Structure of the predual of a JBW^*-triple

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Alfsen and Shultz, in [1], have given several characterizations of the state space of a JB-algebra. Their main result was the following: a compact convex set K is (affinely isomorphic) to the state space of a JB-algebra iff

(i) K has the Hilbert ball property;

(ii) K splits into atomic and non-atomic parts;

(iii) every norm exposed face of K is projective;

(iv) every continuous real affine function on K is the difference of two orthogonal positive continuous affine functions on K.

It was also shown in [1] that (i) and (ii) can be replaced by the following "pure state properties", which are less geometric but more physical:

(v) every extreme point of K is norm exposed;

(vi) every P-projection preserves extreme rays of K;

(vii) for every pair f, g of extreme points of K with support projective units v, u respectively, $f(u) = g(v)$.

In our study of operator algebras without order [7], [8], we proved analogs of each of these properties (except (iv) which has no meaning in our setting) if K is the unit ball of the image of a contractive projection on the dual of a *J^*-algebra*. These results were used to solve the contractive projection problem for J^*-algebras and to characterize the J^*-automorphisms of order 2, [9], [10].

This work on J^*-algebras suggests very strongly that a characterization of the unit ball of the dual of a JB^*-triple exists in terms of these properties. JB^*-triples were introduced by Kaup [13] [14] and can be considered as a Hilbert space-free analog of J^*-algebras.

In order to obtain this characterization it is desirable first to obtain a universal enveloping JBW^*-triple (second dual) of a given JB^*-triple, as well as a representation theory of Gelfand Naimark type. At present, these two goals represent outstanding problems in the analytic theory of infinite dimensional Jordan triple systems.

As the next step in the development of the theory of operator algebras without order, in this paper we study normal functionals on *JBW*-triples*, i.e., *JB**-triples which are Banach dual spaces. Our principal result is a decomposition of the pre-dual of a *JBW**-triple into atomic and non-atomic parts. This implies that U itself decomposes into the direct sum of atomic and non-atomic *JBW**-subtriples.

A major technical tool for establishing our results is the existence and uniqueness of a polar decomposition of an arbitrary normal functional on a *JBW**-triple. Since *JBW**-triples are not concretely represented as operators, we use algebraic and analytic techniques (Peirce decompositions of Jordan triple systems and Siegel domain realizations of bounded symmetric domains) in order to obtain the polar decomposition.

In addition to the atomic decompositions mentioned above we prove the analogs of the other properties mentioned above for the predual of a *JBW**-triple. Noteworthy among these is the extreme ray property.

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§ 1. Peirce projections. Polar decomposition of a normal functional

A *JB*-triple* is a complex Banach space U equipped with a continuous sesquilinear form

$$U \times U \to \mathscr{L}(U), \quad (x, y) \to D(x, y)$$

such that

(1. 1) the triple product $\{xyz\} \equiv D(x, y)\, z$ is symmetric in x and z;

(1. 2) $$\{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}\, vz\} - \{u\{vxy\}\, z\};$$

(1. 3) $D(z, z)$ is hermitian (cf. [3]) with spectrum in $[0, \infty)$;

(1. 4) $$\|D(z, z)\| = \|z\|^2$$

for all x, y, z, u, v in U.

A *JB**-triple also satisfies (cf. [14])

(1. 5) $$\|\{zzz\}\| = \|z\|^3.$$

Let e be a non-zero *tripotent* (i.e., $\{eee\} = \pm e$) in a *JB**-triple U. By (1. 5), $\|e\| = 1$ and by (1. 3), e is a *positive tripotent*: $e = \{eee\}$.

The conditions (1. 1) and (1. 2) are the defining equations of a *Jordan triple system*. Odd powers of elements of U are defined inductively by $x^1 = x$, $x^m = \{xx^{m-2}x\}$ for m odd, $m \geqq 3$. One has $\{x^{m_1}x^{m_2}x^{m_3}\} = x^{m_1+m_2+m_3}$ (cf. [15], § 3. 3). For the convenience of the reader we shall now use some identities valid for all Jordan triple systems to define and establish known properties of the Peirce projections relative to a tripotent. Then we shall use (1. 5) to show that in a JB^*-triple, the Peirce projections are contractive. This provides an important step in the proof of the polar decomposition (see the proof of Lemma 1. 6).

Denote by Q the *quadratic operator* on U: $Q(x)y = \{xyx\}$, for $x, y \in U$. Then define $Q(x, z) = \frac{1}{2}(Q(x+z) - Q(x) - Q(z))$ so that $Q(x, z)y = \{xyz\}$, for $x, y, z \in U$[1]).

We define the *Peirce projections* $P_k(e)$, $k = 0, 1, 2$ relative to a tripotent e by

$$P_2(e) = Q(e)^2, \quad P_1(e) = 2(D(e, e) - Q(e)^2),$$
$$P_0(e) = I - 2D(e, e) + Q(e)^2.$$

Note that $\sum_{j=0}^{2} P_j(e) = I$ and $D(e, e) = P_2(e) + \frac{1}{2} P_1(e)$. It follows from [15], $JP3$, $JP23$, $JP25$, that each $P_j(e)$ is idempotent and that $P_k(e)\,P_j(e) = 0$ if $k \neq j$. Let $U_k(e) = P_k(e)\,U$ be the range of $P_k(e)$, $k = 0, 1, 2$. By [15], $JP1$, $U_k(e)$ is contained in the $\frac{1}{2}k$ eigenspace of $D(e, e)$. From this we get the *Peirce decomposition*

(1. 6) $$U = U_2(e) \oplus U_1(e) \oplus U_0(e)$$

and the fact that $U_k(e)$ is the $\frac{1}{2}k$ eigenspace of $D(e, e)$, $k = 0, 1, 2$.

Now note that by (1. 2), the operator $L = D(e, e)$ is a *derivation* of U in the sense that $L(\{uvz\}) = \{L(u)vz\} - \{uL(v)z\} + \{uvL(z)\}$. Therefore we have the fundamental property:

(1. 7) $$\{U_i(e)\; U_j(e)\; U_k(e)\} \subset U_{i-j+k}(e).$$

The following two facts lie a bit deeper (cf. Loos [15], Satake [16]):

(1. 8) $$\{U_2(e)\; U_0(e)\; U\} = \{U_0(e)\; U_2(e)\; U\} = 0;$$

(1. 9) $U_2(e)$ is a complex Jordan *-algebra, with product $x \circ y = \{xey\}$, unit e, and involution $z^* = \{eze\}$.

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[1]) Loos, in [15], defines $Q(x)y$ to be $\frac{1}{2}\{xyx\}$, and $Q(x, z)$ to be $Q(x+z) - Q(x) - Q(z)$. He also denotes $Q(x)$ by Q_x. Hence some care must be exercised when using the list of identities in [15].

It follows from (1. 2) that $\{\{yex\}ez\}+\{xe\{yez\}\}-\{ye\{xez\}\}=\{x\{eye\}z\}$, and therefore

$$\text{(1. 10)} \qquad \{x\circ y^{\#}\circ z\}=\{xyz\} \quad \text{for} \quad x, y, z\in U_2(e),$$

where the left side of (1. 10) denotes the *Jordan triple product* in $U_2(e)$, i.e.,

$$\{a\circ b\circ c\}=(a\circ b)\circ c-(a\circ c)\circ b+a\circ(b\circ c).$$

Thus, if U is a JB^*-triple, then $U_2(e)$ is a JB^*-algebra (=Jordan C^*-algebra) and hence $\{x\in U_2(E)\colon x=x^{\#}\}$ is a JB-algebra (cf. [18]).

In order to show that in a JB^*-triple the Peirce projections are each of norm one, we shall use the following lemma, in which a one parameter group of isometries is defined for each tripotent.

Lemma 1. 1. *Let e be an arbitrary tripotent in a JB^*-triple U and let $\lambda\in C$, $|\lambda|=1$. Define a linear map $S_\lambda=S_\lambda(e)\colon U\to U$ by*

$$\text{(1. 11)} \qquad S_\lambda=S_\lambda(e)=\lambda^2P_2(e)+\lambda P_1(e)+P_0(e).$$

Then

(a) *S_λ preserves the triple product, i.e.,*

$$S_\lambda(\{xyz\})=\{S_\lambda x, S_\lambda y, S_\lambda z) \quad \text{for} \quad x, y, z\in U;$$

(b) *S_λ is an isometry of U onto U.*

Proof. (a) It suffices, since $2\{xyz\}=Q(x+z)y-Q(x)y-Q(z)y$ and

$$Q(x)y=\frac{1}{4}\sum_{k=0}^{3}(-i)^k(x+i^k y)^3$$

to prove $S_\lambda(x^3)=(S_\lambda x)^3$. Write $x=\sum_{j=0}^{2}x_j$ with $x_j=P_jx$. Then $S_\lambda x=\sum\lambda^j x_j$,

$$\{S_\lambda x, S_\lambda x, S_\lambda x\}=\sum_{i,j,k}\lambda^{i-j+k}\{x_ix_jx_k\},\ \{xxx\}=\sum_{i,j,k}\{x_ix_jx_k\}.$$

By (1. 7), $\{x_ix_jx_k\}\in U_{i-j+k}$. Therefore $S_\lambda(\{xxx\})=\sum_{i,j,k}\lambda^{i-j+k}\{x_ix_jx_k\}$.

(b) $\|S_\lambda(x)\|^3=\|\{S_\lambda(x), S_\lambda(x), S_\lambda(x)\}\|=\|S_\lambda\{xxx\}\|\leqq\|S_\lambda\|\,\|\{xxx\}\|=\|S_\lambda\|\,\|x\|^3$. Thus $\|S_\lambda\|\leqq 1$ and since $S_\lambda S_{\bar\lambda}=I$, S_λ is isometric. □

An alternate proof of (b) of Lemma 1. 1 is to observe that $D(e, e)$ being hermitian, $\exp(2itD(e,e))$ is an isometry of U for all $t\in\mathbb{R}$. But $\exp(2itD(e,e))=S_{\exp(it)}$ since

$$\frac{d}{dt}S_{\exp(it)|t=0}=2iP_2(e)+iP_1(e)=2iD(e,e).$$

On the other hand the elementary proof given above shows that $D(e, e)$ is hermitian. It follows by an approximation argument that the assumption that $D(x, x)$ is hermitian in (1. 3) is redundant.

Corollary 1. 2. *For each tripotent e:*

(a) *The Peirce projections $P_k(e)$ are contractive,* $0 \leqq k \leqq 2$;

(b) $P_2(e)+P_0(e)$ *is a contractive projection.*

Proof. Let P_k denote $P_k(e)$, $0 \leqq k \leqq 2$.

Since $P_2+P_0=\frac{1}{2}(S_1+S_{-1})$, $\|P_2+P_0\| \leqq 1$. Also since $P_1=\frac{1}{2}(S_1-S_{-1})$, $\|P_1\| \leqq 1$. Finally $P_2=\frac{1}{2}(I-S_i)(P_2+P_0)$ and $P_0=\frac{1}{2}(I+S_i)(P_2+P_0)$ imply $\|P_2\| \leqq 1$ and $\|P_0\| \leqq 1$. □

Recall that a J^*-algebra is a norm closed linear subspace M of $\mathscr{L}(H, K)$ (H, K complex Hilbert spaces) which is closed under the triple product $a \to aa^*a$. It is easy to verify that M is a JB^*-triple with $\{abc\}=\frac{1}{2}(ab^*c+cb^*a)$. The tripotents in M are the partial isometries v and the corresponding Peirce projections are given by

$$P_2(v)\,x=lxr, \quad P_1(v)\,x=lx(1-r)+(1-l)\,xr,$$
$$P_0(v)\,x=(1-l)\,x(1-r),$$

where $l=vv^*$ and $r=v^*v$.

The following remark will be used in § 2 (see Corollary 2. 5)

Remark 1. 1. Let u, v be rank one, norm one elements of $\mathscr{L}(H, K)$. Then there is a rank 1, norm 1 element w in $\mathscr{L}(H, K)$ and a scalar λ, $|\lambda|=1$ such that $S_{-1}(w)\,u=\lambda v$. If $H=K$ and u and v are symmetric with respect to a conjugation on H, then w can be chosen symmetric as well.

To see this let $u=\varphi \otimes \psi$, $v=\alpha \otimes \beta$ and solve for $w=\xi \otimes \eta$ in the equation $S_{-1}(w)\,u=v$, where for unit vectors φ, ψ, $\varphi \otimes \psi$ denotes the rank 1, norm 1 operator $\gamma \to (\gamma, \psi)\,\varphi$. One has $l=ww^*=\xi \otimes \xi$, $r=w^*w=\eta \otimes \eta$ so we are to solve

$$4lur-2(lu+ur)+u=v.$$

This reduces to

$$\beta=2(\eta, \psi)\,\eta-\psi, \quad \alpha=2(\varphi, \xi)\,\xi-\varphi$$

which can be solved for ξ and η if (φ, α) and (ψ, β) are real; namely

$$\xi=\frac{\varphi+\alpha}{\|\varphi+\alpha\|}, \quad \eta=\frac{\psi+\beta}{\|\psi+\beta\|}.$$

The second statement follows since $\varphi \otimes \psi$ is symmetric means that $\varphi=\bar{\psi}$ where $\bar{\psi}$ is the conjugation on H.

Since no confusion arises, we shall denote the action of $P_k(e)$ on linear functionals by the same letter: $P_k(e)\,g=g \circ P_k(e)$, for $g \in U'$. The following lemma shows that the ranges of $P_0(e)$ and $P_2(e)$ are M-summands in $U_2(e)+U_0(e) \subset U$ and L-summands in $P_2(e)\,U'+P_0(e)\,U' \subset U'$.

Lemma 1. 3. *Let e be a tripotent in a JB*-triple U. With* $P_k = P_k(e)$, $0 \leqq k \leqq 2$, *we have*

(a) $\|P_2 x + P_0 x\| = \max(\|P_2 x\|, \|P_0 x\|)$, $x \in U$;

(b) $\|P_2 g + P_0 g\| = \|P_2 g\| + \|P_0 g\|$, $g \in U'$;

(c) *if* $\|P_2 g\| = \|g\|$, *then* $P_0 g = 0$;

(d) *if* $\|P_0 g\| = \|g\|$, *then* $P_2 g = 0$.

Proof. By Corollary 1. 2 (a), $\|P_2 x + P_0 x\| \geqq \max(\|P_2 x\|, \|P_0 x\|)$. With

$$y = P_2 x,\ z = P_0 x \quad \text{and} \quad \max(\|y\|, \|z\|) = 1,$$

we have by (1. 5) and (1. 8) for $n \geqq 1$, $\|y+z\| = \|(y+z)^{3^n}\|^{3^{-n}} = \|y^{3^n} + z^{3^n}\|^{3^{-n}} \leqq (2)^{3^{-n}} \to 1$. This proves (a), and (b) follows from (a).

By Corollary 1. 2 (b), $\|P_2 g\| + \|P_0 g\| = \|P_2 g + P_0 g\| \leqq \|g\|$. Therefore (c) and (d) hold. □

Proposition 1 below generalizes [5], Lemma 3. 1, which has been an important tool for dealing with dual spaces of C^*-algebras. It shows that functionals on the subspaces $U_2(e)$ and $U_0(e)$ have unique Hahn-Banach extensions to functionals on U.

To prove Proposition 1 we shall need the following lemma, which provides a relation between elements from $U_2(e)$ or $U_0(e)$ and elements from $U_1(e)$.

Lemma 1. 4. *Let e be a tripotent in a JB*-triple U, let* $x \in U_2(e) \cup U_0(e)$, $y \in U_1(e)$, *and* $t \in \mathbb{C}$. *Then*

$$(1.\ 12) \qquad (x+ty)^{3^n} = x^{3^n} + t2^n D(x^{3^{n-1}}, x^{3^{n-1}}) \cdots D(x^3, x^3)\, D(x, x)\, y + O(|t|^2).$$

Proof. By (1. 7), $\{xyx\} = 0$, and therefore

$$(x+ty)^3 = x^3 + t\{xyx\} + 2t\{xxy\} + O(|t|^2) = x^3 + 2tD(x, x)\, y + O(|t|^2).$$

The result now follows by induction: with

$$x_n = x^{3^n} \text{ and } y_n = 2^n D(x_{n-1}, x_{n-1}) \cdots D(x, x)\, y,$$

we have

$$(x+ty)^{3^{n+1}} = (x_n + ty_n + O(|t|^2))^3 = (x_n + ty_n)^3 + O(|t|^2) = x_n^3 + 2tD(x_n, x_n)\, y_n + O(|t|^2)$$
$$= x_{n+1} + ty_{n+1} + O(|t|^2). \qquad \square$$

Note that by the continuity of $(x, y) \to D(x, y)$, the constant in the $O(|t|^2)$ depends only on n if $\|x\| \leqq 1$ and $\|y\| \leqq 1$.

Proposition 1. *Let U be a JB^*-triple. Let $f \in U'$ and let e be a tripotent in U.*

(a) *If* $\|P_2(e)f\| = \|f\|$, *then* $P_2(e)f = f$.

(b) *If* $\|P_0(e)f\| = \|f\|$, *then* $P_0(e)f = f$.

Therefore every bounded functional on $U_2(e)$ or $U_0(e)$ has a unique Hahn-Banach extension to U.

Proof. (a) By Lemma 1. 3 $P_0(e)f = 0$. It remains to prove that $P_1(e)f = 0$. To this end let $y \in U_1(e)$. We are to prove that $f(y) = 0$. We may assume $\|f\| = 1$, $f(y) \geqq 0$ and $\|y\| \leqq 1$. For $\varepsilon > 0$ choose $x \in U_2(e)$ with $\|x\| = 1$ and $f(x) \geqq 1 - \varepsilon$. Then for $t \in \mathbb{R}$,

$$\|x + ty\| \geqq f(x + ty) = f(x) + tf(y) \geqq 1 - \varepsilon + tf(y).$$

Therefore, by Lemma 1. 4

$$(1 - \varepsilon + tf(y))^{3^n} \leqq \|x + ty\|^{3^n} = \|(x + ty)^{3^n}\|$$
$$\leqq \|x^{3^n}\| + t2^n\|y\| + O(|t|^2)$$

and so

$$(1 - \varepsilon)^{3^n} + 3^n tf(y)(1 - \varepsilon)^{3^n - 1} + O(|t|^2) \leqq 1 + t2^n\|y\| + O(|t|^2).$$

Letting $\varepsilon \to 0$, and dividing by $|t|$ results in

$$f(y) + O(|t|) \leqq \left(\frac{2}{3}\right)^n \|y\| + O(|t|).$$

Letting $t \to 0$, then $n \to \infty$ yields $f(y) = 0$. The proof of (b) is similar. □

Since $U_2(e)$ is a complex Jordan *-algebra, one obtains a positive hermitian form F_2 on $U_2(e)$ with values in $U_2(e)$, by defining $F_2(x, y) = x \circ y^\#$ for $x, y \in U_2(e)$. This form is positive definite if U is a JB^*-triple. Note that $x \circ y^\# = \{xe\{eye\}\} = \{xye\}$ by (1. 2).

We show next that the formula $(x, y) \to \{xye\}$ also defines a positive definite hermitian $U_2(e)$-valued form on $U_1(e)$. (cf. [15], 10. 4, for the finite dimensional case).

Lemma 1. 5. *Let e be a tripotent in a JB^*-triple U and define*

$$F_1 \colon U_1(e) \times U_1(e) \to U_2(e) \quad by \quad F_1(x, y) = \{xye\}.$$

Then

(a) *F_1 is hermitian: $F_1(x, y)^\# = F_1(y, x)$ for $x, y \in U_1(e)$;*

(b) *F_1 is positive definite: $F_1(x, x) \in U_2(e)^+$ and $F_1(x, x) = 0$ implies $x = 0$, for $x \in U_1(e)$.*

Proof. (a) By (1. 2)

$$F_1(x, y)^* = \{e\{xye\}e\} = \{\{yxe\}ee\} + \{ee\{yxe\}\} - \{yx\{eee\}\} = \{yxe\} = F_1(y, x).$$

(b) Let $x \in U_1(e)$.

We may assume $\|x\| \leqq 1$. By (1. 3) and (1. 4) $\mathrm{Sp}(D(x, x)) \subset [0, 1]$ so that $\mathrm{Sp}(D(x, x) - \mathrm{Id}) \subset [-1, 0]$. Since $D(x, x) - \mathrm{Id}$ is hermitian, $\|D(x, x) - \mathrm{Id}\| \leqq 1$ (cf. [3]). Thus $\|\{xxe\} - e\| \leqq 1$ so that the series

$$e + \frac{1}{2}y + \frac{1}{2!}\cdot\frac{1}{2}\left(\frac{1}{2} - 1\right)y^2 + \cdots + \frac{1}{n!}\,\frac{1}{2}\left(\frac{1}{2} - 1\right)\cdots\left(\frac{1}{2} - n + 1\right)y^n + \cdots$$

with $y = \{xxe\} - e$, converges to an element z in the *JB**-algebra $U_2(e)$ with $z^2 = \{xxe\}$. By (a) $\{xxe\}$ is self-adjoint and therefore so is z. Therefore $F_1(x, x) \in U_2(e)^+$. The last statement is proved in [17]. □

The following lemma is a technical result which enables us to introduce a natural order in the set of tripotents. It will be used in the uniqueness part of Proposition 2.

Lemma 1. 6. *Let e be a tripotent in a JB*-triple U, let $x \in U$, $\|x\| = 1$, and $P_2(e)\,x = e$. Then $P_1(e)\,x = 0$.*

Proof. Let $x = x_2 + x_1 + x_0$ be the Peirce decomposition of x with respect to e. By assumption $x_2 = e$ and we wish to prove that $x_1 = 0$.

Let $y = -S_i(e)\,x = e - ix_1 - x_0$. Then $\|y\| = 1$ by Lemma 1. 1 so that

$$z \equiv \frac{1}{2}(x + y) = e + \lambda x_1 \quad \text{with} \quad \lambda = \frac{1}{2}(1 - i)$$

and $\|z\| \leqq 1$. By (1. 5) $\|\{zzz\}\| = \|z\|^3 \leqq 1$. By (1. 7) and Corollary 1. 2,

$$\|e + 2|\lambda|^2\{x_1x_1e\}\| = \|P_2\{zzz\}\| \leqq \|\{zzz\}\| \leqq 1.$$

It follows from Lemma 1. 5 (b) that $x_1 = 0$. □

Lemma 1. 6 can be used to give a description of the natural partial order on the set of tripotents of a *JB**-triple. Recall ([15], § 5) that $e' \leqq e$ for tripotents e' and e in a Jordan triple system means that $e - e'$ is a tripotent orthogonal to e', i.e.,

$$D(e - e', e') = 0.$$

Corollary 1. 7. *For tripotents e and e' in a JB*-triple, $e' \leqq e$ if and only if $P_2(e')\,e = e'$.*

Now suppose U is a *JBW*-triple*, i.e., U is a *JB**-triple which is the dual of a Banach space U_*. We shall assume

(1. 13) *$Q(e)$ is w*-continuous for each tripotent e in U; this implies that $U_2(e) = P_2(e)\,U$ is a JBW*-algebra, i.e., a JB*-algebra which is a dual space.*

In § 2 we shall need to assume, more generally, that

(1. 14) *$D(a, b)$ and $Q(c)$ are w^*-continuous for each a, b, c in U.*

Proposition 2. *Let U be a JBW*-triple satisfying* (1. 13). *Let $f \in U_*$. Then there exists a unique tripotent e in U such that $f = P_2(e)f$ and $f|U_2(e)$ is a faithful normal positive functional on $U_2(e)$.*

Proof. (a) existence: We may assume $\|f\| = 1$. Then $S \equiv \{x \in U: f(x) = \|f\| = \|x\|\}$ is non-empty, convex and w^*-compact. Let e be an extreme point of unit ball of U. By [6] or [13], e is a tripotent of U. Since $\|f\| = f(e) = (P_2(e)f)(e) \leqq \|P_2(e)f\| \leqq \|f\|$ we have $f = P_2(e)f$ by Proposition 1. Set $\varphi = f|U_2(e)$. Since $\|f\| \geqq \|\varphi\| \geqq \varphi(e) = f(e) = \|f\|$, φ is positive. By (1. 13) $U_2(e)$ is a JBW^*-algebra and φ is normal. Let $e_1 \in U_2(e)$ be the support projection of φ. Then e_1 is a tripotent in U, $f = P_2(e_1)f$, and $f|U_2(e_1)$ is a faithful positive normal functional. Indeed, e_1 is a projection in $U_2(e)$, so by (1. 10), $\{e_1 e_1 e_1\} = 2(e_1 \circ e_1) \circ e_1 - (e_1 \circ e_1) \circ e_1 = e_1$ where $\circ$ is the Jordan product in $U_2(e)$. Thus e_1 is a tripotent. Now $f = P_2(e_1)f$ follows from Proposition 1, and the faithfulness of $f|U_2(e_1)$ is obvious since e_1 is the support of φ.

(b) uniqueness: Suppose e_1 and e_2 both satisfy the conditions of the Proposition. Let $\varphi_i = f|U_2(e_i)$, $i = 1, 2$. Then $P_2(e_1)e_2 \in U_2(e_1)$ and

$$\varphi_1(P_2(e_1)e_2) = f(P_2(e_1)e_2) = f(e_2) = \|f\| = \|\varphi_1\|.$$

By [8], Lemma 2. 4 (which is valid for JB^*-algebras), $P_2(e_1)e_2 = e_1$, so that $e_2 \geqq e_1$ by Corollary 1. 7. By symmetry $e_1 \geqq e_2$ and thus $e_1 = e_2$. □

For $f \in U_*$, we shall denote by $e(f)$ the unique tripotent in U given by Proposition 2. Proposition 2 and Lemma 1. 5 can now be combined to construct an inner product space corresponding to each $f \in U_*$.

Corollary 1. 8. *For each $f \in U_*$, where U is a JBW*-triple, $U_1(e)$, with $e = e(f)$ is an inner product space with respect to the inner product*

$$(x, y)_f = f(\{xye\}), \quad x, y \in U_1(e).$$

In Section 2 we shall also need a polar decomposition as well as a spectral decomposition and functional calculus for an arbitrary element of a JBW^*-triple. These results can be obtained easily from [6] or [13] by noting that $\bar{U}_x^{w^*}$, the weak*-closure of the JB^*-triple U_x generated by x, is isomorphic to a (commutative) von Neumann algebra.

Remark 1. 9 (cf. [11], Th. 3. 2). Let x be an arbitrary element of a JBW^*-triple U.

(a) There is a Borel subset $S \subset (0, \infty)$ and a J^*-isomorphism $g \to g(x)$ from the bounded Borel functions on S to U given by

$$g(x) = \int_S g(\lambda)\, dv_\lambda$$

where $\sigma \to v(\sigma)$ is a tripotent-valued spectral measure on the Borel subsets of S.

(b) There is a tripotent $v = v(x)$ in U such that

$$x = \{vvx\} = \{vxv\}.$$

The tripotent v is uniquely determined by the condition that x is a "weakly strictly positive" element of the JBW^*-algebra $U_2(v)$, i.e., $x \in U_2(v)^+$ and for every normal state φ of $U_2(v)$, $\varphi(x) > 0$.

An important technique for dealing with an algebraic structure is to employ commutativity relations for operators thereon. The Peirce projections are fundamental operators on Jordan triple systems. The following lemma gives a sufficient condition for commutativity of the families of Peirce projections corresponding to different tripotents.

Lemma 1. 10. *Let e and v be tripotents in a JB^*-triple U and suppose $e \in U_\mu(v)$ for some $\mu \in \{0, 1, 2\}$. Then $[P_k(e), P_j(v)] = 0$ for all $k, j \in \{0, 1, 2\}$.*

Proof. For $x \in U$, let $x_k = P_k(v)\, x$, $0 \leqq k \leqq 2$. Then (1. 7) implies that

$$P_2(e)\, P_k(v)\, x = \{e\{ex_k e\}e\} \in U_k(v) \quad \text{and} \quad D(e, e)\, P_k(v)\, x = \{eex_k\} \in U_k(v).$$

Therefore

$$P_2(e)\, P_k(v) = P_k(v)\, P_2(e)\, P_k(v), \quad 0 \leqq k \leqq 2; \tag{1. 15}$$

and

$$D(e, e)\, P_k(v) = P_k(v)\, D(e, e)\, P_k(v), \quad 0 \leqq k \leqq 2. \tag{1. 16}$$

It follows from (1. 16), since $D(e, e) - P_2(e) = \frac{1}{2} P_1(e)$, that

$$P_1(e)\, P_k(v) = P_k(v)\, P_1(e)\, P_k(v), \quad 0 \leqq k \leqq 2. \tag{1. 17}$$

Now $P_k(v)\, P_2(e) = P_k(v)\, P_2(e) \left(\sum_{j=0}^{2} P_j(v) \right)$, so (1. 15) implies

$$P_k(v)\, P_2(e) = P_k(v)\, P_2(e)\, P_k(v), \quad 0 \leqq k \leqq 2; \tag{1. 18}$$

and similarly

$$P_k(v)\, P_1(e) = P_k(v)\, P_1(e)\, P_k(v), \quad 0 \leqq k \leqq 2. \tag{1. 19}$$

By (1. 15) and (1. 18)

$$[P_2(e), P_k(v)] = 0, \quad 0 \leqq k \leqq 2; \tag{1. 20}$$

and by (1. 17) and (1. 19)

$$[P_1(e), P_k(v)] = 0, \quad 0 \leqq k \leqq 2. \tag{1. 21}$$

Since $P_0(e) = I - P_2(e) - P_1(e)$, (1. 20) and (1. 21) imply

$$[P_0(e), P_k(v)] = 0, \quad 0 \leqq k \leqq 2. \quad \square \tag{1. 22}$$

Recall that for $f \in U_*$ we denote by $e(f)$ the unique tripotent in U arising from the polar decomposition of f. The Peirce projections corresponding to $e(f)$ will be denoted by $P_k(f)$, $k \in \{0, 1, 2\}$.

Proposition 3. *Let v be a tripotent in a JBW*-triple U and let $f \in U_*$ satisfy $P_\mu(v) f = f$ for some $\mu \in \{0, 1, 2\}$. Then $e \equiv e(f) \in U_\mu(v)$ and therefore $[P_k(f), P_j(v)] = 0$ for all $k, j \in \{0, 1, 2\}$.*

Proof. Define $f_1 = f | U_\mu(v)$. Since $P_\mu(v)$ is contractive, $\|f_1\| = \|f\|$. Let $e_1 = e(f_1)$. Then e_1 is a tripotent in the *JBW**-triple $U_\mu(v)$, which is a *JBW**-subtriple of U. Thus e_1 is a tripotent in U. By Proposition 1, $f = P_2(e_1) f$ and therefore $\varphi \equiv f | U_2(e_1)$ is a positive normal functional on the *JBW**-algebra $U_2(e_1)$. We shall show that φ is faithful. It will follow, by Proposition 2, that $e(f) = e_1 \in U_\mu(v)$. The proof that φ is faithful is divided into 2 cases.

Case 1. $\mu = 0$ or 2. For $x \in U$, by (1. 7) and (1. 8), we have

$$Q(e_1)\, x = \left\{ e_1, \sum_{j=0}^{2} P_j(v)\, x, e_1 \right\} = \{e_1, P_\mu(v)\, x, e_1\}$$

so that

$$P_2(e_1) = P_2(e_1)\, P_\mu(v). \tag{1. 23}$$

By Proposition 2, $f_1 | P_2(e_1)\, P_\mu(v)\, U$ is faithful. Since $P_2(e_1)\, U_\mu(v) \subset U_\mu(v)$, $f | P_2(e_1)\, U_\mu(v)$ is faithful, so by (1. 23), $\varphi = f | U_2(e_1)$ is faithful.

Case 2. $\mu = 1$. We show directly that φ is faithful. Let $x \in (U_2(e_1))^+$ and suppose $\varphi(x) = 0$. By Lemma 1. 10, $P_1(v)$ and $P_2(e_1)$ commute. Therefore $P_1(v)$ is a unital contraction of the *JB**-algebra $U_2(e_1)$ into itself. It follows that $P_1(v)\, x \in (U_2(e_1))^+$. But $f_1(P_1(v)\, x) = f(P_1(v)\, x) = f(x) = f(P_2(e_1)\, x) = \varphi(x) = 0$, and f_1 is faithful on

$$P_2(e_1)\, P_1(v)\, U = P_1(v)\, P_2(e_1)\, U.$$

Thus $P_1(v)\, x = 0$.

To show, finally, that $x = 0$, write $x = \{y e_1 y\}$, where y is the positive square root of x in the *JB**-algebra $U_2(e_1)$, and let $y = y_2 + y_1 + y_0$ be the Peirce decomposition of y with respect to v. Since $[P_k(v), P_2(e_1)] = 0$, we have

$$y_k = P_k(v)\, y = P_k(v)\, P_2(e_1)\, y = P_2(e_1)\, P_k(v)\, y \in U_2(e_1),$$

for $k \in \{0, 1, 2\}$. Now $y = y^* = \{e_1 y e_1\} = \{e_1 y_2 e_1\} + \{e_1 y_1 e_1\} + \{e_1 y_0 e_1\}$, so that by (1. 6) and (1. 7)

$$y_1 = \{e_1 y_1 e_1\} = y_1^\#, \quad y_2 = \{e_1 y_0 e_1\} = y_0^\# \text{ and } y_0 = \{e_1 y_2 e_1\} = y_2^\#.$$

Therefore, by (1. 7) and (1. 8),

$$0 = P_1(v)\, x = P_1(v)\,(\{y e_1 y\}) = 2\{y_2 e_1 y_0\} + \{y_1 e_1 y_1\} = 2 y_2 \circ y_2^\# + y_1 \circ y_1^\#.$$

Therefore $y_1 = y_2 = 0$ and hence $y_0 = y_2^\# = 0$ so that $x = 0$. □

In Proposition 1 it was shown that all functionals on the subspaces $U_0(e)$ and $U_2(e)$ of a JB^*-triple U have unique Hahn Banach extensions to U. This fact is not true for the subspace $U_1(e)$ as the following example shows.

Let $U=M_{2,2}(\mathbb{C})$, $e=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ so that $U_1(e)=\left\{\begin{bmatrix}0 & a\\ b & 0\end{bmatrix}: a, b\in\mathbb{C}\right\}$. Let $f=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $g=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ be elements of U'. Then $\|f\|=\|g\|=2$ and $f|U_1(e)=g|U_1(e)$ has norm 2 also.

The next lemma shows that a Hahn Banach extension of a functional on $U_1(e)$ is uniquely determined by its action on one of the subspaces $U_2(e)$ or $U_0(e)$.

Lemma 1. 11. *Let v be a tripotent in a JBW^*-triple U and suppose $f\in U_*$ satisfies $\|P_1(v)f\|=\|f\|$. Then $P_2(v)f$ and $P_0(v)f$ each uniquely determines the other and $\|P_2(v)f\|=\|P_0(v)f\|$.*

Proof. Let $e_1=e(P_1(v)f)$. By Proposition 3, $e_1\in U_1(v)$ and therefore

$$\|f\|=\|P_1(v)f\|=\langle P_1(v)f, e_1\rangle=f(e_1).$$

By Proposition 1, $f=P_2(e_1)f$ and therefore $\varphi\equiv f|U_2(e_1)$ is a positive functional on $U_2(e_1)$. Since $[P_k(v), P_j(e_1)]=0$, $P_2(v)f=P_2(e_1)P_2(v)f$, and therefore $P_2(v)f$ is determined by its values on $U_2(e_1)$. The same is true for $P_0(v)f$. To complete the proof it suffices to show that

$$(P_2(v)f|U_2(e_1))^{\#}=P_0(v)f|U_2(e_1)$$

where $\#$ denotes the involution on $U_2(e_1)$.

For $z\in U_2(e_1)$, we have, using (1. 7),

$$\begin{aligned}(P_2(v)f|U_2(e_1))^{\#}(z)&=\overline{\langle P_2(v)f, z^{\#}\rangle}=\overline{\langle f, P_2(v)\{e_1 z e_1\}\rangle}\\ &=\overline{\langle f, \{e_1, P_0(v)z, e_1\}\rangle}=\overline{\langle f, (P_0(v)z)^{\#}\rangle}\\ &=\langle (f|U_2(e_1))^{\#}, P_0(v)z\rangle=\langle f|U_2(e_1), P_0(v)z\rangle\\ &=\langle P_0(v)f|U_2(e_1), z\rangle. \quad\square\end{aligned}$$

§ 2. Properties of normal functionals. Main results

In this section we prove, for JBW^*-triples, the non-ordered analogs of the properties (i)—(iii) and (v)—(vii) stated in the introduction for JB-algebras. The analogs of (i)—(iii) are Proposition 5, Theorem 1, and Proposition 8 respectively. The analogs of the pure state properties (v)—(vii) are Proposition 4, Proposition 7, and Lemma 2. 2, respectively. We also prove the analogs of three technical results from [1] in Proposition 6, Lemma 2. 11, and Theorem 2.

Note that in our setting (i.e., absence of an order structure), some of these results appear naturally and are quite easy to prove. Roughly speaking, the more a Jordan algebra property involves order, the more difficult is its formulation and proof in a Jordan triple system. This is especially true of our second main result, Theorem 2. Its proof requires most of the previous results of this paper; but the corresponding fact for *JBW*-algebras is elementary once you know that the projections form a complete lattice.

A tripotent e in a *JB**-triple U is a *minimal tripotent* of U if $P_2(e)\,U = \mathbb{C}e$. As an application of Lemma 1. 10 we have

Lemma 2. 1. *Let v be a minimal tripotent in a JB*-triple U and let e be a tripotent with $e \in U_1(v)$. Then $v \in U_2(e) \cup U_1(e)$.*

Proof. For $0 \leqq j \leqq 2$, $P_j(e)\,v = P_j(e)\,P_2(v)\,v = P_2(v)\,P_j(e)\,v = \lambda_j v$. Thus

$$0 = P_2(e)\,P_1(e)\,v = \lambda_1 P_2(e)\,v = \lambda_1 \lambda_2 v,$$

so either $\lambda_1 = 0$ or $\lambda_2 = 0$. It remains to prove that $\lambda_0 = 0$. Since $P_0(e)v = \lambda_0 v$ we must have $\lambda_0 = 0$ or $\lambda_0 = 1$. If $\lambda_0 = 1$, then $\{eev\} = 0$ and therefore $\{vve\} = 0$ (by [15], 3. 9), and so $e = P_0(v)\,e = P_0(v)\,P_1(v)\,e = 0$, a contradiction. □

The following Proposition clarifies the relation between minimal tripotents of a *JBW**-triple and the atoms of U, where by an *atom* f of a *JBW**-triple U we mean any extremal point of the unit ball U_{*1} of U_*. As a consequence, each atom of U is a norm exposed point of U_{*1}.

Proposition 4. *Let U be a JBW*-triple with predual U_*.*

(a) *If f is an atom of U, then $e \equiv e(f)$ is a minimal tripotent of U, $P_2(f)\,x = \langle f, x \rangle\, e$ for $x \in U$, and $P_2(f)\,g = \langle g, e \rangle f$ for $g \in U_*$.*

(b) *If e is a minimal tripotent of U, then there exists an atom f of U such that $e = e(f)$.*

(c) *The map $f \to e(f)$ is a bijection of the set of atoms of U onto the set of minimal tripotents of U.*

(d) *Each atom of U is a norm exposed point of $U_{*,1}$.*

Proof. (a) $B \equiv U_2(e)$ is a *JBW**-algebra and $\varphi \equiv f|B$ is a faithful normal state on B. Let S be the face in B_*^+ generated by φ, i.e., for $\tau \in S$, there exists $\sigma \in S$ such that φ is a convex combination of $\tilde{\tau} \equiv \|\tau\|^{-1}\tau$ and $\tilde{\sigma} \equiv \|\sigma\|^{-1}\sigma$. Since $f = P_2(e)\,f$, f is a convex combination of $\tilde{\tau} \circ P_2(e)$ and $\tilde{\sigma} \circ P_2(e)$. Since f is extremal,

$$\tau\bigl(P_2(e)\,x\bigr) = \|\tau\|\, f(x) = \langle \tau, e \rangle f(x), \quad \tau \in S, \quad x \in U. \tag{2. 1}$$

By the Emch-King-Iochum extension of the Kadison-Effros-Tomita theorem [5], [12], S is norm dense in B_*^+. Since B_*^+ linearly spans B_*, (2. 1) holds for every $\tau \in B_*$.

Now let $g \in U_*$ be arbitrary. Then $\tau \equiv P_2(e)\,g|B \in B_*$ and so for $x \in U$, $\langle P_2(e)\,g, x \rangle = \langle P_2(e)\,g, P_2(e)\,x \rangle = \langle P_2(e)\,g, e \rangle f(x) = \langle g, e \rangle f(x)$, i.e., $P_2(f)\,g = \langle g, e \rangle f$. It follows from this that $P_2(f)\,x = \langle f, x \rangle\, e$ for all $x \in U$.

(b) Define f by $f(x)\,e = P_2(e)\,x$ for $x \in U$. Since $P_2(e)$ is w^*-continuous, $f \in U_*$. Since $f(e) = 1$, $\|f\| = 1$ and $\langle P_2(e) f, x\rangle = \langle f, P_2(e)\,x\rangle = \langle f, x\rangle \langle f, e\rangle = f(x)$ i.e., $f = P_2(e) f$. Since $f | U_2(e) = f | \mathbb{C}$ is faithful, $e = e(f)$ by Proposition 2. It remains to show that f is an atom of U.

Since $P_2(e)$ is w^*-continuous, the minimality of e implies that $P_2(e)\,g = \langle g, e)\,f$ for $g \in U_*$. Since $\|f\| = 1$, f is an extreme point of $(P_2(e)\,U_*)_1 \cong \mathbb{C}_1$. An application of Proposition 1 then implies that f is an extreme point of $U_{*,1}$.

(c) This follows from the proofs of (a) and (b).

(d) For an atom f, let H be the hyperplane determined by $e = e(f) \in U$, i.e., $H = \{g \in U_* : \operatorname{Re} g(e) = 1\}$. Then $\{f\} = H \cap U_{*,1}$. Indeed it is clear that $f \in H \cap U_{*,1}$ and conversely if $g \in H \cap U_{*,1}$, then by Proposition 1, $g = P_2(e)\,g$. By (a)

$$P_2(e)\,g = \langle g, e\rangle\, f = f. \quad \square$$

The following Lemma is the analog of the property called "symmetry of transition probabilities" in [1].

Lemma 2. 2. *Let f_1, f_2 be atoms of a JBW*-triple U, and let $e_i = e(f_i)$, $i = 1, 2$. Then $f_1(e_2) = \overline{f_2(e_1)}$.*

Proof. By [15], *JP*1, $\{xy\{xzx\}\} = \{x\{yxz\}x\}$ holds for $x, y, z \in U$. We have, by Proposition 4

$$\{e_1 e_2 e_1\} = \overline{f_1(e_2)}\, e_1 \quad \text{and} \quad \{e_2 e_1 e_2\} = \overline{f_2(e_1)}\, e_2.$$

Thus

$$\{e_1 e_2 \{e_1 e_2 e_1\}\} = \overline{(f_1(e_2))^2}\, e_1$$

and

$$\{e_1 \{e_2 e_1 e_2\} e_1\} = f_2(e_1)\, \overline{f_1(e_2)}\, e_1.$$

Thus either $f_1(e_2) = 0$ or $\overline{f_1(e_2)} = f_2(e_1)$. If $f_1(e_2) = 0$ then $f_2(e_1) = 0$ by reversing the roles of e_1 and e_2 in the above argument. $\square$

We shall prove the extreme ray property below in Proposition 7 using Theorem 1. To prove Theorem 1 below we need an analogue of the extreme ray property dealing with minimal tripotents (Proposition 6). To prove Proposition 6 we shall need a few lemmas and another Proposition.

Lemma 2. 3. *Let u and v be minimal tripotents in a Jordan triple system U. The Jordan triple system generated by u and v is of dimension at most 4, being linearly spanned by the elements u, v, $P_1(u)\,v$, $P_1(v)\,u$.*

Proof. Since $\{uuv\} = P_2(u)\,v + \frac{1}{2} P_1(u)\,v$, it suffices to prove that any triple product $\{abc\}$ with $a, b, c \in A = A_1 \cup A_2$ (where A_1 is the set consisting of u and v and A_2 consists of the two elements $\{uuv\}$ and $\{vvu\}$) is a linear combination of u, v, $\{uuv\}$, $\{vvu\}$.

Given $a, b, c \in A$, let i be the number of (not necessarily distinct) elements among a, b, c which belong to A_1.

If $i=3$ there is nothing to prove. If $i=2$, it suffices by minimality and symmetry to consider only the elements x_1, x_2, x_3, x_4, x_5, which can be expanded using [15], *JP*7, 9, 10, 16, 16 respectively:

$$\begin{aligned} x_1 &= \{\{vuu\}uv\} = \{u\{uvu\}v\} + \{v\{uuu\}v\} \\ &= \lambda_1\{uuv\} + \lambda_2 v; \end{aligned}$$

$$\begin{aligned} x_2 &= \{\{vuu\}vu\} = \{u\{uuu\}v\} + \{u\{vvv\}u\} \\ &= \{uuv\} + \lambda_3 u; \end{aligned}$$

$$\begin{aligned} x_3 &= \{u\{vuu\}v\} = \{\{uvu\}uv\} + \{vv\{uuu\}\} \\ &= \lambda_4\{uuv\} + \{vvu\}; \end{aligned}$$

$$\begin{aligned} x_4 &= \{\{vvu\}uu\} = \{u\{vvu\}u\} + \{v\{uuv\}u\} - \{\{uuv\}vu\} \\ &= \lambda_5 u + x_3 - x_2; \end{aligned}$$

$$x_5 = \{\{uuv\}uu\} = \{v\{uuu\}u\} + \{u\{uvu\}u\} - \{\{vuu\}uu\};$$

so

$$x_5 = \frac{1}{2}\{vuu\} + \frac{1}{2}\lambda_6 u.$$

If $i=1$, it suffices to consider only seven non-trivial terms $y_1, \ldots, y_7$ and for $i=0$ there are only three non-trivial terms z_1, z_2, z_3. These terms are listed below and can be expanded by using (1. 2). These calculations are omitted here. The terms are

$$\begin{aligned} y_1 &= \{u\{uuv\}\{uuv\}\}, & y_2 &= \{u\{uuv\}\{vvu\}\}, \\ y_3 &= \{u\{vvu\}\{uuv\}\}, & y_4 &= \{u\{vvu\}\{vvu\}\}, \\ y_5 &= \{\{uuv\}u\{uuv\}\}, & y_6 &= \{\{uuv\}u\{vvu\}\}, \\ y_7 &= \{\{vvu\}u\{vvu\}\}, \end{aligned}$$

$$\begin{aligned} z_1 &= \{\{uuv\}\{uuv\}\{uuv\}\}, \\ z_2 &= \{\{uuv\}\{uuv\}\{vvu\}\}, \\ z_3 &= \{\{uuv\}\{vvu\}\{uuv\}\}. \quad \square \end{aligned}$$

Remark 2. 4. Let J be the Jordan triple system generated by two minimal tripotents u and v.

(a) If J is of rank 1, then J has dimension $\leqq 2$;

(b) J is simple unless u and v are orthogonal.

Proof. (a) J of rank 1 implies $P_0(u)\,v=0=P_0(v)\,u$ and therefore

$$v=P_2(u)\,v+P_1(u)\,v+P_0(u)\,v=\lambda u+P_1(u)\,v.$$

Thus $P_1(u)\,v$ (and by symmetry $P_1(v)\,u$) lies in the linear span of u and v.

(b) If $J=J_1\oplus J_2$, then u, v must each lie in a component, by minimality. □

From Lemma 2. 3, Remark 2. 4, and the classification of simple finite dimensional Jordan triple systems over $\mathbb{C}$ [15], we have:

Proposition 5. *Let u, v be minimal tripotents in a complex Jordan triple system, and let J be the Jordan triple system generated by u and v. Then J is isomorphic to one of the following:*

$$\mathbb{C},\quad M_{1,2}(\mathbb{C}),\quad \mathbb{C}\oplus\mathbb{C},\quad S_2(\mathbb{C}),\quad M_{2,2}(\mathbb{C}).$$

Because of Proposition 4 (c), we may view Proposition 5 as the non-ordered analog of the Hilbert ball property of [1]. Indeed, the rank 1 cases $\mathbb{C}$ and $M_{1,2}(\mathbb{C})$ are Hilbert spaces of dimension 1 and 2; and the rank two cases $\mathbb{C}\oplus\mathbb{C}$, $S_2(\mathbb{C})$, $M_{2,2}(\mathbb{C})$ are complex spin factors with self-adjoint parts which are isomorphic to Hilbert spaces [2], § 7. Here, $S_2(\mathbb{C})$ denotes the symmetric 2 by 2 complex matrices.

Corollary 2. 5. *Let u and v be minimal tripotents in a complex Jordan triple system U which are not orthogonal. Then up to scalar multiples, u and v can be "exchanged by a symmetry" in U; i.e., there is a tripotent w in U and $|\lambda|=1$ such that $S_{-1}(w)\,u=\lambda v$.*

Proof. The Jordan triple system J generated by u and v is isomorphic to either $M_{1,2}(\mathbb{C})$, $M_{2,2}(\mathbb{C})$, or $S_2(\mathbb{C})$. The equation $S_{-1}(w)\,u=\lambda v$ can be solved in J for w and λ by the remark following Lemma 1. 1. □

The next two lemmas give more information about the Jordan triple system J generated by two minimal tripotents u and v in a complex Jordan triple system U.

Lemma 2. 6. *If J is of rank 1, i.e., $J\cong\mathbb{C}$ or $J\cong M_{1,2}(\mathbb{C})$, then $\alpha u+\beta v$ is a scalar multiple of some minimal tripotent of U, for every $\alpha, \beta\in\mathbb{C}$.*

Proof. If $J\cong\mathbb{C}$, then v is a multiple of u so $\alpha u+\beta v$ is a multiple of u. If $J\cong M_{1,2}(\mathbb{C})$, then by Corollary 2. 5, each $b\in J$ with $\|b\|=1$, can be exchanged with u by a symmetry in U. Therefore b is a minimal tripotent of U. Since u and v span J the result follows. □

Lemma 2. 7. *If J is of rank 2, i.e., $J\cong\mathbb{C}\oplus\mathbb{C}$ or $J\cong M_{2,2}(\mathbb{C})$, or $J\cong S_2(\mathbb{C})$, then there exist two orthogonal minimal tripotents e_1, e_2 of U such that $P_2(e_1+e_2)\,a=a$ for all $a\in J$.*

Proof. If $J\cong\mathbb{C}\oplus\mathbb{C}$, we may choose $e_1=u$ and $e_2=v$. Now suppose $J\cong S_2(\mathbb{C})$ or $J\cong M_{2,2}(\mathbb{C})$. Let $e_i\in J$, $1\leqq i\leqq 3$ correspond to $\tilde{e}_i\in S_2(\mathbb{C})\subset M_{2,2}(\mathbb{C})$, $1\leqq i\leqq 3$ where

$$\tilde{e}_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad \tilde{e}_2=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \tilde{e}_3=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

By a simple calculation in $M_{2,2}(\mathbb{C})$, $S_{-1}(\tilde{e}_3)\,\tilde{e}_1=\tilde{e}_2$. Also, trivially, $P_2(\tilde{e}_1+\tilde{e}_2)\,m=m$ for all $m\in M_{2,2}(\mathbb{C})$. Therefore $S_{-1}(e_3)\,e_1=e_2$ and $P_2(e_1+e_2)\,a=a$ for all $a\in J$. Since we may choose $e_1=u$ it follows that e_2 is also a minimal tripotent of U and the lemma follows. □

We are now ready to prove the analog of the extreme ray property dealing with minimal tripotents. Recall that elements, $a, b \in U$ are orthogonal if $D(a, b)=0$ and elements $f, g \in U'$ are orthogonal ($f \perp g$) if $e(f)$ and $e(g)$ are orthogonal.

Proposition 6. *Let e be a tripotent in a JBW^*-triple U and let u be a minimal tripotent of U. Then $P_2(e)\,u$ is a scalar multiple of some minimal tripotent of U.*

Proof. Let S be the symmetry determined by e, $S=S_{-1}(e)$. Since S is a JB^*-triple automorphism, $v \equiv Su$ is also a minimal tripotent of U. By Proposition 5 the Jordan triple system J generated by u and v is isomorphic to one of

$$\mathbb{C},\ M_{1,2}(\mathbb{C}),\ \mathbb{C} \oplus \mathbb{C},\ M_{2,2}(\mathbb{C}),\ S_2(\mathbb{C}).$$

On the other hand by the definition of S_{-1}, we have

$$\frac{1}{2}(u+v)=P_2(e)\,u+P_0(e)\,u. \tag{2.2}$$

In the cases $J \cong \mathbb{C}$ or $J \cong M_{1,2}(\mathbb{C})$ we know by Lemma 2. 6 that $u+v$ is a multiple of some minimal tripotent, say w, of U. Since $\alpha \equiv P_2(e)\,u$ and $\beta \equiv P_0(e)\,u$ are orthogonal, (2. 2) implies that α and β are multiples (possibly zero) of some minimal tripotents of U. Indeed, $\lambda w=\alpha+\beta$ with $\lambda \neq 0$ implies $w=\{www\}=\lambda^{-3}\{\alpha\alpha\alpha\}+\lambda^{-3}\{\beta\beta\beta\}$. This implies that $\frac{\alpha}{\|\alpha\|}$ and $\frac{\beta}{\|\beta\|}$ are tripotents and since w is minimal one of α or β is zero. If $\lambda=0$, then $\alpha=\beta=0$ by orthogonality.

In the cases $J \cong \mathbb{C} \oplus \mathbb{C}$, $J \cong S_2(\mathbb{C})$, or $J \cong M_{2,2}(\mathbb{C})$ let e_1, e_2 be two orthogonal minimal tripotents of U with $P_2(e_1+e_2)a=a$ for all $a \in J$ (by Lemma 2. 7). For notation's sake, let $s=\frac{1}{2}(u+v)$, $\tilde{e}=e_1+e_2$, $\alpha=P_2(e)\,u$, $\beta=P_0(e)\,u$. Then $P_2(\tilde{e})\,u=u$, $P_2(\tilde{e})\,v=v$ and $s=\alpha+\beta$. We claim that $P_2(\tilde{e})\,\alpha=\alpha$ and $P_2(\tilde{e})\,\beta=\beta$. Using the local polar decompositions discussed above in Remark 1. 9, we have tripotents $v_s, v_\alpha, v_\beta \in U$ with $v_s=v_\alpha+v_\beta$ and v_α and v_β orthogonal. Since $s \in U_2(\tilde{e})$, we have $v_2 \in U_2(\tilde{e})$ and therefore $U_2(v_s) \subset U_2(\tilde{e})$ by (1. 7). By orthogonality $v_\alpha \in U_2(v_s)$ and therefore $v_\alpha \in U_2(\tilde{e})$ and $U_2(v_\alpha) \subset U_2(\tilde{e})$. Since $\alpha=\{v_\alpha v_\alpha \alpha\} \in U_2(v_\alpha)$ we have proved that $P_2(\tilde{e})\,\alpha=\alpha$. A similar proof shows that $P_2(\tilde{e})\,\beta=\beta$. Thus α and β belong to the JBW^*-algebra $P_2(\tilde{e})\,U$. Since $\tilde{e}=e_1+e_2$, this JBW^*-algebra has rank 2, i.e., its identity is the sum of two orthogonal minimal (self-adjoint) idempotents. Thus every element γ of norm 1 in $P_2(\tilde{e})\,U$ is either a minimal tripotent of $P_2(\tilde{e})\,U$ or has the property that $P_2(\gamma)$ is the identity on $P_2(\tilde{e})\,U$. Apply this with $\gamma=\frac{\alpha}{\|\alpha\|}$. In the first case, there is a symmetry in the JBW^*-algebra $P_2(\tilde{e})\,U$ which exchanges γ and e_1 (by Corollary 2. 5), and therefore γ is a minimal tripotent of U. In the second case $P_2(\gamma)$ is the identity on J so $P_2(\gamma)\,u=u$. But (1. 7) implies $P_2(\gamma)=P_2(e)\,P_2(\gamma)$ so that

$$P_2(e)u=P_2(e)\,P_2(\gamma)u=P_2(\gamma)u=u. \qquad \square$$

We are now ready to prove the main results of this paper. The following remark is needed in Theorem 1.

Remark 2. 8. Let φ be a normal functional on a JBW^*-triple U. Since $\varphi|P_2(\varphi)\,U$ is a positive normal functional on the JBW^*-algebra $P_2(\varphi)\,U$ we have, by using [1], § 5, a decomposition

$$\varphi=\sum_i \lambda_i f_i+\Psi \tag{2. 3}$$

where $\{f_i\}$ is a sequence of pairwise orthogonal atoms of U, Ψ is orthogonal to each f_i, $\lambda_i \geqq 0$, $\|\varphi\|=\sum \lambda_i+\|\Psi\|$, and $P_2(\Psi)\,U$ is a purely non-atomic JBW^*-algebra.

Theorem 1. *Let U be a JBW^*-triple with predual U_*. Let $\mathscr{A}$ be the norm closure of the linear span of the atoms of U. Then $U_*=\mathscr{A}\oplus_{l_1}\mathscr{N}$ where the unit ball of $\mathscr{N}$ has no extreme points. In fact*

$$\mathscr{N}=\{\Psi\in U_*:\ \Psi \text{ is orthogonal to all atoms of } U\}.$$

Proof. Each $\varphi\in U_*$ has the form $\varphi=h+\Psi$ where $h\in\mathscr{A}$, $P_2(\Psi)\,U$ is a purely non-atomic JBW^*-algebra, Ψ is orthogonal to h, and $\|\varphi\|=\|h\|+\|\Psi\|$.

Let f be an arbitrary atom of U. We shall show that Ψ is orthogonal to f and that the decomposition $\varphi=h+\Psi$ is unique. By Proposition 6, $P_2(\Psi)\,e(f)=\lambda v$ for some minimal tripotent v of U and $\lambda\in\mathbb{C}$. If $\lambda\neq 0$ than $v\in P_2(\Psi)\,U$ which is purely non-atomic, contradiction. Thus $P_2(\Psi)\,e(f)=0$. Suppose now that $\varphi=h_1+\Psi_1=h_2+\Psi_2$ with $h_1, h_2\in\mathscr{A}$ and $P_2(\Psi_i)\,e(f)=0$ for all atoms f of U and $i=1, 2$. Now

$$e(\varphi)=e(h_i)+e(\Psi_i),\quad i=1, 2,$$

so that

$$P_2(\Psi_1)\,e(\varphi)=P_2(\Psi_1)\,\big(e(h_2)+e(\Psi_2)\big)=P_2(\Psi_1)\,e(\Psi_2)$$

and

$$P_2(\Psi_1)\,e(\varphi)=P_2(\Psi_1)\,\big(e(h_1)+e(\Psi_1)\big)=e(\Psi_1).$$

Thus $P_2(\Psi_1)\,e(\Psi_2)=e(\Psi_1)$ so $e(\Psi_2)\geqq e(\Psi_1)$ (by Corollary 1. 7) and by symmetry we have $e(\Psi_2)=e(\Psi_1)$ and therefore $P_2(\Psi_1)=P_2(\Psi_2)$. Thus

$$\Psi_1=P_2(\Psi_1)\,(h_1+\Psi_1)=P_2(\Psi_2)\,(h_2+\Psi_2)=\Psi_2.$$

To prove that Ψ is orthogonal to an arbitrary atom f, consider $f+\Psi$. By Remark 2. 8 we can write $f+\Psi=h_1+\Psi_1$ with $h_1\in\mathscr{A}$ and $h_1\perp\Psi_1$. By the uniqueness just proved, $f=h_1$ and $\Psi=\Psi_1$. In particular Ψ is orthogonal to f. We have proved that $U_*=\mathscr{A}\oplus_{l^1}\mathscr{N}$ where $\mathscr{N}=\{\Psi\in U_*:\ \Psi$ is orthogonal to all atoms of $U\}$. Since the sum is l^1, any extreme point of the unit ball of $\mathscr{N}$ would be extremal in the unit ball of U_*. □

Corollary 2.9. *$\mathcal{N}=\{\varphi \in U_* : \varphi(e(f))=0$ for all atoms f of $U\}$.*

Proof. Each $\Psi \in \mathcal{N}$ obviously satisfies $\Psi(e(f))=0$ for all atoms f since $\Psi \perp f$. Suppose $\varphi \in U_*$ satisfies this condition and let g be an arbitrary atom of U. Write $g+\varphi=h+\Psi$ with $h \in \mathcal{A}$ and $\Psi \in \mathcal{N}$. Since φ and Ψ both vanish on all $e(f)$ (f atom of U), $g(e(f))=h(e(f))$ for all atoms f of U. Now $h(e(g))=g(e(g))=\|g\|=1$ and, writing $h=\sum \lambda_i h_i$ with h_i orthogonal atoms of U, we have

$$\|h\|=h(\sum e(h_i))=\sum h(e(h_i))=g(\sum e(h_i)) \leqq \|g\|=1,$$

the sum $\sum e(h_i)$ converging in the weak *-topology of $P_2(h)\,U$. Therefore $\|h\|=1$. On the other hand, $\|P_2(g)\,h\| \geqq \langle P_2(g)\,h, e(g)\rangle = h(e(g))=1$. Thus $1 \leqq \|P_2(g)\,h\| \leqq \|h\|=1$ so by Proposition 1, $h=P_2(g)\,h$. Since g is an atom of U, Proposition 4 implies $h=P_2(g)\,h=\langle h, e(g)\rangle\, g=g$. Therefore $\varphi=\Psi \in \mathcal{N}$. □

Since $U=(U_*)'$ we may write $U=\mathcal{N}^\perp \oplus_{l_\infty} \mathcal{A}^\perp$ where e.g. $\mathcal{N}^\perp$ denotes the annihilator of $\mathcal{N}$. Let U_0 denote the linear span of the minimal tripotents of U and let $\mathcal{A}_0$ denote the linear span of the atoms of U.

Corollary 2.10. *U_0 is weak*-dense in $\mathcal{N}^\perp$.*

Proof. For $\Psi \in \mathcal{N}$ and $x \in U_0$, $\Psi(x)=0$, so that $U_0 \subset \mathcal{N}^\perp$. Let $\varphi \in U_*$ vanish on U_0. By Corollary 2.9 $\varphi \in \mathcal{N}$ so φ vanishes on $\mathcal{N}^\perp$. □

Lemma 2.11. *There is a linear bijection $\pi_0 : \mathcal{A}_0 \to U_0$ given by*

$$\pi_0\left(\sum_{i=1}^{n} \alpha_i f_i\right)=\sum_{i=1}^{n} \bar{\alpha}_i e(f_i).$$

The map π_0 extends to a contractive linear map π of $\mathcal{A}$ into $\mathcal{N}^\perp$.

Proof. If $\sum_{i=1}^{n} \alpha_i f_i=0$ then $b \equiv \sum_{i=1}^{n} \bar{\alpha}_i e(f_i)$ is in $\mathcal{N}^\perp$. But b vanishes on $\mathcal{A}_0$ by a simple application of Lemma 2.2 so $b \in \mathcal{A}^\perp \cap \mathcal{N}^\perp=(0)$. This implies that the map $\pi_0 : \mathcal{A}_0 \to U_0$ is well defined, linear and onto. Another application of Lemma 2.2 together with Corollary 2.9 shows that if $\sum_{i=1}^{n} \alpha_i e(f_i)=0$ then $\sum_{i=1}^{n} \bar{\alpha}_i f_i \in \mathcal{A} \cap \mathcal{N}=(0)$. Thus π_0 is a linear bijection of $\mathcal{A}_0$ onto U_0.

We now define $\pi : \mathcal{A} \to \mathcal{N}^\perp$ as follows. Each $h \in \mathcal{A}$ has the form $h=\sum \lambda_i h_i$ with $\lambda_i \geqq 0$, $\sum \lambda_i=\|h\|$ and h_i orthogonal atoms of U. Since $e(h_i)$ are orthogonal, $\sum \bar{\lambda}_i e(h_i)$ converges in the w^*-topology of $P_2(h)\,U$. We define $\pi(h)$ to be $\sum \bar{\lambda}_i e(h_i)$. By Lemma 2.2 π is well defined and therefore is linear and extends π_0. Finally

$$\|\pi(h)\|=\|\sum \bar{\lambda}_i e(h_i)\|=\sup_i \lambda_i \leqq \sum_i \lambda_i=\|h\|. \quad □$$

We are now ready to prove the extreme ray property. For convenience we record the following, which is a consequence of Proposition 1 and the proof of Proposition 3.

Remark 2.12. Let $h \in U_*$ and let $u=e(h)$. If w is a tripotent in U, then $P_2(w)\,u=u$ if and only if $P_2(w)\,h=h$.

Proposition 7. *Let e be a tripotent in a JBW*-triple U and let f be an atom of U. Then $P_2(e)\,f$ is a scalar multiple of an atom of U.*

Proof. Set $g=S_{-1}(e)f$, $u=e(f)$, $v=e(g)$ and let J be the Jordan triple system generated by u and v. Since f and g are atoms of U, Proposition 4 implies that we are in one of the five cases enumerated in Proposition 5. We also have with $\varphi_1=P_2(e)f$ and $\varphi_2=P_0(e)f$,

$$\frac{1}{2}(f+g)=\varphi_1+\varphi_2. \tag{2.4}$$

Suppose J is of rank 1, i.e., $J\cong \mathbb{C}$ or $J\cong M_{1,2}(\mathbb{C})$. By Lemmas 2.6 and 2.10 and Proposition 4, every linear combination of f and g is a multiple of some atom of U. In particular $\varphi_1+\varphi_2=\lambda h$ for some atom h of U and $\lambda>0$, and

$$\begin{aligned}\lambda=\|\varphi_1\|+\|\varphi_2\| &\geqq \|P_2(h)\,\varphi_1\|+\|P_2(h)\,\varphi_2\|\\ &\geqq \langle P_2(h)\,\varphi_1, e(h)\rangle+\langle P_2(h)\,\varphi_2, e(h)\rangle=\langle \varphi_1+\varphi_2, e(h)\rangle=\lambda.\end{aligned}$$

By Propositions 1 and 4, $\varphi_i=P_2(h)\,\varphi_i=\langle \varphi_i, e(h)\rangle\, h$.

Suppose next that J is of rank 2, i.e., $J\cong \mathbb{C}\oplus\mathbb{C}$, or $J\cong S_2(\mathbb{C})$, or $J\cong M_{2,2}(\mathbb{C})$. Let e_1 and e_2 be two orthogonal minimal tripotents given by Lemma 2.7 so that $P_2(w)\,a=a$ for all $a\in J$, where $w=e_1+e_2$. Thus $P_2(w)\,u=u$, $P_2(w)\,v=v$ so by Remark 2.12, $P_2(w)\,h=h$ where $h=\frac{1}{2}(f+g)=\varphi_1+\varphi_2$. We have

$$\begin{aligned}\|h\|=\|\varphi_1\|+\|\varphi_2\| &\geqq \|P_2(w)\,\varphi_1\|+\|P_2(w)\,\varphi_2\|\\ &\geqq \langle P_2(w)\,\varphi_1, e(h)\rangle+\langle P_2(w)\,\varphi_2, e(h)\rangle\\ &=\langle \varphi_1, e(h)\rangle+\langle \varphi_2, e(h)\rangle \quad \text{(by Remark 2.12)}\\ &=\|h\|.\end{aligned}$$

By Proposition 1, $P_2(w)\,\varphi_i=\varphi_i$, $i=1,2$. Let $\Psi_i=\varphi_i|U_2(w)$, $i=1,2$. Then Ψ_i is a normal functional on the JBW^*-algebra $U_2(w)$ of rank 2. It follows that $\frac{\Psi_i}{\|\Psi_i\|}$ is either faithful or extremal. Therefore, if $\frac{\varphi_i}{\|\varphi_i\|}$ is not an atom of U, then $P_2(\varphi_i)\,a=a$ for all $a\in U_2(w)$. In particular, $P_2(\varphi_i)\,a=a$ for all $a\in J$. By (1.7) $P_2(e)\,P_2(\varphi_1)=P_2(\varphi_1)$ and therefore $P_2(e)\,a=a$ for all $a\in J$. In particular $P_2(e)\,u=u$ and so $P_2(e)f=f$, contradiction. □

Theorem 2. *Let U be a JBW^*-triple. Then U decomposes into an orthogonal direct sum of JBW^*-ideals A and N where A is the weak*-closure of the linear span of its minimal tripotents, and N has no minimal tripotents.*

Proof. We show first that $A\equiv\mathcal{N}^\perp$ and $N\equiv\mathcal{A}^\perp$ are JBW^*-subtriples of U. Let $a=\sum_{i=1}^{n}\alpha_i e_i\in U_0$, and set $h=\pi_0^{-1}(h)=\sum\bar{\alpha}_i f_i$. By Proposition 7 we may write $h=P_2(h)\,h=\sum\bar{\alpha}_i P_2(h)\,f_i=\sum\beta_i g_i$ where g_i is an atom of U with $P_2(h)\,g_i=g_i$. Working in the JBW^*-algebra $P_2(h)\,U$ we may use [1], § 5, to write $h=\sum\lambda_i h_i$ with $\lambda_i>0$, $\sum\lambda_i=\|h\|$, and h_i orthogonal atoms of U. Now a and $b\equiv\pi(h)=\sum\lambda_i e(h_i)$ belong to $\mathcal{N}^\perp$ and by Lemma 2.2, $a-b\in\mathcal{A}^\perp$. Therefore $a=b$ and $\{aaa\}=\{bbb\}=\sum\lambda_i^3 e(h_i)\in\mathcal{N}^\perp$.

It follows from the standard polarization formulas (cf. Lemma 1. 1) that $\{U_0 U_0 U_0\} \subset \mathcal{N}^\perp$. By (1. 14) and Corollary 2. 10, we get $\{\mathcal{N}^\perp \mathcal{N}^\perp \mathcal{N}^\perp\} \subset \mathcal{N}^\perp$.

We show next that

$$(2.5) \qquad \mathcal{A}^\perp = \bigcap \{P_0(e)\, U : e \text{ a minimal tripotent of } U\},$$

which implies that $\mathcal{A}^\perp$ is a *JBW**-subtriple of U. Let $b \in \mathcal{A}^\perp$, $\|b\| = 1$, and let e be a minimal tripotent of U. Since $U = \mathcal{N}^\perp \oplus_{l_\infty} \mathcal{A}^\perp$ we have $\|e \pm b\| = 1$. Therefore

$$\|e \pm P_2(e)\, b\| \leqq 1$$

and since e is an extreme point of the unit ball of $P_2(e)\, U$, $P_2(e)\, b = 0$.

By Lemma 1. 6 with $x = e + b$, we have $P_1(e)(e+b) = 0$ i.e., $P_1(e)\, b = 0$, and therefore $b = P_0(e) b$ as required for the proof of (2. 5).

Now (2. 5) and Corollary 2. 10 imply that $D(a, b) = 0$ and hence $D(b, a) = 0$ for $(a, b) \in A \times N$. This implies that $\{AUU\} \subset A$, $\{UAU\} \subset A$, $\{NUU\} \subset N$, $\{UNU\} \subset N$, i.e., A and N are ideals in U.

It is easy to see that A is purely atomic and that N has no minimal tripotents. □

We conclude by showing that each norm exposed face in the unit ball K of the predual U_* of a *JBW**-triple U, is projective. Recall that a face $F \subset K$ is *norm exposed* if there is an element $x \in U$ such that $F = H_x \cap K$, where $H_x = \{f \in U_* : Re \langle f, x\rangle = \|x\|\}$. The norm exposed face F is said to be *projective* if the x can be chosen as a tripotent in U.

Proposition 8. *Let K be the unit ball of the predual of a JBW*-triple. Then every norm exposed face in K is projective.*

Proof. Suppose $F = H_x \cap K$ is a norm exposed face, with $\|x\| = 1$ for simplicity. By Remark 1. 9 write $x = x_1(\varepsilon) + x_2(\varepsilon)$ for $\varepsilon > 0$ where

$$x_1 = x_1(\varepsilon) = \int_0^{1-\varepsilon} \lambda\, dv_\lambda \quad \text{and} \quad x_2 = x_2(\varepsilon) = \int_{1-\varepsilon}^{1} \lambda\, dv_\lambda .$$

We shall show that $f(x_2) = 1$ and $f(x_1) = 0$ for every $f \in F$. It will follow that $w \equiv w^*\text{-}\lim_{\varepsilon \to 0} x_2(\varepsilon)$ is a tripotent in U with $\mathrm{Ref}(w) = 1$ for all f in F, and therefore $F \subset H_w \cap K$. Writing $x = y + w$ where $y \equiv w^*\text{-}\lim_{\varepsilon \to 0} x_1(\varepsilon)$ it will follow that $F = H_w \cap K$.

Let v_i be the tripotent occurring in the polar decomposition of x_i, $i = 1, 2$. For every $f \in F$,

$$\begin{aligned} 1 = \mathrm{Re}\,\langle f, x\rangle = \mathrm{Re}\,\langle f, x_1 + x_2\rangle &= \mathrm{Re}\,\langle (P_2(v_1) + P_2(v_2))\, f, x_1 + x_2\rangle \\ &\leqq |\langle P_2(v_1)\, f, x_1\rangle| + |\langle P_2(v_2)\, f, x_2\rangle| \\ &\leqq \|P_2(v_1)\, f\|\,(1-\varepsilon) + \|P_2(v_2)\, f\|. \end{aligned}$$

Now, if $P_2(v_1)\, f \neq 0$ then $1 < \|P_2(v_1)\, f\| + \|P_2(v_2)\, f\| = \|(P_2(v_1) + P_2(v_2))\, f\| \leqq \|f\| = 1$, contradiction, where we have used Lemma 1. 3 (b) since $P_2(v_1)\, f$ and $P_2(v_2)\, f$ are orthogonal.

Thus $f(x_1) = \langle f, P_2(v_1)\, x_1 \rangle = 0$ so that for all ε,

$$1 = \operatorname{Re} \langle f, x_2(\varepsilon)\rangle, \quad \text{and} \quad \operatorname{Re} \langle f, w\rangle = \lim_{\varepsilon \to 0} \operatorname{Re} \langle f, x_2(\varepsilon)\rangle = 1.$$

Finally, if $x = y + w$ with $y = \lim_{\varepsilon \to 0} x_1(\varepsilon)$, we have, with u = the tripotent arising in the polar decomposition of y, and $f \in H_w \cap K$,

$$\|P_2(u)f\| + 1 = \|P_2(u)f\| + \|P_2(w)f\| = \|(P_2(u) + P_2(w))f\| \leqq \|f\| = 1,$$

so that $P_2(u)f = 0$ and $f(y) = \langle f, P_2(u)y\rangle = 0$. Thus $H_w \cap K \subset F$ and the proof is complete. □

Remarks added in proof, January 22, 1984. 1. S. Dineen (The second dual of a JB^* triple system, preprint) has shown by a short elegant argument that the second dual U'' of a JB^* triple is a JB^* triple. His argument has been extended by T. Barton and R. Timoney to show that U'' is a JBW^* triple, i.e. (1. 14) holds. Thus all results in this paper hold automatically for the dual of a JB^* triple.

2. The authors have discovered a proof of Lemma 1. 6 which does not depend on Lemma 1. 5, and is thus independent of the theory of Siegel domains. This elementary proof is based on affine geometric properties of a JBW^* triple which are developed in a forthcoming paper.

3. K. McCrimmon (Pacific J. Math. **103** (1982), 57—102, § 1.8) has a characterization of pairs of tripotents (in any Jordan triple system) for which the families of Peirce projections commute. Our Lemma 1. 10 is included in his result.

References

[1] *E. Alfsen, F. W. Shultz,* State spaces of Jordan algebras, Acta Math. **140** (1978), 155—190.

[2] *E. Alfsen, F. W. Shultz* and *E. Størmer,* A Gelfand-Neumark theorem for Jordan algebras, Adv. in Math. **28** (1978), 11—56.

[3] *F. F. Bonsall* and *J. Duncan,* Numerical Ranges of operators on normed spaces and of elements of normed algebras, Cambridge 1971.

[4] *E. Effros,* Order ideals in a C^*-algebra and its dual, Duke Math. J. **30** (1963), 391—412.

[5] *G. G. Emch* and *W. P. C. King,* Faithful normal states on *JBW*-algebras, Proc. Sympos. Pure Math. **38** (1982), 305—307.

[6] *Y. Friedman* and *B. Russo,* Function representation of commutative operator triple systems, J. Lon. Math. Soc. (2) **27** (1983), 513—524.

[7] *Y. Friedman* and *B. Russo,* Algèbres d'opérateurs sans ordre: solution du problème du projecteur contractif, C. R. Acad. Sci. Paris **296** (1983), 393—396.

[8] *Y. Friedman* and *B. Russo,* Contractive projections on operator triple systems, Math. Scand. **52** (1983) 279—311.

[9] *Y. Friedman* and *B. Russo,* Solution of the contractive projection problem, to appear in J. Funct. Anal.

[10] *Y. Friedman* and *B. Russo,* Conditional Expectation without order, Pac. J. Math. **115** (1984), 351—360.

[11] *L. Harris,* A generalization of C^*-algebras, Proc. Lon. Math. Soc. **42** (1981), 331—361.

[12] *B. Iochum,* Cones autopolaires et algèbres de Jordan, Lecture Notes in Math. **1049,** Berlin-Heidelberg-New York 1984.

[13] *W. Kaup,* Algebraic characterization of symmetric complex Banach manifolds, Math. Ann. **228** (1977), 39—64.

[14] *W. Kaup*, A Riemann Mapping Theorem for bounded symmetric domains in Complex Banach Spaces, Math. Z. **183** (1983), 503—529.
[15] *O. Loos*, Bounded Symmetric Domains and Jordan Pairs, Lecture Notes, Irvine 1977.
[16] *I. Satake*, Algebraic structures of symmetric domains, Publ. Math. Soc. Japan **14** (1980).
[17] *H. Upmeier*, Symmetric Banach Manifolds and Jordan C^*-algebras, to appear.
[18] *J. D. M. Wright*, Jordan C^*-algebras, Mich. Math. J. **24** (1977), 291—302.

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