Complex Analysis
Math 220C—Spring 2008

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Contents

1 Monday March 31, 2008—class cancelled due to the Master's travel plans

2 Wednesday April 2, 2008–Course information; Riemann's removable singularity theorem; Assignment 1

2.1 Course information

- Course: Mathematics 220C MWF 12:00–12:50 MSTB 114
- Instructor: Bernard Russo MSTB 263 Office Hours: Friday 2:30-4 or by appointment (Note: I am usually in my office the hour before class, that is, MWF 11)
- There is a link to this course on Russo's web page: www.math.uci.edu/∼brusso
- Homework: There will be assignments at almost every lecture with at least one week notice before the due date.
- Grading: TBA
- Holidays: May 26
- Text: Robert E. Greene and Steven G. Krantz, "Function Theory of One Complex Variable", third edition.
- Material to be Covered: We will cover chapter 10 and parts of chapters 9 and 15. We shall also provide a comprehensive review of complex analysis in preparation of the qualifying examination in complex analysis. The review will be based on notes from Math 220ABC (1993-94) which will be handed out. These notes follow the book by Conway (see below). Two copies of Conway have been put on one day reserve in the Science Library.
- Some alternate texts that you may want to look at, in no particular order. There are a great number of such texts at the undergraduate and at the graduate level.

  Undergraduate Level

  1. S. Fisher: Complex Variables
  2. R. Churchill and J. Brown; Complex Variables and Applications
  3. J. Marsden and M. Hoffman, Basic Complex Analysis
  4. E. Saff and A. Snider: Fundamentals of Complex Analysis

  Graduate Level

  1. L. Ahlfors; Complex Analysis
  2. J. Conway; Functions of one Complex Variable
  3. J. Bak and D. Newman; Complex Analysis

2.2 Riemann's removable singularity theorem

Theorem 2.1 (Riemann's Removable Singularity Theorem) *Let f be analytic on a punctured disk $B(a,R) - \{a\}$. Then f has an analytic extension to $B(a,R)$ if and only if $\lim_{z\to a}(z-a)f(z)$ exists and equals 0.*

Proof: If the analytic extension g exists, then $\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)g(z) = 0\cdot g(a) = 0$.

Now suppose that $\lim_{z\to a}(z-a)f(z) = 0$. Define a function g by $g(z) = (z-a)f(z)$ for $z \neq a$ and $g(a) = 0$. The function g is analytic for $z \neq a$, and is continuous at a. We shall show using the Triangulated Morera theorem (see below) that g is analytic at a. Assuming for the moment that this is true, let us complete the proof. Since g is analytic and $g(a) = 0$, then by a consequence of the identity theorem, $g(z) = (z-a)h(z)$ where h is analytic in $B(a,R)$. Thus, for $z \neq a$, $(z-a)f(z) = g(z) = (z-a)h(z)$, and thus $f(z) = h(z)$ for $z \neq a$. Thus h is the analytic extension of f to $B(a,R)$.

It remains to prove that g is analytic at a. We first state a generalization of the theorem of Morera.

Theorem 2.2 (Triangulated Morera Theorem) *Let f be continuous on a domain D and suppose that $\int_T f(z)\,dz = 0$ for every triangle T which together with its inside lies in D. Then f is analytic in D.*

Proof: Let $a \in D$ and let $B(a,R) \subset D$. For $z \in B(a,R)$, let $F(z) := \int_{[a,z]} f(s)\,ds$ where $[a,z]$ denotes the line segment from a to z. For any other point $z_0 \in B(a,R)$, by our assumption, $F(z) = \int_{[a,z_0]} f(s)\,ds + \int_{[z_0,z]} f(s)\,ds$. Therefore

$$\frac{F(z)-F(z_0)}{z-z_0} - f(z_0) = \frac{1}{z-z_0}\int_{[z_0,z]}[f(s)-f(z_0)]\,ds$$

and

$$\left|\frac{F(z)-F(z_0)}{z-z_0} - f(z_0)\right| \leq \sup_{s\in[z_0,z]} |f(s)-f(z_0)|.$$

Since f is continuous at z_0, $F'(z_0)$ exists and equals $f(z_0)$ so f is analytic. □

To complete the proof of Riemann's removable singularity theorem, it remains to show that g is analytic using the Triangulated Morera theorem. We must show that if T is any triangle in $B(a,R)$, then $\int_T f(s)\,ds = 0$. There are four possible cases.

Case 1: <u>a **is a vertex of** T</u> : In this case let x and y denote points on the two edges for which a is an endpoint. Then $\int_T f(s)\,ds = \int_{[a,y,x]} f(s)\,ds + \int_{[y,x,b,c]} f(s)\,ds$ where b and c are the other two vertices of T and $[\alpha,\beta,\cdots]$ denotes a polygon with vertices $\alpha,\beta,\cdots$. By the continuity of g at a, the first integral approaches zero as x and y approach a. The second integral is zero by Cauchy's theorem.

Case 2: <u>a **is inside** T</u> : In this case, draw lines from a to each of the vertices of T. Then $\int_T f(s)\,ds$ is the sum of three integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 3: <u>a **lies on an edge of** T</u> : In this case, draw a line from a to the vertex which is opposite to the edge containing a. Then $\int_T f(s)\,ds$ is the sum of two integrals of f over triangles having a as a vertex, each of which is zero by case 1.

Case 4: <u>a **is outside of** T</u> : In this case, $\int_T f(s)\,ds = 0$ by Cauchy's theorem. □

2.3 Examples of analytic continuations

Example 1 (Example 10.1.1 on page 299 of the text) Let $f(z) = \sum_0^\infty z^j$. The radius of convergence of this power series (a geometric series) is 1 and the sum is $1/(z-1)$. Thus the function f on $\{|z| < 1\}$ has an analytic continuation g (that is, an analytic extension) to $\mathbf{C} - \{1\}$, namely $g(z) = 1/(1-z)$, for $z \in \mathbf{C} - \{1\}$.

Example 2 (Example 10.1.2 on page 300 of the text) Let $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ for $\operatorname{Re} z > 0$. Using integration by parts we can see that this holomorphic function continues analytically to $\mathbf{C} - \{0, -1, -2, \ldots\}$. The extension has poles of order 1 at each of the points $0, -1, -2, \ldots$ and the residue at $-k$ is $(-1)^k/k!$ (see Proposition 15.1.4 of the text—we will cover this later in the course)

Example 3(see section 15.2 of the text—we will cover this later in the course) Let $\zeta(z) = \sum_1^\infty n^{-z}$ which converges for $\operatorname{Re} z > 1$ to a holomorphic function, the zeta function. If we define $\eta(z) = \sum_1^\infty (-1)^{n+1} n^{-z}$ then it can be shown that this series converges for $\operatorname{Re} z > 0$ and it is easily seen that $\eta(z) = (1 - 2^{1-z})\zeta(z)$ holds for $\operatorname{Re} z > 1$. Using this formula as a definition, we see that ζ continues analytically to $\operatorname{Re} z > 0$. But in fact, because of the identity

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\sin\left(\frac{1-z}{2}\pi\right)\Gamma(z)\zeta(z), \tag{1}$$

valid for $0 < \operatorname{Re} z < 1$, we can easily see that ζ analytically continues to $\mathbf{C} - \{1\}$. As a matter of fact, ζ has a pole at $z = 1$, and because of (1), it has zeros at the negative even integers. These are called the trivial zeros of the zeta function. The Riemann hypothesis is the (as yet unproven) assertion that all other zeros of the zeta function must lie on the line $\operatorname{Re} z = 1/2$.

Assignment 1 Problems 14 and 15 of chapter 6 of the text. These will de discussed in the informal discussion section on Friday April 4 at 4 pm in MSTB 256. The due date for this assignment is Friday April 11.

3 Friday April 4

Definitions: **function element, direct analytic continuation, analytic continuation** (See Definitions 10.1.4 and 10.1.5 on page 303 of Greene-Krantz.

Example: $\sqrt{z}$ (See Example 10.1.3 on page 301 of Greene-Krantz)

Definition: **analytic continuation along a curve** (See Definition 10.2.1 on page 304 of Greene-Krantz)

First of two uniqueness results:

Proposition 3.1 (Proposition 10.2.2 on page 305 of Greene-Krantz) *Any two analytic continuations of a function element along a curve are equivalent. This means that if* $\{(f_t, U_t) : t \in [0,1]\}$ *and* $\{(\tilde{f}_t, \tilde{U}_t) : t \in [0,1]\}$ *are analytic continuations of* (f, U) *along a curve* γ*, then* $f_t = \tilde{f}_t$ *on* $U_t \cap \tilde{U}_t$ *for all* $t \in [0,1]$.

3.1 Discussion section—Friday April 4

Problems 14 and 15 in chapter 6 of Greene-Krantz were discussed.

4 Monday April 7

Assignment 2 Exercises 9,10,11,12 of chapter 10 of Greene-Krantz. These will be discussed in the discussion section on Friday April 11 at 4 pm in MSTB 256. The due date for this assignment is Friday April 18.

4.1 Proof of Proposition 3.1

UNDER CONSTRUCTION

4.2 The identity theorem and its proof

Theorem 4.1 *Let D be a connected open set and let f be analytic on D. The following are equivalent:*

(a) $f \equiv 0$*, that is,* $f(z) = 0$ *for every* z *in* D*.*

(b) *There exists a point* $z_0 \in D$ *such that* $f^{(n)}(z_0) = 0$ *for every* $n \geq 0$*.*

(c) *The set* $\{z \in D : f(z) = 0\}$ *has a limit point in* D*, that is, there is a sequence of* <u>*distinct*</u> *points* z_k *in* D *such that* $f(z_k) = 0$ *and* $\lim_{k\to\infty} z_k$ *exists and* <u>*belongs*</u> *to* D*.*

Proof: (a) implies (c) is trivial.

<u>(c) implies (b)</u>: Let z_0 be a limit point of $\{z \in D : f(z) = 0\}$ and suppose $z_0 \in D$. Since D is open, $\exists R > 0$ such that $B(z_0, R) \subset D$. Let us assume that (b) does not hold for any point of D. Then $\exists n \geq 1$ such that $0 = f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0)$ and $f^{(n)}(z_0) \neq 0$. Expanding f is a Taylor series about the point z_0, we have $f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots = (z - z_0)^n(a_n + a_{n+1}(z - z_0) + \cdots) = (z - z_0)^n g(z)$, where g is analytic and $g(z_0) = a_n = f^{(n)}(z_0)/n! \neq 0$. We have now reached a contradiction as follows. Since g is continuous and $g(z_0) \neq 0$, $\exists r,\ 0 < r \leq R$ with $g(z) \neq 0$ for $|z - z_0| < r$. Hence $\{z \in D : f(z) = 0\} \cap B(z_0, r) = \{z_0\}$. This contradicts the fact that z_0 is a limit point of $\{z \in D : f(z) = 0\}$, and thus completes the proof of (c) implies (b).

<u>(b) implies (a)</u>: Let $A = \{z \in D : \forall n \geq 0,\ f^{(n)}(z) = 0\}$. By assumption $A \neq \emptyset$. We shall prove that both $D - A$ and A are open sets. It will follow from the connectedness that $D = A$ and therefore f is identically zero in D.

<u>A is open</u>: Let $a \in A$. Since D is open, $\exists R > 0$ with $B(a, R) \subset D$. Write f in a Taylor series $f(z) = \sum_0^\infty a_n(z - a)^n$ for $|z - a| < R$ with $a_n = f^{(n)}(a)/n!$. Since $a \in A$, each $a_n = 0$ and so f is identically zero on $B(a, R)$. This means that $B(a, R) \subset A$ and so A is an open set.

<u>$D - A$ is open</u>: If $z \in D - A$, then there exists n_0 with $f^{(n_0)}(z) \neq 0$. Since $f^{(n_0)}$ is a continuous function, by "persistence of sign", there exists $r > 0$ such that $f^{(n_0)}$ never vanishes on $B(z, r)$. This says that $B(z, r) \subset D - A$ showing that $D - A$ is an open set. □