# A GEOMETRIC SPECTRAL THEOREM[1]

*By* YAAKOV FRIEDMAN *and* BERNARD RUSSO

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## § 0. Introduction

IN this paper we discuss the problem of characterizing geometrically those Banach spaces which admit an algebraic structure.

For ordered Banach spaces well known results of Alfsen–Schultz [**3**] and Alfsen–Schultz–Hanche–Olsen [**1**] give geometric characterizations of the state spaces of Jordan operator algebras and $C^*$-algebras respectively. Several of the properties occurring in these characterizations are natural assumptions for the state space of a physical system. This gives added importance to the problem we are considering.

Let's examine two known mathematical models for quantum mechanics (cf. e.g. [**6**]). In the *Hilbert space model,* states are unit vectors on a separable complex Hilbert Space $\mathcal{H}$, identified modulo the unit circle, and the observables are self-adjoint operators on $\mathcal{H}$. The spectral decomposition $A = \int_{\mathbf{R}} \lambda \, dE_\lambda$ of observable $A$ yields the distribution of $A$ and its expected value in the state $\Psi$ via the formulas

$$P_\Psi\{A \leq \lambda\} = (E_\lambda \Psi, \Psi) = \|E_\lambda \Psi\|^2; \tag{0.1}$$

$$E_\Psi(A) = \int_{\mathbf{R}} \lambda \, d(E_\lambda \Psi, \Psi) = \int_{\mathbf{R}} \lambda \, d\, \|E_\lambda \Psi\|^2. \tag{0.2}$$

In the *algebraic model,* the set of observables is assumed to be equipped with two algebraic structures, namely sum and square. This leads to a Jordan algebra structure in which the states are now positive functionals of norm 1. The fact that the classification theorem of Jordan–von Neumann–Wigner has now been extended to infinite dimensions [**8**], has lead to renewed interest in this approach.

In order to avoid some of the unnatural algebraic assumptions in these models we propose here a geometric model for quantum mechanics. Our starting point will be the assumption that the states of a physical system are the unit vectors of some normed space $Z$. We shall impose some natural axioms on the geometry of the unit ball of $Z$ which involve its facial structure and certain symmetries of $Z$. The observables will be elements of the dual space $Z^*$ and so a spectral theorem is needed for elements of $Z^*$. This requires that an analog of "spectral projection" be defined and that an appropriate notion of orthogonality be formulated.

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For ordered Banach spaces, a complete theory exists (Alfsen–Schultz [2]) which can be used to construct a Jordan algebra structure on $Z^*$ under appropriate additional assumptions.

Let's see why an order structure on $Z$ is needed in order to obtain a binary product structure on $Z^*$. For simplicity we temporarily assume $Z$ is finite dimensional.

Generally, the building blocks for the affine geometric structure of any convex subset $K$ of a Banach space are the extremal points of $K$, or more generally, extremal subsets or faces of $K$.

As shown originally by Effros [5] and Prosser [9], the norm closed faces of the normal state space of a von Neumann algebra $M$ are in one-to-one correspondence with the projections (self-adjoint idempotents) of $M$, which are the building blocks for the algebraic structure of $M$. Moreover, orthogonality of faces (defined geometrically) corresponds to orthogonality of projections (defined by having zero product).

In principle then, a spectral decomposition of an element $x \in Z^*$ should have the form $x = \sum_{i=1}^{k} \alpha_i u_i$ where $\alpha_i$ is a scalar and $\{u_1, \ldots, u_k\}$ is an orthogonal (in an appropriate sense) family of "$n$-potents", i.e., for an $n$-ary product, $u^n = u$. The "$n$-potents" should form a distinguished subset of $Z^*$ which is in one-one correspondence with the set of norm exposed faces of a convex subset of the unit ball $Z_1$ of $Z$.

It is natural to assume that any algebraic structure should be real linear in each component. We ignore unary operations and consider first the construction of a binary product on the real Banach space $Z^*$. As noted earlier, each face $F$ in $Z_1$ should correspond uniquely to an idempotent ($=2$-potent) $e$ in $Z^*$, $e^2 = e$. But $-F$ is also a face, corresponding to $-e$, which is not an idempotent: $(-e)^2 = e \neq -e$. In order to distinguish $F$ from $-F$, geometry alone is not enough. One needs a mechanism for picking out faces $F$ which corresponds to idempotents. That mechanism is given by an order structure on $Z$, i.e., a convex cone $P$ with $P \cap (-P) = (0)$ and $Z = P - P$.

We conclude that if one wishes to construct a binary structure, one must begin with an ordered Banach space. Stated another way, in order to construct an algebraic structure on a Banach space without order, one must consider $n$-ary operations, with $n \geq 3$. Fortunately, $n = 3$ should be enough, as suggested below.

If a triple product is to exist, then $n$-potents become "tripotents", i.e., $e^3 = e$, and if $e$ corresponds to a face $F$ then $-e$, which corresponds to $-F$, is also a tripotent: $(-e)^3 = -e$. Furthermore, upon moving to the case of a complex Banach space, the faces $\alpha F$ with $|\alpha| = 1$, $\alpha$ complex, correspond to $\alpha e$ and in order to have $(\alpha e)^3 = \alpha e$, our triple product must be conjugate linear in one component, which we may assume to be the "middle" variable, and complex linear in the two outer variables.

Returning to a binary structure for a moment, a binary product can be defined as follows: if $x = \sum \alpha_i u_i$ is the spectral decomposition of $x \in Z^*$, then by the orthogonality of $u_i$, and the bilinearity of the product, one has $x^{(2)} = \sum \alpha_i^2 u_i$, and finally

$$x \circ y = \frac{(x+y)^{(2)} - x^{(2)} - y^{(2)}}{2}.$$

In a similar way, in the ternary context, as soon as one has a spectral theorem together with the appropriate notion of orthogonality, one can define the "cube" of $x = \sum_{i=1}^{n} \lambda_i u_i$ as $x^{(3)} = \sum_{i=1}^{n} \lambda_i \bar{\lambda}_i \lambda_i u_i$. It follows that if $\langle x, y, z \rangle$ denotes our triple product, then, by the linearity and conjugate linearity,

$$\langle xyz \rangle + \langle zyx \rangle = \frac{1}{8} \sum_{\substack{\alpha^4=1 \\ \beta^2=1}} \alpha\beta(x + \alpha y + \beta z)^{(3)}. \tag{0.3}$$

Thus if you have a triple product which is complex linear in two variables and conjugate linear in the third, only the symmetrized version of it can be defined in terms of cubes. Stated otherwise, a triple product, which is to be derived from geometry alone must be symmetric in the 2 variables in which it is linear.

In order to motivate our main axiom, let's return again to the geometric setting appropriate for binary products. In this case there is a set of projection operators on $Z$ (called $P$-projections, and occurring in pairs $P$, $P'$) in one-one correspondence with the sets of idempotents and norm exposed faces [2: § 2]. Thus each norm exposed face gives rise to a symmetry (a surjective linear map of order 2), given by $S_F = 2(P_F + P'_F) - I$ (where the face $F$ corresponds to the $P$-projection $P_F$), and the fixed point set of $S_F$ is generated by $F$ and its complementary face $F'$. This situation also prevails in the non-ordered context considered by the authors in [7], i.e., for any norm exposed face of the unit ball of the predual of a $JBW^*$-triple, there exists an isometric symmetry which fixes precisely $\overline{sp}F$ and the set of elements which are orthogonal to each element of $F$.

In the present paper, we shall postulate the existence of such a symmetry corresponding to each norm exposed face of the unit ball $Z_1$. With this axiom alone we are able to give an abstract definition of tripotent corresponding to each norm exposed face, and prove the one-to-one correspondence of these two sets. This is done by showing that the existence of a symmetry $S_F$ corresponding to the face $F$ gives rise to a family $P_0(F)$, $P_1(F)$, $P_2(F)$ of contractive projections, which are analogs of the Peirce projections corresponding to a tripotent in a Jordan triple system, (cf. [7]).

In Section 1 we develop the notion of orthogonality for elements and faces of $Z$, and establish the one-to-one correspondence between generalized tripotents (the building blocks in the spectral theorem) and norm exposed faces.

In Section 2 we define orthogonality for elements of $Z^*$, and discuss properties relating orthogonality of elements and projective units in $Z^*$ with norm exposed faces in $Z$. The main result (Theorem 1) gives, for a reflexive space $Z$ satisfying the symmetry axiom, the existence and uniqueness of a spectral decomposition of an arbitrary element of $Z^*$.

This paper is completely self contained. All of the proofs used only elementary functional analysis, except for some remarks which are included for motivational purposes only.



## § 1. Orthogonality and projective units

We begin by making precise what is meant by orthogonality.

PROPOSITION 1.1. *Let $Z$ be a real or complex normed space, and let $f, g \in Z$. The following are equivalent*:

(a) $\|g+f\| = \|g-f\| = \|g\| + \|f\|$;

(b) $\|\alpha g \pm \beta f\| = \alpha\,\|g\| + \beta\,\|f\|$ *for all* $\alpha, \beta$ *with* $\alpha > 0, \beta > 0, \alpha + \beta = 1$;

(c) $\|\alpha g + \beta f\| = |\alpha|\,\|g\| + |\beta|\,\|f\|$ *for all* $\alpha, \beta \in \mathbb{R}$.

*If* $f \neq 0$ *and* $g \neq 0$, *we may add*

(d) *There exist* $u, v \in Z^*$ *such that* $\|u\| = \|v\| = 1 = \|u \pm v\|$, $f(u) = \|f\|$, $g(v) = \|g\|$, $f(v) = g(u) = 0$.

*Proof.* (a)$\Rightarrow$(b): $\|f\| + \|g\| = \|f \pm g\| = \|(\alpha+\beta)f \pm (\alpha+\beta)g\| = \|(\alpha f \pm \beta g) + (\beta f \pm \alpha g)\| \leq \|\alpha f \pm \beta g\| + \|\beta f \pm \alpha g\| \leq \alpha\,\|f\| + \beta\,\|g\| + \beta\,\|f\| + \alpha\,\|g\| = \|f\| + \|g\|$.

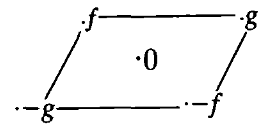
(b)$\Rightarrow$(c): Let $\gamma = |\alpha| + |\beta|$ and $\tilde{f}, \tilde{g}$ denote $\pm f$, $\pm g$ respectively. Then $\|\alpha g + \beta f\| = \gamma\,\||\alpha|\,\gamma^{-1}\tilde{g} + |\beta|\,\gamma^{-1}\tilde{f}\,\| = (\gamma|\alpha|\,\gamma^{-1}\,\|\tilde{g}\| + |\beta|\,\gamma^{-1}\,\|\tilde{f}\|) = |\alpha|\,\|g\| + |\beta|\,\|f\|$.

(c)$\Rightarrow$(a): Trivial.

(a)$\Rightarrow$(d): We may assume $\|f\| = \|g\| = 1$. Choose $x, y \in Z^*$ of norm 1 with $f(x) + g(x) = 2 = f(y) - g(y)$. Then $u = \frac{1}{2}(x+y)$, $v = \frac{1}{2}(x-y)$ satisfy the requirements.

(d)$\Rightarrow$(a): $\|g\| + \|f\| = g(v) + f(u) = (g \pm f)(v \pm u) \leq \|g \pm f\| \leq \|g\| + \|f\|$.

We shall say $f$ and $g$ are *orthogonal* if they satisfy one, hence all of the conditions of Proposition 1.1. The zero vector is orthogonal to all $f$. Note that, by (b), if $\|f\| = \|g\| = 1$ and $f$ is orthogonal to $g$, the four line segments connecting $f$ to $g$ and $-g$, and $-f$ to $g$ and $-g$, all lie in the boundary of the unit ball $Z_1$, which we denote by $\partial Z_1$. Thus $Z_1 \cap \mathrm{sp}_{\mathbf{R}}\{f, g\}$ is the closed parallelogram with vertices $\pm f$, $\pm g$.




We shall write $f \diamond g$ to indicate that $f$ is orthogonal to $g$. For a subset $S$ of $Z$ we let $S^{\diamond} = \{f \in Z\colon f \diamond g \text{ for all } g \in S\}$ and call $S^{\diamond}$ the orthogonal complement of $S$. It is obvious that $S \subset T \Rightarrow S^{\diamond} \supset T^{\diamond}$; and $S \subset S^{\diamond\diamond}$. Hence $S^{\diamond} = S^{\diamond\diamond\diamond}$. It follows that $S \subset T^{\diamond}$ if and only if $T \subset S^{\diamond}$. In this case we say that $S$ and $T$ are orthogonal and write $S \diamond T$.

We note that $S^{\diamond}$ is invariant under real scalar multiplication but we shall show that in general, $S \neq S^{\diamond\diamond}$ and $S^{\diamond}$ is not additive or complex linear.

If $Z$ is an $L^1$ space, it is well known that our notion of orthogonality corresponds precisely to disjointness of the supports of the real functions $f$ and $g$. A similar result holds if $Z$ is the self-adjoint part of the pre-dual of a von-Neumann algebra. It is easy to extend this result, via the polar decomposition to all (complex valued) normal functionals. The following remark is for motivational purposes only and will not be used in the sequel.

*Remark* 1.2. Let $f$ and $g$ be normal functionals on a $JBW^*$-triple $U$ (cf. [**7**]). Then $f \diamond g$ if and only if $e(f)$ and $e(g)$ are orthogonal tripotents of $U$, where $e(f)$ is the tripotent occurring in the polar decomposition of $f$ [**7**: Prop. 2].

*Proof.* We may assume $\|f\| = \|g\| = 1$. Suppose first that $e(f)$ and $e(g)$ are orthogonal tripotents. Then $e_{\pm} := e(f) \pm e(g)$ is of norm 1, and $(f \pm g)(e_{\pm}) = 2$ so $\|f \pm g\| = 2 = \|f\| + \|g\|$. Conversely if $f \diamond g$, set $w = e(\tfrac{1}{2}(f+g))$. Then $2 = (f+g)(w) = f(w) + g(w) \leq |f(w)| + |g(w)| \leq \|f\| + \|g\| = 2$ so that $f(w) = g(w) = 1$. It follows that $\|P_2(w)f\| = \|f\|$ and $\|P_2(w)g\| = \|g\|$, so by [**7**: Prop. 1] $f$ and $g$ may be identified with normal states on the $JBW^*$-algebra $U_2(w)$. We still have $\|f \pm g\| = 2$, so by the Jordan decomposition in $JBW$-algebras, $f$ and $g$ are orthogonal in the $JBW^*$-algebra $U_2(w)$, and hence orthogonal.

Now let $Z = M_{2,3}(\mathbb{C})$ with the trace norm, which is the predual of the

$JBW^*$-triple $M_{3,2}(\mathbb{C})$ and let $S = \{(a_{ij}) \in Z\colon\ a_{13} = a_{23} = 0\}$. Then $S^\diamond = \{0\}$ and $S^{\diamond\diamond} = Z$, so $S \neq S^{\diamond\diamond}$.

We now consider orthogonality of faces of the unit ball $Z_1$ of $Z$.

Let $K$ be a convex set. A *face* of $K$ is a non-empty subset $F$ of $K$ with the following property: if $f \in F$ and $g, h \in K$ satisfy $f = \lambda g + (1-\lambda)h$ for some $\lambda \in (0, 1)$, then $g, h \in F$. An important example for us is: $K = Z_1 =$ the unit ball of $Z$ and $F = \{f \in K\colon\ f(x) = 1\}$ for some element $x \in Z^*$ of norm 1. We shall denote this $F$ which is either empty, or a face (called a *norm exposed face*), by $F_x$.

We now have the following consequence of Proposition 1.1.

COROLLARY 1.3. (a) *Let $F_x$ and $F_y$ be norm exposed faces of $Z_1$ with $\|x \pm y\| = 1$. Then $F_x \diamond F_y$.*

(b) *Let $f, g$ be unit vectors in $Z$ with $f \diamond g$. Then there exist orthogonal norm exposed faces $F_x$, $F_y$ of $Z_1$ with $f \in F_x$, $g \in F_y$.*

*Proof.* (a) If $\rho \in F_x$, then $1 \geq |\rho(x \pm y)| = |1 \pm \rho(y)|$ and so $p(y) = 0$. Similarly if $\sigma \in F_y$, then $\sigma(x) = 0$. By (d) of Proposition 1.1, $\rho \diamond \sigma$.

(b) This follows immediately from Proposition 1.1(d) and (a).

We next consider some examples of unit balls in $\mathbb{R}^3$ which illustrate the concepts just introduced.

EXAMPLE 1. Let $C$ be a double cone in $\mathbb{R}^3$ with circular base and let $x$ be the apex (Fig. 1). Since $C$ is convex and symmetric it is the unit ball $Z_1$ for some norm on $Z = \mathbb{R}^3$. Then $\{x\}^\diamond \cap \partial Z_1$ is the base circle so $\{x\}^\diamond$ is a linear space.

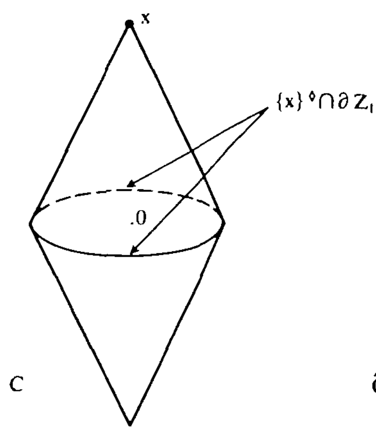



FIG. 1

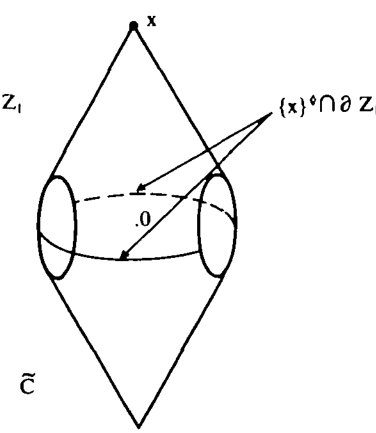



FIG. 2

EXAMPLE 2. Chop $C$ by two parallel planes to define $\tilde{C}$, a convex symmetric set which is the unit ball $Z_1$ of some norm on $Z = \mathbb{R}^3$ (Fig. 2). In this case $\{x\}^\diamond \cap \partial Z_1$ consists of two arcs and so $\{x\}^\diamond$ is not a linear space.

We shall call an element $u \in Z^*$ a *projective unit* if $\|u\| = 1$ and $\langle u, F_u^\diamond \rangle = 0$. Note that this implies $F_u \neq \emptyset$ so that $F_u$ is a norm exposed face in $Z_1$, and $F_u$ is "parallel" to $F_u^\diamond$, i.e., $\langle u, F_u \rangle = 1$, $\langle u, F_u^\diamond \rangle = 0$.

In Examples 1 and 2, $\{x\}$ is a face and the unique plane passing through $x$ which is parallel to the base determines a projective unit.

Let $\mathcal{F}$ and $\mathcal{U}$ denote the collections of proper norm exposed faces of $Z_1$ and projective units in $Z^*$, respectively. The map $\mathcal{U} \ni u \mapsto F_u \in \mathcal{F}$ is not onto in general (see Example 4).

EXAMPLE 3. Consider a "straight" tent of height 1 sitting on a frozen lake (Fig. 3). This is the unit ball for some norm on $Z = \mathbb{R}^3$. The face $F = [-1, 1] \times \{0\} \times \{1\}$ is of the form $F_u$ for the projective unit $u = (0, 0, 1) \in Z^*$ since $\langle u, F \rangle = 1$ and $\langle u, F^\diamond \rangle = 0$.

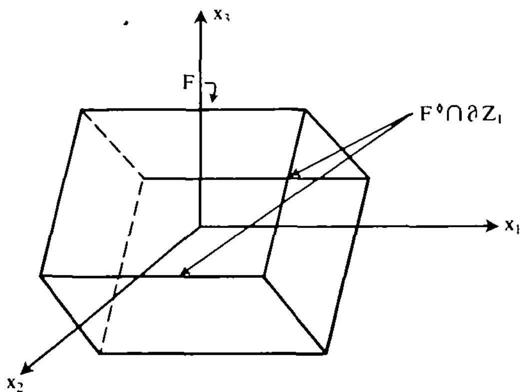



FIG. 3

EXAMPLE 4. Consider a "crooked" tent, extended downward so as to remain convex and symmetric (Figure 4). It is clear that $F$ is norm exposed but not of the form $F_u$ for any $u \in \mathcal{U}$, since $F$ and $F^\diamond$ are not parallel.

In order to motivate the next definition, consider the following. Let $Z$ be the pre-dual of a $JBW^*$-triple. As follows from [7], the set $\mathcal{U}$ of projective units coincides with the set of tripotents and the map $u \to F_u$ is a bijection of $\mathcal{U}$ onto the set of all norm exposed faces of $Z_1$. Moreover, the Peirce projections $P_k(u)$, $k = 0, 1, 2$ corresponding to a tripotent $u$ give rise to a symmetry $S_u = P_2(u) - P_1(u) + P_0(u)$ which makes the norm exposed face $F_u$ into a symmetric face, which is defined as follows.

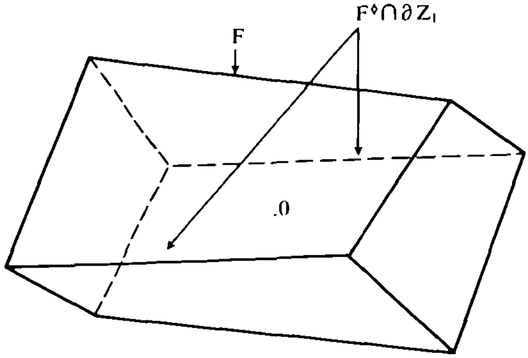



FIG. 4

DEFINITION. A norm exposed face $F$ of the unit ball $Z_1$ of a complex normed space $Z$ is a *symmetric face* if there is a unique linear isometry $S_F$ of $Z$ onto $Z$ of order 2 whose fixed point set is $\overline{\mathrm{sp}}_{\mathbb{C}}\, F \oplus F^{\diamond}$.

Note that if $F^{\diamond}$ is not a linear space, then $F$ is not a symmetric face. For instance the face $\{x\}$ in Example 2 is not symmetric.

Symmetries of this type occur naturally in measuring processes in quantum mechanics. With any measurement we can associate a filtering projection $p_f$, a face $f$, which consists of states which "pass" through the filter with probability 1, and a projective unit exposing this face (cf. Araki [**4**]). Each filtering projection has a complementary filtering projection $p_f^{\#} = p_{f^{\#}}$ corresponding to particles which do not pass the filter. The mapping $2(p_f + p_f^{\#} - \mathrm{id.})$ is a symmetry which fixes span $f$ and span $f^{\#}$. Under appropriate assumptions this leads to a Jordan algebra structure on the set of observables.

Our model differs from the Jordan algebra model in the following way. The latter model implies uniqueness of the complementary filtering projection, which may be questionable from a physical standpoint. In our model this is not the case, as seen from the example of a $JBW^*$-triple. Here, the complementary projection corresponding to a face of $F$ (namely $P_0(F)$) is the smallest projection containing all possible complementary filtering projections.

We shall now construct a family of projections, called *generalized Peirce projections*, corresponding to each symmetric face.

Let $\mathscr{E}(T, \lambda)$ denote the eigenspace of an operator $T$ corresponding to the eigenvalue $\lambda$. If $F$ is a symmetric face, we have

$$\mathscr{E}(S_F, 1) = \overline{\mathrm{sp}}_{\mathbb{C}}\, F \oplus F^{\diamond}. \tag{1.1}$$

For each symmetric face $F$, we may define contractive projections $P_k(F)$, $k=0, 1, 2$ on $Z$ as follows. First, $P_1(F):=\frac{1}{2}(I-S_F)$ is the projection on $\mathscr{E}(S_F, -1)$. Secondly, because of (1.1) we may define $P_2(F)$ as the composition of the projection $\tilde{P}(F):=\frac{1}{2}(I+S_F)(=\text{the projection}$ on $\mathscr{E}(S_F, 1))$ followed by the projection of $\mathscr{E}(S_F, 1)$ onto $\overline{\text{sp}}_{\mathbf{C}}\,F$. Similarly $P_0(F)$ is the projection with range $F^{\diamond}$. Note that $P_2(F)+P_0(F)=\frac{1}{2}(I+S_F)$ is a contractive projection. We also have

$$\|P_2(F)\rho\|+\|P_0(F)\rho\|=\|P_2(F)\rho+P_0(F)\rho\|, (\rho\in Z);$$
$$P_2(F)+P_1(F)+P_0(F)=I;$$
$$P_2(F)-P_1(F)+P_0(F)=S_F.$$

In the geometric framework appropriate to ordered Banach spaces, there are one-to-one correspondences between three collections of objects: certain faces, certain projections (called $P$-projections) and certain elements (called projective units), cf. [**2**: § 2].

The following proposition gives a one-to-one correspondence between generalized tripotents and symmetric faces, analogous to [**2**: Corollary 2.18].

DEFINITION. A *generalized tripotent* is a projective unit $u\in\mathscr{U}$ with the property that $F_u$ is a symmetric face and $S^*_{F_u}u=u$.

PROPOSITION 1.4. *The map $u\to F_u$ is a bijection of the set of generalized tripotents and the set of norm exposed symmetric faces of $Z_1$.*

*Proof.* Let $F$ be a symmetric face and suppose $F=F_x$ for some $x\in Z^*$ with $\|x\|=1$. Set $u:=P_2(F)^*x$. Then $u$ is a tripotent and $F_u=F$. Indeed, let $\rho\in F$. Then $\langle u, \rho\rangle=\langle P_2(F)^*x, \rho\rangle=\langle x, P_2(F)\rho\rangle=\langle x, \rho\rangle=1$. This shows that $F\subset F_u$ and $\|u\|\geq 1$. Since $P_2(F)$ is contractive, $\|u\|=1$.

Now suppose $\rho\in F_u$. Then $\langle P_2(F)\rho, x\rangle=\langle\rho, P_2(F)^*x\rangle=\langle\rho, u\rangle=1$ and so $P_2(F)\rho\in F$, and $\|P_2(F)\rho\|=1$. Then $1=\|\rho\|\geq\|P_2(F)\rho+P_0(F)\rho\|=\|P_2(F)\rho\|+\|P_0(F)\rho\|=1+\|P_0(F)\rho\|$ and so $P_0(F)\rho=0$. Hence $\frac{1}{2}(\rho+S_F\rho)=P_2(F)\rho\in F$ and since $F$ is a face, $\rho\in F$. This proves $F_u\subset F$, so $F=F_u$ is symmetric and $\langle u, F_u^{\diamond}\rangle=\langle P_2(F)^*x, F^{\diamond}\rangle=\langle x, P_2(F)F^{\diamond}\rangle=0$. Since obviously $S^*_{F_u}u=(P_2(F)^*-P_1(F)^*+P_0(F)^*)\times P_2(F)^*x=u$, $u$ is a generalized tripotent, and the map $u\to F_u$ is onto.

Suppose $u_1$ and $u_2$ are tripotents and $F_{u_1}=F_{u_2}=F$ say. By definition of tripotent, $P_1(F)^*u_i=0$ for $i=1, 2$. By definition of projective unit, $\langle P_0(F)^*u_i, Z\rangle=\langle u_i, F_{u_i}^{\diamond}\rangle=0$, i.e., $P_0(F)^*u_i=0$, $i=1, 2$. Therefore, for arbitrary $\rho\in Z$, $\langle u_i, \rho\rangle=\langle(P_0(F)^*+P_1(F)^*+P_2(F)^*)u_i, \rho\rangle=\langle P_2(F)^*u_i, \rho\rangle=\langle u_i, P_2(F)\rho\rangle$. But $P_2(F)Z=\overline{\text{sp}}_{\mathbf{C}}\,F$ and $\langle u_i, F\rangle=1$, so $u_1=u_2$.

## § 2. Spectral theorem

In this section we shall prove, for a certain class of normed spaces, the existence and uniqueness of a spectral decomposition for each element $x$ in the dual.

Let $Z$ be any normed space. Elements $a, b \in Z^*$ are *orthogonal* if there is a symmetric face $F \subset Z_1$ such that either

(i) $a \in \operatorname{im} P_2(F)^*$ and $b \in \operatorname{im} P_0(F)^*$;

or

(ii) $a \in \operatorname{im} P_0(F)^*$ and $b \in \operatorname{im} P_2(F)^*$.

We shall write $a \diamondsuit b$ or $b \diamondsuit a$ to indicate this relation.

LEMMA 2.1. *Let $a, b \in Z^*$ and suppose $a \diamondsuit b$. Then*

(i) $\|a+b\| = \max(\|a\|, \|b\|)$

(ii) $F_a \subset F_{a+b}$ if $\|a\| = 1$ and $\|b\| \leqslant 1$.

(iii) $F_a \diamondsuit F_b$ if $\|a\| = 1$ and $\|b\| = 1$.

*Proof.* Without loss of generality, we may assume that $a \in \operatorname{im} P_2(F)^*$ and $b \in \operatorname{im} P_0(F)^*$ for some symmetric face $F$. If $M$ denotes $\max(\|a\|, \|b\|)$, and $\rho \in Z$, we have $|\langle a+b, \rho\rangle| = |\langle P_2(F)^*a, \rho\rangle + \langle P_0(F)^*b, \rho\rangle| \leqslant \|a\|\,\|P_2(F)\rho\| + \|b\|\,\|P_0(F)\rho\| \leqslant M\,\|(P_2(F) + P_0(F))\rho\| \leqslant M\,\|\rho\|$, so that $\|a+b\| \leqslant M$. On the other hand since $P_2(F)^*(a+b) = a$, $P_0(F)^*(a+b) = b$, we have $\|a+b\| \geqslant M$.

Now suppose $\|a\| = 1$ and $\|b\| \leqslant 1$. Then $\|a \pm b\| = 1$, and $|\langle a \pm b, \rho\rangle \leqslant \|\rho\|$ for $\rho \in Z$. This implies $F_a \subseteq F_{a+b}$.

The last statement follows from Corollary 1.3(a) since $\|a \pm b\| = 1$ by (i).

COROLLARY 2.2. *If $Z$ is a reflexive Banach space then every family of pairwise orthogonal elements of $Z^*$ is finite.*

*Proof.* If the conclusion were false we would have an infinite sequence of pairwise orthogonal non-zero elements in $Z^*$. By the lemma their span would be a copy of $l^\infty$ in $Z^*$. Therefore $Z^*$ and hence $Z$ cannot be reflexive.

DEFINITION. Let $Z$ be a normed space. We call $Z$ a *facially symmetric normed space* if each norm exposed face $F$ in the unit ball $Z_1$ is strongly symmetric, i.e., $F$ is symmetric and for each $y \in Z^*$ of norm one with $F \subseteq F_y$, we have $S_F^* y = y$, where $S_F$ is the symmetry corresponding to $F$.

From the theory of $JB^*$-triples developed in [7] we know that if $Z$ is the dual of a $JB^*$-triple, or more generally, if $Z$ is the pre-dual of a $JBW^*$-triple, then $Z$ is a facially symmetric Banach space. In particular, the dual of a $C^*$-algebra (or a $JB^*$-algebra) or more generally the predual of a von Neumann algebra (or a $JBW^*$-algebra) are examples of facially symmetric Banach spaces.

Note that Example 2 shows that, in general, not every projective unit is a tripotent. However, for facially symmetric spaces we have:

*Remark* 2.3. Let $Z$ be a facially symmetric normed space. Then every projective unit in $Z^*$ is a generalized tripotent. Hence by Proposition 1.4, the map $u \to F_u$ is a bijection of the set of projective units in $Z^*$ and the set of norm exposed faces of $Z_1$.

Throughout the rest of this section we shall assume for convenience that $Z$ is a facially symmetric normed space. By Remark 2.3 we can denote by $v_F$ the unique projective unit with the property that $F_{v_F} = F$ where $F$ is any norm exposed face of $Z_1$.

LEMMA 2.4. *Let $S$ be a linear isometry of $Z$ onto $Z$. Then $S^{-1}(F) = F_{S^*(v_F)}$ and $S^*(v_F) = v_{S^{-1}(F)}$ for each norm exposed face $F$ of $Z_1$.*

*Proof.* Let $G$ denote the face $S^{-1}(F)$. Then $\langle S^*(v_F), G\rangle = \langle v_F, F\rangle = 1$, so $G \subset F_{S^*(v_F)}$. Conversely, $\rho \in F_{S^*(v_F)}$ implies $1 = \langle S^*(v_F), \rho\rangle = \langle v_F, S\rho\rangle$, i.e., $S\rho \in F$ or $\rho \in S^{-1}(F)$. Hence $G = F_{S^*(v_F)}$.

We next show that $\langle S^*(v_F), G^\diamond\rangle = 0$. If $\rho \in G^\diamond$, then $S\rho \in S(G^\diamond) = S(G)^\diamond = F^\diamond$, and so $\langle S^*(v_F), \rho\rangle = \langle v_F, S\rho\rangle = 0$.

We have shown that $S^{-1}(F) = F_{S^*(v_F)}$ and that $S^*(v_F) \in \mathcal{U}$. By Remark 2.3 the lemma follows.

The following gives equivalent conditions for orthogonality of projective units in the dual of a facially symmetric Banach space.

LEMMA 2.5. *For $u, v \in \mathcal{U}$, the following are equivalent:*

(1) $u \diamond v$

(2) $F_u \diamond F_v$

(3) $v \in \operatorname{im} P_0(u)^*$

(4) $u \in \operatorname{im} P_0(v)^*$

(5) $u \pm v \in \mathcal{U}$.

*Proof.* (1)$\Rightarrow$(2): By Lemma 2.1, $\|u \pm v\| = 1$. By Corollary 1.3(a), (2) follows.

(2)$\Rightarrow$(3): Since $F_v \subseteq F_u^\diamond = \operatorname{im} P_0(u)$, we have $S_u(F_v) = F_v$ where $S_u$ denotes the symmetry corresponding to $F_u$. By Lemma 2.4, $S_u^*(v) = S_u^*(v_{F_v}) = v_{S_u(F_v)} = v_{F_v} = v$, i.e., $P_1(u)^*v = 0$.

To prove (3) it remains to show that $P_2(u)^*v = 0$. Since $F_u \subset F_v^\diamond$ and $\langle v, F_v^\diamond\rangle = 0$, we have $\langle v, F_u\rangle = 0$. Now let $\rho \in Z$. Then $\langle P_2(u)^*v, \rho\rangle = \langle v, P_2(u)\rho\rangle = 0$ since $P_2(u)\rho \in \operatorname{im} P_2(u) = \overline{\mathrm{sp}}\, F_u$. Hence $P_2(u)^*v = 0$.

Similarly, (2)$\Rightarrow$(4).

(3)$\Rightarrow$(1): trivial.

(4)$\Rightarrow$(1): trivial.

Thus (1), (2), (3), (4) are equivalent.

(1)$\Rightarrow$(5): By Lemma 2.1, (1) implies that $\|u \pm v\| = 1$ and $F_u \cup F_v \subset F_{u\pm v}$. Therefore $F_u^\diamond \cap F_v^\diamond \supset F_{u\pm v}^\diamond$ and $\langle u \pm v, F_{u\pm v}^\diamond\rangle = \langle u, F_{u\pm v}^\diamond\rangle \pm \langle v, F_{u+v}^\diamond\rangle = 0$.

(5)$\Rightarrow$(2): By Corollary 1.3(a).

COROLLARY 2.6. *Let $u_1, u_2, u_3 \in \mathcal{U}$ and suppose $u_1 \diamond u_2$ and $u_3 \diamond (u_1 + u_2)$. Then $u_3 \diamond u_1$ and $u_3 \diamond u_2$.*

*Proof.* By Lemma 2.1, $F_{u_1} \subset F_{u_1+u_2}$. Therefore $F_{u_1}^\diamond \supset F_{u_1+u_2}^\diamond$. Now $u_3 \diamond (u_1 + u_2) \Rightarrow F_{u_3} \diamond F_{u_1+u_2} \Rightarrow F_{u_3} \subset F_{u_1+u_2}^\diamond \subset F_{u_1}^\diamond \Rightarrow F_{u_3} \diamond F_{u_1} \Rightarrow u_3 \diamond u_1$.

The following lemmas describe connections between faces and elements exposing them.

LEMMA 2.7. *Let $u \in \mathcal{U}$, $x \in Z^*$, $\|x\| = 1$ and suppose $F_u \subset F_x$. Then either $F_u = F_x$ or $F_u^\diamond \cap F_x \neq \emptyset$.*

*Proof.* Suppose $F_u \neq F_x$ and fix $\rho \in F_x$ with $\rho \notin F_u$. Since $Z$ is facially symmetric, $\langle S_u\rho, x\rangle = \langle \rho, S_u^* x\rangle = \langle \rho, x\rangle = 1$ and therefore $P_2(u)\rho + P_0(u)\rho = (\rho + S_u\rho)/2 \in F_x$.

*Case 1.* $P_2(u)\rho = 0$; then $P_0(u)\rho \in F_u^\diamond \cap F_x$.

*Case 2.* $P_0(u)\rho \neq 0$ and $P_2(u)\rho \neq 0$; then since $P_2(u)\rho + P_0(u)\rho \in F_x$, $1 = \|P_2(u)\rho\| + \|P_0(u)\rho\|$ and

$$\|P_2(u)\rho\| \left(\frac{P_2(u)\rho}{\|P_2(u)\rho\|}\right) + \|P_0(u)\rho\| \left(\frac{P_0(u)\rho}{\|P_0(u)\rho\|}\right) \in F_x$$

By definition of face, $\dfrac{P_0(u)\rho}{\|P_0(u)\rho\|} \in F_x \cap F_u$.

*Case 3.* $P_0(u)\rho = 0$; then $P_2(u)\rho \in F_x$. But $P_2(u)\rho \in \operatorname{im} P_2(u) = \overline{\operatorname{sp}}\, F_u$ implies $P_2(u)\rho = \lim_\alpha \sum \lambda_i^\alpha \sigma_i^\alpha$ with $\sigma_i^\alpha \in F_u$. Since $F_u \subset F_x$, $\sigma_i^\alpha \in F_x$ and $\rho(u) = \langle P_2(u)\rho, \quad u\rangle = \lim_\alpha \sum \lambda_i^\alpha \sigma_i^\alpha(u) = \lim_\alpha \sum \lambda_i^\alpha \sigma_i^\alpha(x) = \langle P_2(u)\rho, \quad x\rangle = 1$,so $\rho \in F_u$ a contradiction, and this case does not occur.

LEMMA 2.8. *Let $u \in \mathcal{U}$, $x \in Z^*$, $\|x\| = 1$ and suppose $F_u \subset F_x$. Then $x = u + P_0(u)^*x$. Moreover, if $Z$ is reflexive, then $F_u = F_x$ if and only if, $x = u + P_0(u)^*x$, with $\|P_0(u)^*x\| < 1$.*

*Proof.* By definition, since $F_u \subset F_x$, $S_u^* x = x$, i.e., $P_1(u)^* x = 0$.

Now let $y = P_2(u)^* x$ so that $\|y\| \leq 1$. Let $\rho \in F_u$. Then $\langle y, \rho \rangle = \langle P_2(u)^* x, P_2(u)\rho \rangle = \langle x, \rho \rangle = 1$ since $F_u \subset F_x$. Thus $F_u \subset F_y$ and $\|y\| = 1$. Now $\langle y, F_u^\diamond \rangle = \langle P_2(u)^* x, P_0(u)Z \rangle = 0$ and since $F_y^\diamond \subset F_u^\diamond$ we have $y \in \mathcal{U}$.

We next show that $F_u = F_y$. If $F_u \neq F_y$ then by Lemma 2.7 we can find $\sigma \in F_u^\diamond \cap F_y$. But $\sigma \in F_u^\diamond$ implies $\langle y, \sigma \rangle = \langle P_2(u)^* x, P_0(u)\sigma \rangle = 0$ contradicting $\sigma \in F_y$. Therefore $F_u = F_y$, $u = y = P_2(u)^* x$.

Suppose now that $Z$ is reflexive and that $F_u = F_x$. By the first statement, $x = u + P_0(u)^* x$. If $c := P_0(u)^* x$ has norm one, then by reflexivity $F_c \neq \varnothing$. By Lemma 2.1 then $F_c \subseteq F_x$ which is a contradiction since $F_c \nsubseteq F_u$.

Conversely, suppose $\|c\| < 1$. If $F_u \neq F_x$ then by Lemma 2.7 there exists $\rho \in F_u^\diamond \cap F_x$, so $1 = \rho(x) = \rho(c) < 1$, contradiction.

The following is the key step in the proof of our spectral theorem.

LEMMA 2.9. *Suppose that $Z$ is reflexive and let $u \in \mathcal{U}$, $b \in \operatorname{im} P_0(u)^*$, $\|b\| = 1$. Then there exists $w \in \mathcal{U}$, $w \diamond u$, such that $b = w + c$, with $\|c\| < 1$ and $c \diamond (u + w)$.*

*Proof.* Choose $w \in \mathcal{U}$ such that $F_b = F_w$. By Lemma 2.8, $b = w + c$ with $\|c\| < 1$ and $c \in \operatorname{im} P_0(w)^*$. By Lemma 2.1(iii) $F_b \diamond F_u$ and therefore $F_w \diamond F_u$. By Lemma 2.5, $u \diamond w$. It remains to prove that $c \in \operatorname{im} P_0(u + w)^*$.

Notice that by Lemma 2.1, $\|b + u\| = \|w + u\| = 1$. We will show that $F_{b+u} = F_{w+u}$. To do this consider $\rho \in Z$, $\|\rho\| = 1$ and $a \in Z^*$, $\|a\| = 1$ with $a \diamond u$. Then $\rho(a + u) = \rho(a) + \rho(u) = \langle P_0(u)\rho, a \rangle + \langle P_2(u)\rho, u \rangle \leq \|P_0(u)\rho\| + \|P_2(u)\rho\| \leq \|\rho\| = 1$. Thus

$$\rho \in F_{b+u} \Leftrightarrow \rho(b + u) = 1 \Leftrightarrow \frac{P_0(u)\rho}{\|P_0(u)\rho\|} \in F_b$$

and

$$\frac{P_2(u)\rho}{\|P_0(u)\rho\|} \in F_u \Leftrightarrow \frac{P_0(u)\rho}{\|P_0(u)\rho\|} \in F_w$$

and

$$\frac{P_2(u)\rho}{\|P_2(u)\rho\|} \in F_u \Leftrightarrow \rho(w + u) = 1.$$

Therefore $F_{b+u} = F_{w+u}$.

Finally, by Lemma 2.8, $b + u = w + u + P_0(w + u)^*(b + u)$ or $c = b - w \in \operatorname{im} P_0(u + w)^*$.

THEOREM 1 (Spectral Theorem). *Let $Z$ be a reflexive facially symmetric Banach space. Then for each non-zero $x$ in $Z^*$, there exist a unique family of pairwise orthogonal generalized tripotents $u_1, \ldots, u_n$ and real numbers*

$\lambda_1, \ldots, \lambda_n$ *such that*

$$x = \sum_{i=1}^{n} \lambda_i u_i \quad \textit{and} \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0.$$

*Proof.* Let $x$ be given. We shall prove by induction the following proposition:

$$(*)\begin{cases}\text{For any integer } k \geq 1, \text{ either}\\ \text{(1) the theorem holds for some } n \leq k, \text{ or}\\ \text{(2) there is a family of pairwise orthogonal generalized tripotents}\\ \quad u_1, \ldots, u_k, \text{ real numbers } \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0, \text{ and non-zero}\\ \quad d_k \in Z^* \text{ such that } x = \sum_{i=1}^{k} \lambda_i u_i + d_k,\ d_k \diamondsuit (u_1 + \cdots + u_k), \text{ and}\\ \quad \|d_k\| < \lambda_k.\end{cases}$$

Consider first the case $k = 1$. Let $b_1 = x/\|x\|$. Since $Z$ is reflexive, $F_{b_1}$ is a norm exposed face. Let $u_1$ be the generalized tripotent corresponding to this face, i.e., $F_{u_1} = F_{b_1}$. By Lemma 2.8 $b_1 = u_1 + c_1$, where $c_1 \diamondsuit u_1$ and $\|c_1\| < 1$. Let $\lambda_1 = \|x\|$, $d_1 = \lambda_1 c_1$. Then $x = \lambda_1 u_1 + d_1$, $d_1 \diamondsuit u_1$ and $\|d_1\| < \lambda_1$.

Suppose now that $(*)$ holds for $k = l - 1$. If (1) holds for $k = l - 1$, then it holds for $k = l$. If (2) holds for $k = l - 1$, we define $\lambda_l = \|d_{l-1}\|$ and $b_l = \lambda_l^{-1} d_{l-1}$. By Lemma 2.9 (since $b_l \in \operatorname{im} P_0(u_1 + \cdots + u_{l-1})^*$), we have $u_l \in \mathcal{U}$ with $u_l \diamondsuit (u_1 + \cdots + u_{l-1})$, $b_l = u_l + c_l$, $c_l \diamondsuit (u_1 + \cdots + u_l)$ and $\|c_l\| < 1$. This implies $x = \lambda_1 u_1 + \cdots + \lambda_l u_l + \lambda_l c_l$. With $d_l = \lambda_l c_l$ we have that $d_l$ satisfies $(*)$. By Corollary 2.6, $u_1, \ldots, u_l$ is a pairwise orthogonal family. Thus $(*)$ holds for $k = l$.

By Corollary 2.2, for sufficiently large $k$, only (1) can occur. This proves existence.

Suppose now that $x$ has two such decompositions, say $x = \sum_{i=1}^{n} \lambda_i u_i$ and $x = \sum_{j=1}^{m} \mu_j w_j$, with $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, $\mu_1 > \mu_2 > \cdots > \mu_m > 0$, $u_i$, $w_j \in \mathcal{U}$, $u_1, \ldots, u_n$ pairwise orthogonal, $w_1, \ldots, w_m$ pairwise orthogonal.

If, for some $k \geq 1$ we have $\lambda_i = \mu_i$ and $u_i = w_i$ for all $i \leq k - 1$, then set $y := x - \sum_{i=1}^{k-1} \lambda_i u_i = x - \sum_{j=1}^{k-1} \mu_j w_j$. By Lemma 2.1 $\lambda_k = \|y\| = \mu_k$. Moreover $F_{u_k} = F_{y/\|y\|} = F_{w_k}$ so by Remark 2.3, $u_k = w_k$. The uniqueness follows by induction.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*Department of Mathematics*
*University of California*
*Los Angeles, California 90024*
*USA*

*Department of Mathematics*
*University of California*
*Irvine, California 92717*
*USA*