k\pi)|^2). \tag{17}$$

Then $\{\phi_{j,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for V_j for $j \in \mathbf{Z}$, where $\phi_{j,n}(t) := 2^{-j/2}\phi(2^{-j}t - n)$.

Proof: We shall prove that ϕ can be chosen in V_0 such that (17) holds and that $\{\phi_{0,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for V_0. The rest follows from property 3 of the definition of MRA.

By the structure of V_0, we need to first determine the coefficients $a[n]$ such that with $\phi(t) := \sum_n a[n]\theta(t - n)$, both (17) and

$$(\phi(\cdot - n)|\phi(\cdot - p)) = \delta_{n,p} \tag{18}$$

hold. This will show the orthonormality of the set $\{\phi_{0,n}\}_{n\in\mathbf{Z}} \subset V_0$.

Note first that for our provisional ϕ, for any $(n,p) \in \mathbf{Z}^2$

$$(\phi_{0,n}|\phi_{0,p}) = \int \phi(t-n)\overline{\phi(t-p)}\,dt = \phi * \phi^*(p-n).$$

[12]Wait a minute. We still need Exercise 6

[13]Since the denominator $\sum_{-\infty}^{\infty} |\hat{\theta}(\omega + 2k\pi)|^2$ is bounded below (by Proposition 11.2), ϕ exists as an element of L^2

So to prove (18), it suffices to find ϕ such that $\phi * \phi^*(n) = \delta_{n,0}$ for every $n \in \mathbf{Z}$. It is equivalent to prove the equality of the two distributions $\sum_n \phi * \phi^*(n)\delta(t-n)$ and $\sum_n \delta_{n,0}\delta(t-n) = \delta(t)$, the Dirac delta function.

By Proposition 5.2, the Fourier transform of $\sum_n \phi * \phi^*(n)\delta(t-n)$ is $\sum_{-\infty}^{\infty} \widehat{\phi * \phi}^*(\omega + 2k\pi) = \sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2k\pi)|^2$. Thus, we need to find ϕ such that $\sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1$. Since $\hat{\phi}(\omega) = g(\omega)\hat{\theta}(\omega)$ where $g \in L^2[0, 2\pi]$ and $\hat{g}(n) = a[n]$, it suffices to choose $g(\omega) = [\sum_{-\infty}^{\infty} |\hat{\theta}(\omega + 2k\pi)|^2]^{-1/2}$, which achieves (17) and (18) with $\phi \in V_0$.

The completeness of $\{\phi_{0,n}\}_{n\in\mathbf{Z}}$ is now proved as follows. Suppose $\psi \in V_0$ and $\psi \perp \phi_{0,m}$ for every $m \in \mathbf{Z}$. Write $\phi = \sum_n a[n]\theta(\cdot - n)$ and $\psi = \sum_n b[n]\theta(\cdot - n)$. We know that $\hat{\psi}(\omega) = h(\omega)\hat{\theta}(\omega)$, where $h \in L^2[0, 2\pi]$ and $\hat{f}(n) = b[n]$. Similarly, since $\phi_{0,m}(t) = \phi(t - m)$, $\hat{\phi}_{0,m}(\omega) = g(\omega)e^{im\omega}\hat{\theta}(\omega)$ where $g \in L^2[0, 2\pi]$ and $\hat{g}(n) = a[n]$. Then

$$0 = 2\pi(\psi|\phi_{0,m}) = (\hat{\psi}|\hat{\phi}_{0,m}) = \int h(\omega)\overline{g(\omega)}|\hat{\theta}(\omega)|^2 e^{-im\omega}\, d\omega$$

This says that $h(\omega)g(\omega) = 0$ a. e. , and since $g(\omega) = [\sum_{-\infty}^{\infty} |\hat{\theta}(\omega + 2k\pi)|^2]^{-1/2}$, it follows that $h = 0$ a. e. , and hence $\psi = 0$, as required. □

13 Thursday May 11, 2006—Characterization of scaling function

13.1 Properties of scaling function

Theorem 13.1 ([10, Theorem 7.2, p. 229, part I—properties of scaling function]) *Let $\phi \in L^2 \cap L^1$ be a scaling function for a multiresolution approximation $\{V_j, \theta\}$ and let h be the 2π-periodic function with Fourier coefficients $\hat{h}(n) = (2^{-1/2}\phi(t/2)|\phi(t-n))$. Then* [14]

$$|h(\omega)|^2 + |h(\omega + \pi)|^2 = 2, \ \forall \omega \in \mathbf{R} \tag{19}$$

and

$$h(0) = \sqrt{2}. \tag{20}$$

Proof: $\phi = \phi_{0,0} \in V_0$ so that $\phi(t/2) \in V_1$. Thus, there exist $a \in \ell^2$ such that $2^{-1/2}\phi(t/2) = \sum_n a[n]\phi(t-n)$, from which it follows that

$$\hat{\phi}(\omega) = 2^{-1/2}h(\omega/2)\hat{\phi}(\omega/2), \tag{21}$$

where $h \in L^2[0, 2\pi]$ with $\hat{h}(n) = a[n]$. Since for any scaling function ϕ, $\sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1$ for a. e. ω, we have

$$2 \;=\; \sum_{k-\infty}^{\infty} |h(\omega/2 + k\pi)|^2 |\hat{\phi}(\omega/2 + k\pi)|^2$$

[14] $h(0)$ is defined since h is continuous at 0, as shown by (21) and the fact, shown in the proof, that $\hat{\phi}(0) \neq 0$

$$= \sum_{p=-\infty}^{\infty} |h(\omega/2)|^2 |\hat{\phi}(\omega/2 + 2p\pi)|^2 + \sum_{p=-\infty}^{\infty} |h(\omega/2+\pi)|^2 |\hat{\phi}(\omega/2 + \pi + 2p\pi)|^2$$

$$= |h(\omega/2)|^2 + |h(\omega/2+\pi)|^2 \text{ which proves (19) .}$$

To prove (20), all we have to do is prove that $\hat{\phi}(0) \neq 0$. Indeed, once this is done, we appeal to the continuity of $\hat{\phi}$ together with $\hat{\phi}(0) = 2^{-1/2} h(0)\hat{\phi}(0)$. In fact, we shall show that $|\hat{\phi}(0)| = 1$. This will be achieved by computing first a formula for $\widehat{P_{V_j} f}$ for arbitrary f and then applying it to $\hat{f} = 1_{[-\pi,\pi]}$. In the first place, property 5 of MRA gives

$$\lim_{j\to-\infty} \|\hat{f} - \widehat{P_{V_j} f}\| = (2\pi)^{-1} \lim_{j\to-\infty} \|f - P_{V_j} f\| = 0. \tag{22}$$

We note next that

$$f * \overline{\phi_{j,0}}(2^j n) = \int f(t)\overline{\phi_{j,0}(t - 2^j n)}\, dt = \int f(t)\overline{\phi_{j,n}(t)}\, dt = (f|\phi_{j,n})$$

It follows that

$$P_{V_j} f = \sum_n (f|\phi_{j,n})\phi_{j,n} = \sum_n f * \overline{\phi_{j,0}}(2^j n)\phi_{j,n}.$$

which in turn implies that

$$P_{V_j} f = \phi_{j,0} * \left(\sum_n f * \overline{\phi_{j,0}}(2^j n)\delta(t - 2^j n) \right). \tag{23}$$

To verify this last assertion, let F denote the distribution $\sum_n f * \overline{\phi_{j,0}}(2^j n)\phi_{j,n}$, let G denote the distribution $\sum_n f * \overline{\phi_{j,0}}(2^j n)\delta(t - 2^j n)$, and let φ be a test function. Then

$$\langle \phi_{j,0} * G, \varphi \rangle = \langle G, \phi_{j,0} * \varphi \rangle = \sum_n f * \overline{\phi_{j,0}}(2^j n)(\phi_{j,0} * \varphi)(2^j n) = \sum_n (f|\phi_{j,n})(\phi_{j,n}|\overline{\varphi})$$

and

$$\langle F, \varphi \rangle = \int F(t)\varphi(t)\, dt = (F|\overline{\varphi}) = \sum_n (f|\phi_{j,n})(\phi_{j,n}|\overline{\varphi}).$$

We now take the Fourier transform of (23), noting that $\widehat{\phi_{j,0}}(\omega) = 2^{-j/2} 2^j \hat{\phi}(2^j \omega)$ and $\widehat{f * \overline{\phi_{j,0}}} = \hat{f}(\omega) 2^{-j/2} 2^j \overline{\hat{\phi}(2^j \omega)}$ and using Proposition 5.2 to get

$$\widehat{P_{V_j} f}(\omega) = \hat{\phi}(2^j \omega) \sum_k \hat{f}\left(\omega - \frac{2k\pi}{2^j}\right) \overline{\hat{\phi}}\left(2^j \left[\omega - \frac{2k\pi}{2^j}\right]\right). \tag{24}$$

We now take $f = 1_{[-\pi,\pi]}$ in (24) and let j be negative and very large in absolute value. You get $\widehat{P_{V_j} f}(\omega) = |\hat{\phi}(2^j \omega)|^2$ for $|\omega| \leq \pi$ and $\widehat{P_{V_j} f}(\omega) = 0$ for $|\omega| > \pi$. Then from (22),

$$\int_{-\pi}^{\pi} |1 - |\hat{\phi}(2^j \omega)|^2|^2\, d\omega = \|\hat{f} - \widehat{P_{V_j} f}\|^2 \to 0$$

as $j \to -\infty$. As ϕ is assumed to be integrable, $\hat{\phi}$ is continuous and so by dominated convergence $\int_{-\pi}^{\pi} |1 - |\hat{\phi}(0)|^2|^2 \, d\omega = 0$. $\Box$

Examples- For the Shannon MRA, $\phi = \theta = (\sin \pi t)/\pi t$ and so $\hat{\phi} = 1_{[-\pi,\pi]}$ and the "conjugate mirror filter" h can be computed from the equation $\hat{\phi}(2\omega) = 2^{-1/2} h(\omega)\hat{\phi}(\omega)$ to be $h(\omega) = \sqrt{2} 1_{[-\pi/2,\pi/2]}$

For the piecewise constant MRA, $\phi = \theta = 1_{[0,1]}$ and h is computed via $\hat{h}(n) = 2^{-1/2}(\phi(t/2)|\phi(t-n)) = 2^{-1/2}|[0,2] \cap [n, n+1]| = 2^{-1/2}$ if $n = 0$ or 1 and 0 otherwise, so that $h(\omega) = 2^{-1/2}(1 + e^{i\omega})$.

13.2 Characterization of scaling function

Theorem 13.2 ([10, Theorem 7.2, p. 229, part II—characterization of scaling function]) *Let $h \in L^2[0, 2\pi]$ be continuously differentiable in a neighborhood of 0, and suppose h satisfies (19), (20), and $\inf_{|\omega| \le \pi/2} |h(\omega)| > 0$. Let ϕ be defined by $\hat{\phi}(\omega) = \Pi_{p=1}^{\infty}(h(2^{-p}\omega)/\sqrt{2})$. Then ϕ is a scaling function for some MRA $\{V_j\}_{j \in \mathbf{Z}}$.*

14 Monday May 15, 2006—Proof of Theorem 13.2

Exercise 11 *Prove that $\lim_{k\to\infty} \frac{h(\omega/2)h(\omega/2^2)\cdots h(\omega/2^k)}{2^{k/2}}$ exists.* (Going to the original source [9] for a clue doesn't help very much. However it does suggest considering absolute values and proving that if $0 \le a_k \le 1$ and $a_k \to 1$, then $\lim_{k\to\infty} a_1 \cdots a_k$ exists.)

Assuming the validity of Exercise 11, the function ϕ exists. We will show that $\phi \in L^2$ and that $\{\phi(t-n)\}_{n \in \mathbf{Z}}$ is an orthonormal set, hence an orthonormal basis for the space V_0 which it generates.

Consider a function $\hat{\phi}_k$ defined by

$$\hat{\phi}_k(\omega) = \left(\Pi_{p=1}^{k} \frac{h(2^{-p}\omega)}{\sqrt{2}}\right) 1_{[-2^k\pi, 2^k\pi]}(\omega)$$

and let $I_k[n] := \int |\hat{\phi}_k(\omega)|^2 e^{in\omega} \, d\omega$. We shall show that $I_k[n] = 2\pi\delta_{n,0}$ for every $k \ge 1$.

Making the change of variable $\omega' = \omega + 2^k\pi$ and noting that since h is 2π-periodic, $|h(2^{-p}[\omega' - 2^k\pi])|^2 = |h(2^{-p}\omega')|^2$ for $p < k$, and then using

$$|h(2^{-k}(\omega' - 2^k\pi))|^2 + |h(2^{-k}\omega')|^2 = 2,$$

we have

$$\int_{-2^k\pi}^{0} \left(\Pi_1^k \frac{|h(2^{-p}\omega)|^2}{2}\right) e^{in\omega} \, d\omega =$$

$$\begin{aligned}
&\int_0^{2^k\pi}\left(\Pi_1^{k-1}\frac{|h(2^{-p}\omega')|^2}{2}\right)\frac{|h(2^{-k}(\omega'-2^k\pi))|^2}{2}e^{in\omega'}\,d\omega' \\
=\ &\int_0^{2^k\pi}\left(\Pi_1^{k-1}\frac{|h(2^{-p}\omega')|^2}{2}\right)e^{in\omega'}\,d\omega'-\int_0^{2^k\pi}\left(\Pi_1^{k-1}\frac{|h(2^{-p}\omega')|^2}{2}\right)\frac{|h(2^{-k}\omega')|^2}{2}e^{in\omega'}\,d\omega'.
\end{aligned}$$

Thus

$$\begin{aligned}
I_k[n] &= \int_{-2^k\pi}^{0}\left(\Pi_1^{k}\frac{|h(2^{-p}\omega)|^2}{2}\right)e^{in\omega}\,d\omega+\int_0^{2^k\pi}\left(\Pi_1^{k}\frac{|h(2^{-p}\omega)|^2}{2}\right)e^{in\omega}\,d\omega \\
&= \int_0^{2^k\pi}\left(\Pi_1^{k-1}\frac{|h(2^{-p}\omega')|^2}{2}\right)e^{in\omega'}\,d\omega'.
\end{aligned}$$

Since $\Pi_{p=1}^{k-1}(|h(2^{-p}\omega)|^2/2)e^{in\omega}$ is $2^k\pi$-periodic and the interval $[-2^{k-1}\pi,2^{k-1}\pi]$ is of length 2^k, we can write

$$I_{k-1}[n]=\int_{-2^{k-1}\pi}^{2^{k-1}\pi}\Pi_{p=1}^{k-1}(|h(2^{-p}\omega)|^2/2)e^{in\omega}\,d\omega=\int_0^{2^k\pi}\Pi_{p=1}^{k-1}(|h(2^{-p}\omega)|^2/2)e^{in\omega}\,d\omega=I_k[n],$$

and so $I_k[n]=I_{k-1}[n]=\cdots=I_1[n]=\int_0^{2\pi}e^{in\omega}\,d\omega=2\pi\delta_{n,0}$ is proved.

The fact that $\hat\phi\in L^2$ follows from Fatou's lemma:

$$\int|\hat\phi(\omega)|^2\,d\omega\le\lim_k\int|\hat\phi_k(\omega)|^2\,d\omega=2\pi.$$

To prove that $\{\phi(t-n)\}_{n\in\mathbf{Z}}$ is an orthonormal set, we calculate

$$\begin{aligned}
(\phi|\phi(\cdot-n)) &= \int\phi(t)\overline{\phi(t-n)}\,dt=(2\pi)^{-1}\int|\hat\phi(\omega)|^2e^{in\omega}\,d\omega \\
&= (2\pi)^{-1}\lim_k\int|\hat\phi_k(\omega)|^2e^{in\omega}\,d\omega=\delta_{n,0},
\end{aligned}$$

where in order to justify the use of the dominated convergence theorem in this calculation, we need to prove the existence of a constant C such that

$$|\hat\phi_k(\omega)|^2\le C|\hat\phi(\omega)|^2\ \text{for a. e. }\omega. \tag{25}$$

If $|\omega|>2^k\pi$, then $\hat\phi_k(\omega)=0$, so in (25) we only need to worry about $|\omega|\le2^k\pi$. For any ω, we have from the definition of $\hat\varphi$ as an infinite product

$$\begin{aligned}
|\hat\phi(\omega)|^2 &= |2^{-1/2}h(\omega/2)\hat\phi(\omega/2)|^2=|2^{-1/2}h(\omega/2)2^{-1/2}h(\omega/4)\hat\phi(\omega/4)|^2 \\
&= \cdots=|\hat\phi_k(\omega)|^2|\hat\phi(2^{-k}\omega)|^2. \qquad (26)
\end{aligned}$$

The inequality (25) for $|\omega|>2^k\pi$ will follow from

$$|\hat\phi(\omega)|^2\ge\frac{1}{C}\ \text{for }|\omega|\le\pi. \tag{27}$$

Indeed, by (26) and (27), $|\hat\phi_k(\omega)|^2=|\hat\phi(\omega)|^2/|\hat\phi(2^{-k}\omega)|^2\le C|\hat\phi(\omega)|^2$ since $|\omega|\le2^k\pi$ implies that $|2^{-k}\omega|\le\pi$. We proceed to the proof of (27). Since

0 is a critical point for the function $\log |h(\omega)|^2$, the derivative vanishes at 0. Hence

$$\left| \frac{|\log |h(\omega)|^2 - \log 2|}{\omega} \right| \leq 1 \text{ if } |\omega| \leq \epsilon.$$

which implies $\log(|h(\omega)|^2/2) \geq -|\omega|$ for $|\omega| \leq \epsilon$. Thus for $|\omega| \leq \epsilon$,

$$|\hat{\phi}(\omega)|^2 = \Pi_1^\infty \frac{|h(2^{-p}\omega)|^2}{2} = \exp \sum_1^\infty \log \frac{|h(2^{-p}\omega)|^2}{2} \geq e^{-|\omega|} \geq e^{-\epsilon}. \qquad (28)$$

This proves part of (27). Now suppose $\epsilon < |\omega| \leq \pi$ and pick l such that $2^{-l}\pi \leq \epsilon$. With $K := \inf_{|\omega| \leq \pi/2} |h(\omega)|$, we have $|\hat{\phi}(2^{-l}\omega)|^2 = \Pi_{p=1}^\infty |h(2^{-(p+l)}\omega)|^2/2$ and so

$$|\hat{\phi}(\omega)|^2 = |\hat{\phi}(2^{-l}\omega)|^2 \Pi_{p=1}^l (|h(2^{-p}\omega)|^2/2) \geq e^{-\epsilon} K^{2l}/2^l := 1/C,$$

completing the proof of (27) and the orthonormality of $\{\phi(t-n)\}_{n \in \mathbf{Z}}$.

We now show that if V_j is defined to be the space generated by the orthonormal set $\{\phi_{j,n}\}_{n \in \mathbf{Z}}$, where $\phi_{j,n}(t) = 2^{-j/2}\phi(2^{-j}t - n)$, then $\{V_j\}_{j \in \mathbf{Z}}$ is an MRA.

Since $\phi_{j,n}(t - 2^j k) = \phi_{j,k+n}(t)$, property 1 holds.

Since $\phi_{j+1,n} = \sum_{-\infty}^\infty \hat{h}(n-2p)\phi_{j,n}$, which will be proved later, property 2 holds.

Since $\phi_{j,n}(t/2) = 2^{1/2}\phi_{j+1,n}(t)$, property 3 holds.

Since $\{f \in L^\infty :$ the support of f is compact$\}$ is dense, it suffices to prove property 4 for an f with $|f(t)| \leq A$ and with support in $[-2^J, 2^J]$. For such f and for $j > J$, we have

$$\begin{aligned}
\|P_{V_j} f\|^2 &= \sum |(f|\phi_{j,n})|^2 = \sum |\int f(t) 2^{-j/2} \overline{\phi}(2^{-j}t - n)\, dt|^2 \\
&\leq 2^{-j} A^2 \sum \left(\int_{-2^J}^{2^J} 1 \cdot |\phi(2^{-j}t - n)|\, dt \right)^2 \\
&\leq A^2 \sum \left(\int_{-2^J}^{2^J} 1\, dt \right) \left(\int_{-2^J}^{2^J} |\phi(2^{-j}t - n)|^2 2^{-j}\, dt \right) \\
&\leq A^2 2^{J+1} \int_{S_j} |\phi(t)|^2\, dt = A^2 2^{J+1} \int |\phi(t)|^2 1_{S_j}(t)\, dt
\end{aligned}$$

where $S_j = \cup_{n \in \mathbf{Z}} [n - 2^{J-j}, n + 2^{J-j}]$. Since for $t \notin \mathbf{Z}$, $1_{S_j}(t) \to 0$ as $j \to \infty$, by dominated convergence $\|P_{V_j} f\|^2 \to 0$, proving property 4.

Property 5 is equivalent to showing that $\|f - P_{V_j} f\| \to 0$ as $j \to -\infty$. Again by a density argument, it suffices to prove this for f with $\hat{f}$ supported in $[-2^J\pi, 2^J\pi]$.

If $j < -J$, then the supports of $\hat{f}(\cdot - 2k\pi/2^j)$ are disjoint for different values of k. Thus, from (24), we can write $\|P_{V_j} f\|^2 = A_j + B_j$, where

$$A_j = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 |\hat{\phi}(2^j\omega)|^4 \, d\omega$$

and

$$B_j = \frac{1}{2\pi} \int \sum_{k \neq 0} |\hat{f}\left(\omega - \frac{2k\pi}{2^j}\right)|^2 |\hat{\phi}(2^j\omega)|^2 |\hat{\phi}\left(2^j\left[\omega - \frac{2k\pi}{2^j}\right]|^2\right) \, d\omega.$$

We have already shown that $|\hat{\phi}(\omega)| \leq 1$ and that $|\hat{\phi}(\omega)| \geq e^{-|\omega|}$ for $|\omega|$ small, so $\lim_{\omega \to 0} |\hat{\phi}(\omega)| = 1$, so that by dominated convergence $A_j \to \|f\|^2$. From $\|f\|^2 \geq \|P_{V_j} f\|^2 = A_j + B_j$ we have $\|P_{V_j} f\|^2 \to \|f\|^2$ and $B_j \to 0$, so that $\|f - P_{V_j} f\|^2 = \|f\|^2 - \|P_{V_j} f\| \to 0$. □

15 Wednesday May 17, 2006—Orthogonal wavelets

Theorem 15.1 ([10, Theorem 7.3, p. 236]) *Let ϕ be a scaling function for some MRA $\{V_j\}_{j \in \mathbf{Z}}$ and let h be the associated conjugate mirror filter. Let ψ be determined by $\hat{\psi}(\omega) = 2^{-1/2} g(\omega/2)\hat{\phi}(\omega/2)$ where $g(\omega) := e^{-i\omega}\overline{h(\omega + \pi)}$. Then $\{\psi_{j,n}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $W_j := V_{j-1} \ominus V_j$, where $\psi_{j,n}(t) = 2^{-j/2}\psi(2^{-j}t - n)$. Thus $\{\psi_{j,n}\}_{j,n \in \mathbf{Z}^2}$ is an orthonormal basis for $L^2(\mathbf{R})$.*

Proof: Recall that ϕ and h are related by $2^{-1/2}\phi(t/2) = \sum_n \hat{h}(n)\phi(t-n)$, which lead to $\hat{\phi}(\omega) = 2^{-1/2} h(\omega/2)\hat{\phi}(\omega/2)$. Analogously, we define ψ by $2^{-1/2}\psi(t/2) = \sum_n \hat{g}(n)\phi(t-n)$ where g is to be determined[15], and this leads to $\hat{\psi}(\omega) = 2^{-1/2} g(\omega/2)\hat{\phi}(\omega/2)$. Steps 1, 2 and 3 ares concerned with the space W_0' generated by $\{\psi_{0,n}\}_{n \in \mathbf{Z}}$. Step 4 completes the proof easily from this.

Step 1 $\{\psi(t-n)\}_{n \in \mathbf{Z}}$ is an orthonormal base for the space W_0' it generates if and only if $|g(\omega)|^2 + |g(\omega + \pi)|^2 = 2$ a. e.

Proof: We know from the proof of Theorem 12.1 that $\{\psi(t-n)\}_{n \in \mathbf{Z}}$ is an orthonormal set if and only if $I(\omega) := \sum_{-\infty}^{\infty} |\hat{\psi}(\omega + 2k\pi)|^2 = 1$ and hence $\sum_{-\infty}^{\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1$. We have

$$\begin{aligned}
I(\omega) &= \sum_k |2^{-1/2} g(\frac{\omega + 2k\pi}{2})\hat{\phi}(\frac{\omega + 2k\pi}{2})|^2 \\
&= \frac{1}{2}|g(\omega/2)|^2 \sum_p |\hat{\phi}(\omega/2 + 2p\pi)|^2 + \frac{1}{2}|g(\omega/2 + \pi)|^2 \sum_p |\hat{\phi}(\omega/2 + \pi + 2p\pi)|^2 \\
&= \frac{1}{2}|g(\omega/2)|^2 + \frac{1}{2}|g(\omega/2 + \pi)|^2, \text{ proving Step 1.}
\end{aligned}$$

[15] We don't need the choice $g(\omega) = e^{-i\omega}\overline{h(\omega + \pi)}$ until near the end of the proof

Step 2 The space W_0' is orthogonal to V_0 if and only if $g(\omega)\overline{h}(\omega) + g(\omega + \pi)\overline{h}(\omega + \pi) = 0$ a. e.

Proof: $\phi(t-n) \perp \psi(t-m)$, $\forall (m,n) \in \mathbf{Z}^2 \Leftrightarrow \psi * \phi^*[n] = (\psi(t)|\phi(t-n)) = 0$, $\forall n \in \mathbf{Z}$. By Proposition 5.2, $\sum_{n\in\mathbf{Z}} \psi \underline{*} \phi^* \, \delta(t-n)$ has Fourier transform equal to $J(\omega) := \sum_{-\infty}^{\infty} \hat{\psi}(\omega + 2k\pi)\overline{\hat{\phi}}(\omega + 2k\pi)$. Breaking the sum into even and odd pieces as before, we get

$$\begin{aligned} J(\omega) &= \sum_{-\infty}^{\infty} 2^{-1/2} g(\omega/2 + k\pi)\hat{\phi}(\omega/2 + k\pi)\overline{2^{-1/2}h(\omega/2 + k\pi)\hat{\phi}(\omega/2 + k\pi)} \\ &= \sum 2^{-1} g(\omega/2 + k\pi)\overline{h(\omega/2 + k\pi)}|\hat{\phi}(\omega/2 + k\pi)|^2 \\ &= 2^{-1}[g(\omega/2)\overline{h(\omega/2)} + g(\omega/2 + \pi)\overline{h(\omega/2 + \pi)}], \text{ proving Step 2.} \end{aligned}$$

Step 3 With the choice $g(\omega) := e^{-i\omega}\overline{h(\omega + \pi)}$, $V_{-1} = V_0 \oplus W_0$.

Proof: By Step 2 and the choice of g, $V_0 \perp W_0'$. We now check that $V_0 \oplus W_0' \subset V_{-1}$. Of course $V_0 \subset V_{-1}$ because of property 2 of MRA. By construction, $\psi(t/2) \in V_0$ so that $\psi \in V_{-1}$ by property 3, and $\psi(t - 2^{-1}k) \in V_{-1}$ by property 1. Since this is true for all $k \in \mathbf{Z}$, $\psi(t-n) \in V_{-1}$ for all $n \in \mathbf{Z}$ and therefore $W_0' \subset V_1$.

We now prove that $V_{-1} \subset V_0 \oplus W_0'$. This will show that $W_0' = W_0$ and prove Step 3. Since $\{\sqrt{2}\phi(2t-n)\}_{n\in\mathbf{Z}}$ is an orthonormal basis for V_{-1}, it suffices to prove that given $a \in \ell^2$, $\exists b, c \in \ell^2$ such that

$$\sum_n a[n] 2^{1/2} \phi(2(t - 2^{-1}n)) = \sum_n b[n]\phi(t-n) + \sum_n c[n]\psi(t-n),$$

or equivalently, by taking Fourier transforms[16],

$$2^{-1/2}\hat{a}(\omega/2)\hat{\phi}(\omega/2) = \hat{b}(\omega)\hat{\phi}(\omega) + \hat{c}(\omega)\hat{\psi}(\omega). \tag{29}$$

Since $\hat{\phi}(\omega) = 2^{-1/2}h(\omega/2)\hat{\phi}(\omega/2)$ and $\hat{\psi}(\omega) = 2^{-1/2}g(\omega/2)\hat{\phi}(\omega/2)$, the equation (29) will follow from

$$\hat{a}(\omega/2) = \hat{b}(\omega)h(\omega/2) + \hat{c}(\omega)g(\omega/2). \tag{30}$$

If you now define

$$\hat{b}(2\omega) = \frac{1}{2}(\hat{a}(\omega)\overline{h(\omega)} + \hat{a}(\omega + \pi)\overline{h(\omega + \pi)})$$

and

$$\hat{c}(2\omega) = \frac{1}{2}(\hat{a}(\omega)\overline{g(\omega)} + \hat{a}(\omega + \pi)\overline{g(\omega + \pi)})$$

[16] It seemed wise to use Mallat's notation here, namely $\hat{a}$ is the 2π-periodic function with Fourier series $\sum_n a[n]e^{in\omega}$, and similarly for b and c

and substitute into (30), the right side of (30) becomes

$$\frac{1}{2}\hat{a}(\omega/2)|h(\omega/2)|^2 + \frac{1}{2}\hat{a}(\omega/2+\pi)\overline{h(\omega/2+\pi)}h(\omega/2)$$

$$+\frac{1}{2}\hat{a}(\omega/2)|g(\omega/2)|^2 + \frac{1}{2}\hat{a}(\omega/2+\pi)\overline{g(\omega/2+\pi)}g(\omega/2).$$

If you now replace $|h(\omega/2)|^2$ by $2 - |h(\omega/2+\pi)|^2$ and use $g(\omega) := e^{-i\omega}\overline{h(\omega+\pi)}$ in three places, this reduces to $\hat{a}(\omega/2)$, proving (29) and Step 3.

Step 4 For all $j \in \mathbf{Z}$, $\{\psi_{j,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis for W_j and $\{\psi_{j,n}\}_{j,n\in\mathbf{Z}^2}$ is an orthonormal basis of $L^2(\mathbf{R})$.

Proof: We know that $\{\psi_{0,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis of W_0, $\{\phi_{0,n}\}_{n\in\mathbf{Z}}$ is an orthonormal basis of V_0 and $V_0 \oplus W_0 = V_{-1}$. Since $(\psi_{j,n}|\psi_{j,p}) = (\psi_{0,n}|\psi_{0,p}) = \delta_{n,p}$ and $(\psi_{j,n}|\phi_{j,p}) = (\psi_{0,n}|\phi_{0,p}) = 0$, we just need to show that $W_j \subset V_{j-1}$ and $V_{j-1} \subset V_j \oplus W_j$. Since $\psi \in V_{-1}$, $\psi(t/2^j) \in V_{j-1}$ and $\psi((t-2^{j-1}k)/2^j) \in V_{j-1}$. Since this is true for all $k \in \mathbf{Z}$, $\psi(2^{-j}t-n) \in V_{j-1}$ for all $n \in \mathbf{Z}$, proving $W_j \subset V_{j-1}$. Finally, if $f \in V_{j-1}$, then $f(2^j t) \in V_{-1} = V_0 \oplus W_0$ which implies $f \in V_j \oplus W_j$.

To prove the second statement, it suffices to show that $L^2 = \oplus_{-\infty}^{\infty} W_j$. In the first place, if $j < l$, then $W_l \subset V_{l-1} \subset V_j$ and since $W_j \perp V_J$, we have $W_l \perp W_j$. Next, if $L < J$, then $V_L = V_J \oplus \oplus_{j=L+1}^{J} W_j$, so for all $f \in L^2$,

$$P_{V_L} f = P_{V_J} f + \sum_{j=L+1}^{J} P_{W_j} f.$$

Now let $J \to \infty$ and $L \to -\infty$ to get $f = \sum_{-\infty}^{\infty} P_{W_j} f$. □

16 Thursday May 18, 2006

16.1 Classes of Wavelet Bases

The philosophy of what follows is contained in the following quotation.

> The design of ψ must be optimized to produce a maximum number of wavelet coefficients $(f|\psi_{j,n})$ that are close to zero.

Theorem 16.1 ([10, Theorem 7.4, p. 241—vanishing moments]) *If ψ is a wavelet with scaling function ϕ with respect to an MRA, and if $|\psi(t)| = O((1+t^2)^{-p/2-1})$ and $|\phi(t)| = O((1+t^2)^{-p/2-1})$, then the following are equivalent:*

(i) *ψ has p vanishing moments: $\int t^k \psi(t)\, dt = 0,\ 0 \le k < p$*

(ii) *$\hat{\psi}(\omega)$ and its first $p-1$ derivatives are zero at $\omega = 0$*

(iii) $\hat{h}(\omega)$ *and its first* $p-1$ *derivatives are zero at* $\omega = \pi$

(iv) *For* $0 \le k < p$, $q_k(t) := \sum_{-\infty}^{\infty} n^k phi(t-n)$ *is a polynomial of degree at most* k

Proof: Will be inserted later.

Proposition 16.2 ([10, Prop. 7.2, p. 243]) *A scaling function* ϕ *has compact support if and only if for the associated conjugate mirror filter* h, $\hat{h}$ *has compact support. The two supports are equal, say* $[N_1, N_2]$, *and the support of* ψ *is then* $[\frac{N_1-N_2+1}{2}, \frac{N_2-N_1+1}{2}]$.

Proof: Will be inserted later.

Examples:

Shannon wavelet $\hat{\phi} = 1_{[-\pi,\pi]}$, $h = \sqrt{2}\,1_{[-\pi/2,\pi/2]}$, $\hat{\psi}(\omega) = e^{-i\omega/2}$ if $\pi \le |\omega| \le 2\pi$, and zero otherwise.

$$\psi(t) = \frac{\sin 2\pi(t-1/2)}{2\pi(t-1/2)} - \frac{\sin \pi(t-1/2)}{\pi(t-1/2)}$$

Meyer wavelets Let $h \in C^n$ be equal to $\sqrt{2}$ on $[-\pi/3, \pi/3]$, zero on $\pi/3 \le |\omega| \le 2\pi/3$ and satisfy $|h(\omega)|^2 + |h(\omega+\pi)|^2 = 2$. Then

$$\hat{\phi}(\omega) = \Pi_1^{\infty} 2^{-1/2} h(2^{-p}\omega) = \begin{cases} 2^{-1/2} h(\omega/2) & |\omega| \le 4\pi/3 \\ 0 & |\omega| > 4\pi/3. \end{cases}$$

and, with $g(\omega) = e^{-i\omega}\overline{h(\omega+\pi)}$,

$$\hat{\psi}(\omega) = \begin{cases} 0 & |\omega| \le 2\pi/3 \\ 2^{-1/2} g(\omega/2) & 2\pi/3 \le |\omega| \le 4\pi/3 \\ 2^{-1/2} e^{-i\omega/2} h(\omega/4) & 4\pi/3 \le |\omega| \le 8\pi/3 \\ 0 & |\omega| > 8\pi/3 \end{cases}$$

Haar wavelet $\phi = 1_{[0,1]}$, $\hat{h}(n) = 2^{-1/2}$ for $n = 0$ or 1 and zero otherwise, so

$$\frac{1}{\sqrt{2}}\psi(t/2) = \sum (-1)^{1-n} \hat{h}((1-n)\phi(t-n) = \frac{1}{\sqrt{2}}[\phi(t-1) - \phi(t)]$$

and

$$\psi(t) = \begin{cases} -1 & 0 \le t < 1/2 \\ 1 & 1/2 \le t < 1 \\ 0 & \text{otherwise .} \end{cases}$$

16.2 Fast orthogonal wavelet transform

The philosophy of what follows is contained in the following quotation.

> A fast wavelet transform decomposes successively each approximation $P_{V_j}f$ into a coarser approximation $P_{V_{j+1}}f$ plus the wavelet coefficients $P_{W_{j+1}}f$. In the other direction, the reconstruction from wavelet coefficients recovers each $P_{V_j}f$ from $P_{V_{j+1}}f$ and $P_{W_{j+1}}f$.

The projections on V_j and W_j are characterized by $a_j[n] = (f|\phi_{j,n})$ and $d_j[n] = (f|\psi_{j,n})$. More notation: $\overline{x}[n] = x[-n]$, and $\check{x}[n] = \begin{cases} x[p] & n = 2p \\ 0 & \text{otherwise} \end{cases}$.

Theorem 16.3 ([10, Theorem 7.7, p. 255])

(a) *At the decomposition ($P_{V_j}f \to P_{V_{j+1}}f$ and $P_{W_{j+1}}f$)*

$$a_{j+1}[p] = a_j * \overline{h}[2p] \tag{31}$$

$$d_{j+1}[p] = a_j * \overline{g}[2p] \tag{32}$$

(b) *At the reconstruction ($P_{V_{j+1}}f$ and $P_{W_{j+1}}f \to P_{V_j}f$))*

$$a_j[p] = \check{a}_{j+1} * h[p] + \check{d}_{j+1} * g[p] \tag{33}$$

Proof: Will be inserted later.

17 Wednesday May 31, 2006—Transition to Daubechies [5]

17.1 Continuous wavelet transform—inversion revisited

Notation from [5]: $\psi \in L^2$, $C_\psi := 2\pi \int \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi$, $\psi^{a,b} = |a|^{-1/2}\psi(\frac{x-b}{a})$ for $a, b \in \mathbf{R}$ and $a \neq 0$, $\|\psi^{a,b}\| = \|\psi\|$ (assumed to be 1),

$$T^{\text{wav}} f(a,b) = (f|\psi^{a,b}) = \int f(x)\overline{\psi^{a,b}(x)}\, dx.$$

Proposition 17.1 ([5, Prop. 2.4.1, p. 24]) *For $f, g \in L^2$,*

$$\int\int T^{wav} f(a,b)\overline{T^{wav} g(a,b)}\, \frac{da\, db}{a^2} = C_\psi (f|g).$$

Proof: [17]

$$
\begin{aligned}
(f|\psi^{a,b}) &= \int f(x)\overline{\psi\left(\frac{x-b}{a}\right)}|a|^{-1/2}\,dx \\
&= \int \hat{f}(\xi)|a|^{1/2}e^{-ib\xi}\overline{\hat{\psi}}(a\xi)\,d\xi = (2\pi)^{1/2}[\hat{f}(\cdot)\overline{\hat{\psi}}(a\cdot)|a|^{1/2}]\hat{}(b).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int\int T^{\mathrm{wav}}f(a,b)\overline{T^{\mathrm{wav}}g(a,b)}\,\frac{da\,db}{a^2} = \int\int (f|\psi^{a,b})\overline{(g|\psi^{a,b})}\,\frac{da\,db}{a^2} = \\
&\quad \int\int [(2\pi)^{1/2}[\hat{f}(\cdot)\overline{\hat{\psi}}(a\cdot)|a|^{1/2}]\hat{}(b)]\overline{[(2\pi)^{1/2}[\hat{g}(\cdot)\overline{\hat{\psi}}(a\cdot)|a|^{1/2}]\hat{}(b)]}\,\frac{da\,db}{a^2} \\
&= \quad 2\pi\int\int \hat{f}(\xi)\overline{\hat{g}}(\xi)|\hat{\psi}(a\xi)|^2\,\frac{da}{|a|}\,d\xi = C_\psi(\hat{f}|\hat{g}) = C_\psi(f|g).
\end{aligned}
$$

17.2 Frames of wavelets—A necessary condition

For $\psi \in L^2$, let $\psi_{m,n}(x) = a_0^{-m/2}\psi(a_0^{-m} - nb_0)$ where $m, n \in \mathbf{Z}, a_0 > 1, b_0 > 0$.

Theorem 17.2 ([5, Theorem 3.3.1, p. 63—a necessary condition]) *If $\psi_{m,n}$ with $m, n \in \mathbf{Z}$ is a frame for $L^2(\mathbf{R})$ with bounds A and B, then*

$$
\frac{b_0\log a_0}{2\pi}A \le \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi}\,d\xi \le \frac{b_0\log a_0}{2\pi}B
$$

and

$$
\frac{b_0\log a_0}{2\pi}A \le \int_{-\infty}^0 \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\,d\xi \le \frac{b_0\log a_0}{2\pi}B
$$

Proof:

Step 1 Every positive trace-class operator C on L^2 has the form $C = \sum c_l(\cdot|u_l)u_l$ for some orthonormal set $\{u_l\}$, where $c_l \ge 0$ and $\operatorname{Tr} C = \sum_l c_l$, since by definition, the trace of C is equal to $\sum_k (Cv_k|v_k)$ for any orthonormal basis $\{v_k\}$. Applying the frame inequality $A\|f\|^2 \le \sum_{m,n}|(f|\psi_{m,n})|^2 \le B\|f\|^2$ to each u_l and summing on l yields

$$
A\sum_l c_l \le \sum_l\sum_{m,n}|(u_l|\psi_{m,n})|^2 \le B\sum_l c_l.
$$

On the other hand, since $C\psi_{m,n} = \sum_l c_l(\psi_{m,n}|u_l)u_l$, we have $(C\psi_{m,n}|\psi_{m,n}) = \sum_l c_l|(u_l|\psi_{m,n})|^2$, so that

$$
A(\operatorname{Tr} C) \le \sum_{m,n}(C\psi_{m,n}|\psi_{m,n}) \le B(\operatorname{Tr} C). \tag{34}
$$

[17] In the definition of the Fourier integral in [5], the factor $(2\pi)^{-1/2}$ is used; in particular $\|f\| = \|\hat{f}\|$

Step 2 Let $h \in L^2$ denote any function[18] with Fourier transform supported in $[0,\infty)$ and $\int_0^\infty \xi^{-1}|\hat{h}(\xi)|^2\,d\xi < \infty$. Let $c = c(a,b)$ be a bounded positive function which is integrable: $\int_0^\infty \int c(a,b)\,\frac{da\,db}{a^2} < \infty$. The bilinear form

$$B(f,g) = \int_0^\infty c(a,b)(f|h^{a,b})(h^{a,b}|g)\,\frac{da\,db}{a^2}$$

satisfies $|B(f,g)| \le \|c\|_{L^1}\|f\|\|g\|\|h\|^2$ and so by the Riesz representation theorem defines a bounded operator C_0 with the property that $(C_0 f|g) = B(f,g)$. Moreover, C_0 is positive and since for any orthonormal basis $\{v_k\}$, $\|h\|^2 = \|h^{a,b}\|^2 = \sum_k |(v_k|h^{a,b})|^2$, we have

$$\operatorname{Tr} C_0 = \sum_k (C_0 v_k|v_k) = \int_0^\infty c(a,b)\,\frac{da\,db}{a^2}\|h\|^2.$$

Now let w be non-negative[19] and in $L^1(\mathbf{R})$, and choose $c(a,b) = w(|b|/a)$ if $1 \le a \le a_0$ and $b \in \mathbf{R}$, and $c(a,b) = 0$ otherwise. We may now write

$$C_0 = \int_1^{a_0}\int w(|b|/a)(\cdot|h^{a,b})h^{a,b}\,\frac{da\,db}{a^2}$$

and, since $\int w(|b|/a)\,db = 2a\int_0^\infty w(b)\,db$,

$$\operatorname{Tr} C_0 = \int_1^{a_0}\int w(|b|/a)\,db\,\frac{da}{a^2}\|h\|^2 = 2\log a_0[\int_0^\infty w(s)\,ds]\|h\|^2. \quad (35)$$

18 Monday June 5, 2006

18.1 Completion of the proof of Theorem 17.2

Step 3 We first calculate that

$$\begin{aligned}(\psi_{m,n}|h^{a,b}) &= a_0^{-m/2}a^{-1/2}\int \psi(a_0^{-m}x - nb_0)\overline{h}\left(\frac{x-b}{a}\right)dx \\ &= a_0^{m/2}a^{-1/2}\int \psi(y)\overline{h}\left(\frac{y+nb_0-ba_0^{-m}}{aa_0^{-m}}\right)dy \\ &= (\psi|h^{a_0^{-m},a_0^{-m}b-nb_0}).\end{aligned}$$

Using this fact and making the changes of variable $a' = a_0^{-m}a$ and $b' = a_0^{-m}b$ (the latter first) we have

$$(C_0\psi_{m,n}|\psi_{m,n}) = \int_1^{a_0}\frac{da}{a^2}\int w(|b|/a)|(\psi_{m,n}|h^{a,b})|^2\,db$$

[18] h will disappear at the end of the proof; all we need to know is that such a function exists

[19] In Step 3, w will be chosen to be $w(s) = \lambda e^{-\lambda^2\pi^2 s^2}$ for some $\lambda > 0$ and then λ will approach 0

$$= \int_1^{a_0} \frac{da}{a^2} \int w(|b|/a) |(\psi | h^{a_0^{-m}, a_0^{-m} b - n b_0})|^2 \, db$$

$$= \int_{a_0^{-m}}^{a_0^{-m+1}} \frac{da'}{a'^2} \int w(|b'|/a') |(\psi | h^{a', b' - n b_0})|^2 \, db'.$$

Then

$$\sum_{m,n} (C_0 \psi_{m,n} | \psi_{m,n}) = \int_0^\infty \frac{da}{a^2} \int |(\psi | h^{a,b})|^2 \sum_n w\left(\frac{|b + n b_0|}{a}\right). \qquad (36)$$

We now interject a lemma from [4].

Lemma 18.1 ([4, Lemma 2.2, p. 975]) *Let f be a positive, continuous, bounded function on $\mathbf{R}$, with $f(x) \to 0$ as $|x| \to \infty$. Assume that f has a finite number of local maxima at x_j, $j = 1, \ldots, N$. Define*

$$\Delta_j = \sup_{0 \le \delta \le 1} \int_{x_j - \delta}^{x_j + \delta} f(x) \, dx.$$

Then

$$\int f(x) \, dx - \sum_{j=1}^N \Delta_j \le \sum_{n \in \mathbf{Z}} f(n) \le \int f(x) \, dx + \sum_{j=1}^N f(x_j).$$

From this lemma, and with the choice $w(s) = \lambda e^{-\lambda^2 \pi^2 s^2}$ $(\lambda > 0)$, we can state that for any $\alpha \in \mathbf{R}$ and $\beta > 0$,

$$\int w(x) \, dx - \beta w_{\max} \le \beta \sum_{n \in \mathbf{Z}} w(\alpha + n\beta) \le \int w(x) \, dx + \beta w_{\max}.$$

Setting $\alpha = b/a$ and $\beta = b_0/a$, we have

$$1 - \frac{b_0}{a} w(0) \le \sum_n w\left(\left|\frac{b}{a} + \frac{b_0}{a} n\right|\right) \le 1 + \frac{b_0}{a} w(0),$$

or $\sum_n w\left(\left|\frac{b}{a} + \frac{b_0}{a} n\right|\right) = \frac{a}{b_0}{}_\rho(a,b)$ with $|\rho(a,b)| \le w(0) = \lambda$.

We now have from (36),

$$\sum_{m,n} (C_0 \psi_{m,n} | \psi_{m,n}) = \frac{1}{b_0} \int_0^\infty \frac{da}{a} \int |(\psi | h^{a,b})|^2 \, db + R \qquad (37)$$

where $R = \int_0^\infty \frac{da}{a^2} \int \rho(a,b) |(\psi | h^{a,b})|^2 \, db$ so that $|R| \le \lambda C_h \|\psi\|^2$. Since $(\psi | h^{a,b}) = \int_0^\infty \hat{\psi}(\xi) a^{1/2} \overline{\hat{h}}(a\xi) e^{ib\xi} \, d\xi = (2\pi)^{1/2} [\hat{\psi}(\cdot) \overline{\hat{h}}(a \cdot)]\widehat{\ }(-b)$, the inner integral in the first term in (37) is equal to $2\pi \int |\hat{\psi}(\xi) \overline{\hat{h}}(a\xi)|^2 \, d\xi$ so that the first term in (37) is equal to

$$\frac{2\pi}{b_0} \int_0^\infty da \int_0^\infty |\hat{\psi}(\xi)|^2 |\hat{h}(a\xi)|^2 \, d\xi = \frac{2\pi}{b_0} \|h\|^2 \int_0^\infty \xi^{-1} |\hat{\psi}(\xi)|^2 \, d\xi. \qquad (38)$$

Step 4 From (35), (37), (38) and $\int_0^\infty w(s)\,ds = 1/2$, we now have

$$A\|h\|^2 \log a_0 \le \frac{2\pi}{b_0}\|h\|^2 \int_0^\infty \frac{|\hat{\psi}(\xi)|^2}{\xi}\,d\xi + R \le B\|h\|^2 \log a_0$$

from which the first inequality in the theorem follows upon letting $\lambda \to 0$. The second inequality is proved similarly. □

19 Friday June 16, 2006—10:00-11:20 Processing a Radar Signal

19.1 The Gabor condition

Let $f \in L^2(\mathbf{R})$ and let α and β be fixed positive numbers. For integers p and q, define $f_{p,q}(t) = e^{2\pi iq\alpha t} f(t - p\beta)$. We say f satisifes the Gabor condition for scales α and β if $\{f_{p,q} : p, q \in \mathbf{Z}\}$ spans a dense subset of L^2.

The sole objective of today's class is to prove the following theorem, which is a necessary condition in the context of windowed Fourier transform, analagous to the necessary condition for being a frame. Here, the frame inequality is replaced by the Gabor condition. Also, the critical constant is 1 instead of 2π because of the particular normalization used.

Theorem 19.1 ([1, Theorem 1.3, p. 196]) *Let $\gamma := \alpha\beta > 1$. Then f does not satisfy the Gabor condition for scales α and β.*

19.2 Some representations of the discrete Heisenberg group

The *discrete Heisenberg group* is the set G of all matrices of the form

$$\begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix},$$

where $p, q, r \in \mathbf{Z}$. The center of G is the subgroup Z determined by the equations $p = q = 0$ and an abelian subgroup H is determined by the equation $p = 0$.

For positive scales α and β, set $\gamma = \alpha\beta$, and define operators $\rho^{\alpha,\beta}_{p,q,r}$ on $L^2(\mathbf{R})$ by

$$\rho^{\alpha,\beta}_{p,q,r} f(t) = e^{-2\pi ir\gamma} e^{2\pi iq\alpha t} f(t - p\beta).$$

For each real t, define a character $\chi^{\gamma,t}$ of the subgroup H by

$$\chi^{\gamma,t}(q, r) = e^{2\pi iqt} e^{-2\pi ir\gamma}$$

and let $\sigma^{\gamma,t}$ denote the representation $\mathrm{ind}_H^G \chi^{\gamma,t}$ of G "induced" from $\chi^{\gamma,t}$, that is, $\sigma^{\gamma,t}$ acts in ℓ^2 and is given by

$$\sigma^{\gamma,t}_{p,q,r} f(n) = e^{-2\pi r\gamma} e^{2\pi iqt} e^{2\pi iq\gamma n} f(n-p).$$

For each $\delta > 0$, define $D^{\gamma,\delta}$ to be the "direct integral" representation

$$D^{\gamma,\delta} = \int_{[0,\delta)}^{\oplus} \sigma^{\gamma,t}\, dt.$$

and write $M^{\gamma,\delta}$ for the von Neumann algebra generated by the image of $D^{\gamma,\delta}$.

Exercise 12 *Verify that the induced representation $\sigma^{\gamma,t}$ is unitarily equivalent to the induced representation $\sigma^{\gamma,t+\gamma}$, and hence that $D^{\gamma,n\gamma}$ is unitarily equivalent to $n \cdot D^{\gamma,\gamma}$.*

Theorem 19.2 ([1, Theorem 2.1, p. 198])

1. *The map $(p,q,r) \mapsto \rho^{\alpha,\beta}_{p,q,r}$ is a unitary representation of G.*
2. *The unitary representation $\rho^{\alpha,\beta}$ is unitarily equivalent to the representation $D^{\gamma,\gamma}$.*
3. *$D^{\gamma,1}$ is equivalent to the representation $\mathrm{ind}_Z^G \phi$ of G induced from the character ϕ of the center Z, where $\phi(r) = e^{-2\pi ir\gamma}$.*
4. *$M^{\gamma,1}$ and its commutant $(M^{\gamma,1})'$ each have a cyclic vector.*
5. *$M^{\gamma,1}$ and its commutant $(M^{\gamma,1})'$ are finite von Neumann algebras.*
6. *$D^{\gamma,\delta}$ is unitarily equivalent to $D^{\gamma',\delta'}$ if and only if $\gamma = \gamma'$ and $\delta = \delta'$.*
7. *For any two positive numbers $\gamma < \delta$, $M^{\gamma,\gamma}$ and $M^{\gamma,\delta}$ are isomorphic von Neumann algebras.*

Proof:

1. This follows easily from the definition
2. Check that the map $U^{\alpha,\beta,\gamma} : L^2(\mathbf{R}) \to L^2([0,\gamma) \times \mathbf{Z})$ defined by

$$U^{\alpha,\beta,\gamma} f(t,n) = f\left(\frac{t}{\alpha} + n\beta\right)$$

is an isometry onto and satisfies $U^{\alpha,\beta,\gamma} \circ \rho^{\alpha,\beta}_{p,q,r} = D^{\gamma,\gamma} \circ U^{\alpha,\beta,\gamma}$.

3. Using Exercise 13 below as well as two functorial properties of induced representations ("inducing commutes with the formation of direct integrals", and "inducing in stages"), and letting "=" denote unitary equivalence,

$$\begin{aligned}\operatorname{ind}_Z^G \phi &= \operatorname{ind}_H^G(\operatorname{ind}_Z^H \phi) = \operatorname{ind}_H^G(\Delta_Q \times \phi) \\ &= \operatorname{ind}_H^G\Big(\int_{[0,1)}^{\oplus} \chi^{\gamma,t}\, dt\Big) = \int_{[0,1)}^{\oplus} \operatorname{ind}_H^G \chi^{\gamma,t}\, dt \\ &= \int_{[0,1)}^{\oplus} \sigma^{\gamma,t}\, dt = D^{\gamma,1}\end{aligned}$$

Exercise 13 *Let Δ_Q denote the left regular representation of the subgroup Q of G defined by the conditions that $p = r = 0$. Show that*

- *$\Delta_Q \times \phi$ is unitarily equivalent to $\int_{[0,1)}^{\oplus} \chi^{\gamma,t}\, dt$.*
- *$\Delta_Q \times \phi$ is unitarily equivalent to $ind_Z^H \phi$.*

4. This proof depends on facts about multiplier representations of abelian groups and is beyond the scope of this course.

5. This proof also depends on facts about multiplier representations of abelian groups and is beyond the scope of this course.

6. (sketch only) The restriction of $D^{\gamma,\delta}$ to Z is a scalar multiple of $e^{-2\pi i r\gamma}$. Thus if $D^{\gamma,\delta}$ is unitarily equivalent to $D^{\gamma',\delta'}$, then $\gamma = \gamma'$. Next, use the finiteness of the von Neumann algebra $(M^{\gamma,1})'$ to show that if $D^{\gamma,\delta}$ is unitarily equivalent to $D^{\gamma,\delta'}$ with $\delta < \delta' = 1$, then $\delta = 1$. Finally, reduce the case $\delta' \neq 1$ to the case $\delta' = 1$ just proved.

7. Since $D^{\gamma,\gamma}$ is a sub-representation of $D^{\gamma,\delta}$, it follows that the restriction mapping from $M^{\gamma,\delta}$ to $M^{\gamma,\gamma}$ is an onto homomorphism. Now choose n so that $n\gamma \geq \delta$ and therefore $D^{\gamma,\delta}$ is a subrepresentation of $D^{\gamma,n\gamma} = n \cdot D^{\gamma,\gamma}$ and $M^{\gamma,n\gamma}$ is isomorphic to $M^{\gamma,\gamma}$. As above, the restriction mapping from $M^{\gamma,n\gamma}$ to $M^{\gamma,\delta}$ is an onto homomorphism. Since there is now an onto homomorphism from $M^{\gamma,\delta}$ onto $M^{\gamma,\gamma}$ and from $M^{\gamma,\gamma}$ onto $M^{\gamma,\delta}$, it follows that $M^{\gamma,\delta}$ and $M^{\gamma,\gamma}$ are isomorphic. □

19.3 Proof of Theorem 19.1

Theorem 19.1 follows immediately from the following theorem.

Theorem 19.3 ([1, Theorem 3.3, p. 202]) *Let α and β be arbitrary positive scales and suppose $\gamma = \alpha\beta > 1$. Then the representation $\rho^{\alpha,\beta}$ is not a cyclic representation, that is, no vector $f \in L^2(\mathbf{R})$ is a cyclic vector for $\rho^{\alpha,\beta}$.*

Proof: Assume, by way of contradiction, that $\rho^{\alpha,\beta}$ has a cyclic vector. Since $\rho^{\alpha,\beta}$ is unitarily equivalent to $D^{\gamma,\gamma}$, $M^{\gamma,\gamma}$ has a cyclic vector. It is easy to check that $\rho^{\alpha,\beta}_{p,q,r}$ commutes with $\rho^{1/\beta,1/\alpha}_{p',q',r'}$ so that $M^{\gamma,\gamma} \subset (M^{1/\gamma,1/\gamma})'$, $(M^{\gamma,\gamma})' \supset M^{1/\gamma,1/\gamma}$, and the cyclic vector for $M^{\gamma,\gamma}$ is also a cyclic vector for $(M^{1/\gamma,1/\gamma})'$. Now $1/\gamma < 1$ so that $D^{1/\gamma,1/\gamma}$ is a subrepresentation of $D^{1/\gamma,1}$, which is cyclic. A subrepresentation of a cyclic representation is a cyclic representation; therefore $M^{1/\gamma,1/\gamma}$ has a cyclic vector, which is also a cyclic vector for $(M^{\gamma,\gamma})'$.

It is a famous theorem due to Murray and von Neumann that if two finite von Neumann algebras are isomorphic, and each of them as well as their commutants have a cyclic vector, then they are spatially isomorphic, that is, there is a unitary operator betweeb the underlying Hilbert spaces which implements the isomorphism. Applying this to the von Neumann algebras $M^{1/\gamma,1/\gamma}$ and $M^{1/\gamma,1}$ yields the unitary equivalence of $D^{1/\gamma,1/\gamma}$ and $D^{1/\gamma,1}$, which contradicts Theorem 19.2 which says that $1/\gamma = 1$. □

19.4 Unfinished business from [1]

Theorem 19.4 ([1, Theorem 1.2, p. 196]) *Let $\gamma := \alpha\beta = 1$. Then f satisifes the Gabor condition for scales α and β if and only if*

$$\sum_{n=-\infty}^{\infty} f(t+n\beta)e^{2\pi ins} \neq 0$$

for almost all t and s.

For any $\gamma > 0$, consider the representation T^γ acting on $L^2([0,1) \times [0,1))$ given by

$$T^\gamma_{p,q,r} f(t,s) = e^{-2\pi ir\gamma} e^{2\pi iqt} e^{2\pi ips} f(\langle t - p\gamma\rangle, s),$$

where $\langle x\rangle$ denotes the fractional part of x.

Theorem 19.4 follows from part 3 of the following theorem.

Theorem 19.5 ([1, Theorem 3.1, p. 200]) *Let α and β be arbitrary positive scales and write $\gamma = \alpha\beta$.*

1. *T^γ is unitarily equivalent to $D^{\gamma,1}$.*
2. *A function $f(t,s)$ is a cyclic vector for T^γ if and only if for almost all $0 \le t < 1$, the sequence of functions $\{f^{n,t}(s)\}$ spans a dense subspace of $L^2([0,1))$, where*
$$f^{n,t}(s) = e^{2\pi ins} f(\langle t - n\gamma\rangle, s).$$
3. *If $\gamma = 1$, than an $f \in L^2(\mathbf{R})$ satisfies the Gabor condition for scales α and β if and only if*
$$\sum_{n=-\infty}^{\infty} f(t+n\beta)e^{2\pi ins} \neq 0$$
for almost all t and s.

20 Unfinished business from [5]

20.1 Frames for the wavelet transform—[5, Prop. 3.3.2, p. 69;a sufficient condition]

20.2 Frames for the windowed Fourier transform—[5, Section 3.4; a necessary condition and a sufficient condition]

20.3 Time Frequency localization—[5, Theorem 3.5.1, p. 88; wavelet transform]

20.4 Frames for the limiting case $\omega_0 t_0 = 2\pi$ have bad localization properties—[5, Theorem 4.1.1 (Balian-Low), p. 108; Zak transform]

20.5 Orthonormal wavelet bases with good time-frequency localization—[5, Section 4.2]

20.6 Regularity of orthonormal wavelet bases—[5, Theorem 5.5.1, p. 153]

References

[1] Larry Baggett, *Processing a radar signal and representations of the discrete Heisenberg group*, Colloq. Math. **60/61** (1990), 195–203.

[2] Ola Bratteli and Palle Jorgensen, "Wavelets through a Looking Glass" Birkhäuser 2002

[3] J. W. Cooley and J. W. Tukey, *An algorithm for machine computation of complex Fourier series*, Math. Comput. **19** (1965). 297–301.

[4] Ingrid Daubechies, *The Wavelet transform, time-frequency localization and signal analysis* I.E.E.E. Trans. Inform. Theory **36** (1990), 961–1005.

[5] Ingrid Daubechies "Ten Lectures on Wavelets" SIAM 1992.

[6] P. Duhamel and M. Vetterli, *Fast Fourier transforms: a tutorial review and a state of the art*, Signal Proc. **19**(4) (1990), 259–299.

[7] G. B. Folland and A. Sitaram, *The uncertainty principle: a mathematical survey*, J. Fourier Anal. Appl. 3(3) (1997), 207–238.

[8] Karlheinz Gröchenig, "Foundations of Time-Frequency Analysis" Birkhäuser 2001

[9] Stéphane Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$, Trans. Amer. Math. Soc., 315 (1989), 69–87.

[10] Stéphane Mallat, "A Wavelet Tour of Signal Processing" Second Edition, Academic Press 1999

[11] H. J. Nussbaumer, "Fast Fourier transforms and Convolution algorithms," Springer-Verlag 1982.

[12] Elias M. Stein and Guido Weiss, "An introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press 1971

[13] Robert Strichartz, "A Guide to Distribution Theory and Fourier Transforms," CRC Press 1994

[14] J. Yeh, "Lectures on Real Analysis," World Scientific 2000.