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Conditional Expectation and Bicontractive Projections on Jordan C^*-algebras and Their Generalizations*

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On a Banach space X a projection $P \in \mathscr{L}(X)$ is called *bicontractive* if $\|P\| \leqq 1$ and $\|I-P\| \leqq 1$. Such projections may be constructed as follows. Let θ be an isometry of X onto X of order 2. Then

$$P = \tfrac{1}{2}(I+\theta) \tag{0.1}$$

is a bicontractive projection. It would be of interest to characterize the class $\mathscr{S}$ of those Banach spaces for which every bicontractive projection is of the above form. Evidently $X \in \mathscr{S}$ if X', its dual, belongs to $\mathscr{S}$.

The purpose of this paper is to show that if X is a JB^*-triple, then $X \in \mathscr{S}$. It follows that all JB^*-algebras (= Jordan C^*-algebras) and their duals and preduals (when existing) belong to $\mathscr{S}$.

By a theorem of Kaup [14, 18], JB^*-triples are precisely those Banach spaces (within isometric isomorphism) for which the open unit ball is a bounded symmetric domain (within biholomorphic equivalence). For the precise definition and basic properties of JB^*-triples we refer to [11, 14, 18].

In Bernau-Lacey [3] it is shown that $X = L_p(\mu)$, $1 \leqq p < \infty$ belongs to $\mathscr{S}$. Earlier Byrne-Sullivan [4] proved the special cases $1 < p < \infty$ and μ a probability measure. As a bi-product of the classification of all contractive projections on the space C_1 of all trace class operators on a separable Hilbert space, Arazy-Friedman show in [1] that C_1, and therefore its predual (the space of all compact operators), belongs to $\mathscr{S}$.

A study of unit preserving bicontractive projections on unital Jordan (operator) algebras was begun in Robertson-Youngson [16]. Using some ideas from [16] and the classification of type I Jordan factors, Stormer [17] proved that every unital bicontractive projection on an arbitrary C^*-algebra is of the form (0.1) with θ a Jordan automorphism of order 2.

In [9] the authors showed that every C^*-algebra belongs to $\mathscr{S}$. This result generalized the above mentioned results of Arazy-Friedman and of Stormer.

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The proof was based on the authors' detailed study of contractive projections in a setting of operator algebras without order [7, 8, 10]. It was actually proved for the class of J^*-algebras and therefore also showed that all JC^*-algebras belong to $\mathscr{S}$.

The proof of our main result below depends only on global properties of JB^*-triples and contractive projections thereon, as developed in [11, 15] and Sects. 1 and 2 below. As such it includes as true corollaries all of the above mentioned results of Arazy-Friedman, Robertson-Youngson, Stormer, and the authors on bicontractive projections.

Recall that a J^*-algebra is a concrete example of a JB^*-triple. Other examples of JB^*-triples which are not J^*-algebras are the two exceptional JB^*-triples of dimensions 16 and 27, which we denote by C^5 and C^6. The structure of an arbitrary JB^*-triple has been studied by the authors in [11] and [12]. As a result of this study some problems for JB^*-triples can be reduced to the corresponding problem for J^*-algebras and the two exceptional JB^*-triples. The former act on Hilbert space and the latter are finite dimensional.

It is natural to attempt a direct proof that C^5 and C^6 belong to $\mathscr{S}$ and to use this, together with the structure theorem mentioned above, in order to show that an arbitrary JB^*-triple belongs to $\mathscr{S}$.

We found that this approach seems to require showing the invariance of the special and exceptional summands of a JBW^*-triple under the projection. To prove this would require much of the fine structure of a contractive projection, as developed in [7] and [10] for J^*-algebras. Hence it is more efficient to follow the outline that solved the bicontractive projection problem for J^*-algebras [9].

This paper is organized as follows. In Sect. 1 we prove some commutativity formulas involving a contractive projection and the Peirce projections associated with an element in the range of the dual projection. The corresponding formulas for the J^*-algebra case played important roles in the study of a contractive projection, and the same is true here. The main tools used in the proofs of these formulas are Propositions 1, 2 and 3 of [11]. In Sect. 2 we exploit two results of Kaup [15] to develop the fine structure of a contractive projection P on a JB^*-triple U. The main results, Theorems 2 and 3, give a concrete realization of $P(U)$ in the second dual U'', and a new conditional expectation formula for P. Our final result, that JB^*-triples belong to the class $\mathscr{S}$, is proved in Sect. 3.

The following are some of the notational conventions used in this paper. If X is a Banach space, X' denotes its normed dual and X_* denotes a predual of X, i.e., $(X_*)' \approxeq X$ (isometric). We use the same notation, namely $P_k(v)$, $k=0,1,2$, for the Peirce projections associated with a tripotent v, and their adjoints. For a normal functional f on a JBW^*-triple U, $e(f)$ denotes the tripotent occuring in the polar decomposition of f. We write $P_k(f)$ for $P_k(e(f))$. Two functionals f, g are orthogonal, denoted by $f \perp g$, if $e(f) \in P_0(g)\,U$. The symbol $\perp$ will also denote orthogonality of elements of U.

The following consequence of [7: Lemma 2.4] and [11: Cor. 1.6] will be used several times:

(0.2) If P is a contractive projection on a JB^*-triple U and $f \in P'(U')$, then with $v = e(f)$ we have $P''v = v + P_0(f)P''v$.

More generally:

(0.3) If $x \in U''$, $\|x\| = 1$, and $f(x) = \|f\|$, for some $f \in U'$, then $x = v + P_0(f)x$.

1. Commutativity Formulas for Contractive Projections

In this section we use freely the notation and results of Friedman-Russo [11: Sect. 1]. By Dineen [5] and Barton-Timoney [2], if U is a JB^*-triple, we may regard U' as the predual of the JBW^*-triple U''.

Lemma 1.1. *Let U be a JB^*-triple and let e be a tripotent of U''. For each $g \in P_2(e)U'$, let $T(g)$ denote the restriction of g to $P_2(e)U''$. Then T is an isometric isomorphism of $P_2(e)U'$ onto $(P_2(e)U'')_*$.*

Proof. The map T is clearly linear, norm decreasing and takes $P_2(e)U'$ into $(P_2(e)U'')_*$. By [11: Prop. 1] T is onto and isometric. □

The proof of the following proposition is the same as [7: Prop. 3.3], with Lemma 1.1 replacing [7: Remark 3.2].

Proposition 1.2. *Let P be a contractive projection on a JB^*-triple U and let $f \in P'(U')$. Then $P'P_2(f) = P_2(f)P'P_2(f)$.*

Proof. Let $B := (P_2(f)U'')_*$, T: $P_2(f)U' \to B$ as in Lemma 1.1 and let V_f be the face generated by $T(f)$ in B^+. By Emch-King [6, 13] V_f is norm dense in B^+. Since B^+ linearly spans B, it will suffice, by Lemma 1.1, to prove that $P'(T^{-1}(V_f)) \subseteq P_2(f)U'$.

For $\tau \in V_f$, write $T(f) = \alpha\tau + \sigma$ for some $\alpha > 0$ and $\sigma \in V_f$. Then $f = \alpha\tilde{\tau} + \tilde{\sigma}$ where $\tilde{\tau} = T^{-1}(\tau)$ and $\tilde{\sigma} = T^{-1}(\sigma)$, and

$$\begin{aligned}\|f\| &= \|T(f)\| = \alpha\|\tau\| + \|\sigma\| = \alpha\|\tilde{\tau}\| + \|\tilde{\sigma}\| \\ &\geqq \alpha\|P'\tilde{\tau}\| + \|P'\tilde{\sigma}\| \geqq \alpha\|P_2(f)P'\tilde{\tau}\| + \|P_2(f)P'\tilde{\sigma}\| \geqq \|f\|\end{aligned}$$

since $f = P_2(f)P'f = P_2(f)P'(\alpha\tilde{\tau} + \tilde{\sigma})$. Therefore $\|P'\tilde{\tau}\| = \|P_2(f)P'\tilde{\tau}\|$ so by [11: Prop. 1] $P'\tilde{\tau} = P_2(f)P'\tilde{\tau}$. □

In order to prove our second commutativity formula we need a lemma which generalizes [7: Lemma 3.4].

Lemma 1.3. *Let P be a contractive projection on a JB^*-triple U and let $f \in P'(U')$. Then $P_0(f)P'P_1(f) = 0$.*

Proof. With $v = e(f)$ consider the map π: $P_1(f)U'' \to U'$ defined by $\pi(y) = D(v, y)f$. We shall show that π is a linear bijection of $P_1(f)U''$ onto a norm dense subspace S of $P_1(f)U'$. Then we shall show that $P_0(f)P'g = 0$ for all $g \in S$, completing the proof.

We show first that $S \subseteq P_1(f)\,U'$. Let $x \in U''$. By the Peirce rules [11: (1.7)], and the fact that $f = P_2(f)\,f$,

$$\langle D(v,y)\,f, x\rangle = \langle f, P_2(f)\,\{v\,y\,x\}\rangle = \langle f, \{v\,y\,P_1(f)\,x\}\rangle$$
$$= \langle P_1(f)\,D(v,y)\,f, x\rangle.$$

Thus $S \subseteq P_1(f)\,U'$.

Now let $z \in U''$ satisfy $\langle z, S\rangle = 0$, so that $\langle z_1, S\rangle = 0$ where $z_1 = P_1(f)\,z$. Then $f\{v\,y\,z_1\} = 0$ for all $y \in P_1(f)\,U''$ and in particular $f\{v\,z_1\,z_1\} = 0$. Since, by [11: Sect. 1], $\{v\,z_1\,z_1\}$ is positive in $P_2(f)\,U''$ and f is faithful there, $\{v\,z_1\,z_1\} = 0$ and so $z_1 = 0$. Thus $\langle z, P_1(f)\,U'\rangle = 0$, proving that S is norm dense in $P_1(f)\,U'$.

Next, let $g = D(v,y)\,f \in S$ for some $y \in P_1(f)\,U''$. We shall show that $P_0(f)\,P'g = 0$. Since $D(y,v) + D(v,y)$ is hermitian

$$|\langle f, (\exp i\,t(D(y,v) + D(v,y)))\,x\rangle| \leqq 1$$

for all $x \in U''$ with $\|x\| \leqq 1$ and all $t \in R$. Therefore

$$|f(x) + i\,t\,f(\{y\,v\,x\} + \{v\,y\,x\}) + O(t^2)| \leqq 1,$$

and since by the Peirce rules

$$f\{y\,v\,x\} = 0, \qquad |f(x) + i\,t\,f\{v\,y\,x\}| \leqq 1 + O(t^2).$$

Thus

$$\|f + i\,t\,g\| \leqq 1 + O(t^2).$$

Set $w_t = P'(f + i\,t\,g)$. Since $P''v = v + P_0(f)\,P''v$,

$$w_t(v) = \langle P'f, v\rangle + i\,t\langle P'g, v\rangle = 1.$$

Therefore $1 \leqq \|P_2(f)\,w_t\| \leqq \|w_t\| \leqq 1 + O(t^2)$.

Finally, since by Proposition 1.2 $P_0(f)\,P'f = 0$, we have

$$1 + t\,\|P_0(f)\,P'g\| \leqq \|P_2(f)\,w_t\| + \|P_0(f)\,P'(f + i\,t\,g)\|$$
$$= \|(P_2(f) + P_0(f))\,w_t\| \leqq \|w_t\| = 1 + O(t^2),$$

which forces $P_0(f)\,P'g = 0$. □

By using this lemma and Proposition 1.2, the following proposition can be proved exactly as [7: Prop. 3.5].

Proposition 1.4. *Let P be a contractive projection on a JB*-triple U and let $f \in P'(U')$. Then $P_0(f)\,P' = P_0(f)\,P'\,P_0(f) = P'\,P_0(f)\,P'$ and $P_0(f)\,P'$ and $P_1(f)\,P'$ are projections.*

Proof. Writing P_k for $P_k(f)$ we have

$$P_0\,P' = P_0\,P'(P_2 + P_1 + P_0) = P_0\,P'\,P_2 + P_0\,P'\,P_1 + P_0\,P'\,P_0 = P_0\,P'\,P_0$$

by Proposition 1.2 and Lemma 1.3. Also $P_0\,P'\,P_0\,P' = P_0\,P'\,P' = P_0\,P'$ is a projection.

Let $g \in U'$. Then $\|P_0 P' g\| = \|P_0 P' P_0 P' g\| \leqq \|P' P_0 P' g\| \leqq \|P_0 P' g\|$. By [11: Prop. 1] $P' P_0 P' g = P_0 P' P_0 P' g = P_0 P' g$. Finally,

$$P_1 P' = P_1 P'(P_2 + P_1 + P_0) P' = P_1(P' P_2) P' + P_1(P' P_0 P') + P_1 P' P_1 P' = P_1 P' P_1 P'. \quad \square$$

To establish our final commutativity formula we need a lemma which generalizes and considerably simplifies [7: Cor. 4.2].

Lemma 1.5. *Let v be a tripotent in U'' where U is a JB^*-triple and suppose $f \in P_2(v) U'$ and $g \in P_1(v) U'$. If $f \neq 0$, then $\|f+g\| > \|g\|$.*

Proof. Suppose that $\|f+g\| \leqq \|g\|$. Let $u = e(g)$. Then by [11: Prop. 3], $u \in P_1(v) U''$ and $[P_k(v), P_j(u)] = 0$ for all j, k. Thus $\langle f+g, u\rangle = \langle g, u\rangle = \|g\| \geqq \|f+g\|$. By [11: Prop. 1], $f+g$ and g belong to $P_2(u) U'$, implying $f \in P_2(u) U'$. Let $x \in U''$ be such that $f(x) \neq 0$ and set $y = P_2(v) P_2(u) x$. Denote by $\#$ the involution on the JBW^*-algebra $P_2(u) U''$. Then by Peirce rules with respect to v, $y^{\#} = \{u\, y\, u\} \in P_0(v) U''$. Since $g(y) = g(P_1(v) y) = 0$ and $f+g$ is positive, hence self-adjoint on $P_2(u) U''$, we have

$$\overline{f(x)} = \overline{f(y)} = \overline{\langle f+g, y\rangle} = \langle f+g, y^{\#}\rangle = 0, \quad \text{a contradiction.} \quad \square$$

The proof of our final commutativity formula is the same as [7: Prop. 4.3] with Lemma 1.5 in place of [7: Cor. 4.2].

Proposition 1.6. *Let P be a contractive projection on a JB^*-triple U and let $f \in P'(U')$. Then $P_1(f) P' = P' P_1(f) P'$, $P_2(f) P' = P' P_2(f) P'$, and $P_2(f) P'$ is a projection.*

Proof. We have

$$P' P_1 P' = (P_2 + P_1 + P_0) P' P_1 P' = P_2 P' P_1 P' + P_1 P' P_1 P' + P_0 P' P_1 P' = P_2 P' P_1 P' + P_1 P'.$$

Therefore for arbitrary $g \in U'$,

$$\|P_2(P' P_1 P' g) + P_1(P' g)\| = \|P' P_1 P' g\| \leqq \|P_1(P' g)\|,$$

and by Lemma 1.5, $P_2 P' P_1 P' g = 0$, so $P' P_1 P' = P_1 P'$.

Furthermore,

$$\begin{aligned} P_2 P' &= (1 - P_1 - P_0) P' = P' - P_1 P' - P_0 P' \\ &= P' - P' P_1 P' - P' P_0 P' = P'(1 - P_1 - P_0) P' = P' P_2 P'. \end{aligned}$$

Finally $P_2(P' P_2 P') = P_2 P_2 P' = P_2 P'$. $\square$

We summarize the results of this section in the following theorem.

Theorem 1. *Let P be a contractive projection on a JB^*-triple U and let $f \in P'(U')$. With $P_k = P_k(f)$, $k = 0\,1, 2$ we have*

(a) *On U', $P' P_2 = P_2 P' P_2$, $P_0 P' = P_0 P' P_0 = P' P_0 P'$,*

$$P_1 P' = P' P_1 P', \qquad P_2 P' = P' P_2 P';$$

(b) *On U'', $P_2 P'' = P_2 P'' P_2$, $P'' P_0 = P_0 P'' P_0 = P'' P_0 P''$,*

$$P'' P_1 = P'' P_1 P'', \qquad P'' P_2 = P'' P_2 P''.$$

2. Conditional Expectation Property of a Contractive Projection

In this section we shall use the following two results of Kaup [15] concerning a contractive projection P on a JB^*-triple U.

(2.1) $P\{PabPc\}=P\{PaPbPc\}$, for $a, b, c \in U$;

(2.2) $P(U)$ is a JB^*-triple in the triple product $\{xyz\}_{P(U)}:=P\{xyz\}$, for $x, y, z \in P(U)$.

As noted earlier in (0.2) if $f \in P'(U')$, then $P''v=v+b$ where $v=e(f)$ and b is orthogonal to v. The next lemma, which will be needed in Lemma 2.6, shows that in fact b is orthogonal to $e(g)$ where g is an *arbitrary* element of $P'(U')$.

Lemma 2.1. *Let P be a contractive projection on a JB^*-triple U, let $f \in P'(U')$ and let $v=e(f)$ and $b=P''v-v$. Then for any $g \in P'(U')$ we have $b \in P_0(g)U''$.*

Proof. Let $u=e(g)$. Set $c=P''u-u$. Then $b \perp v$ and $c \perp u$. We calculate $\lambda: =g\{P''v, P''v, P''u\}$ in two ways. First, by (2.1) applied to P'',

$$\lambda=g(P''\{P''v, P''v, P''u\})=g(P''\{P''v, v, P''u\})$$
$$=g(\{v+b, v, P''u\})=g(\{vvP''u\}).$$

Second, $\lambda=g(\{v+b, v+b, P''u\})=g\{vvP''u\}+g\{bbP''u\}$. Therefore $g\{bbP''u\}=0$, i.e., $g\{bbu\}+g\{bbc\}=0$.

Let $b=b_2+b_1+b_0$ be the Peirce decomposition of b with respect to u. Then by the "Peirce rules for multiplication" and the fact that $g=P_2(u)g$, we have $g\{bbc\}=0$ (so that $g(\{bbu\})=0$), and $g\{bbu\}=g\{b_2b_2u\}+g\{b_1b_1u\}$. Since $\{b_2b_2u\}$ and $\{b_1b_1u\}$ belong to $(P_2(u)U'')^+$ and g is positive we have $g\{b_1b_1u\}=g\{b_2b_2u\}=0$. Moreover g is faithful on $P_2(u)U''$, so that $\{b_1b_1u\}=\{b_2b_2u\}=0$. Finally by [11: p. 73] we have $b_1=b_2=0$. □

It follows from (2.2) that $M:=P''(U'')$ is a JBW^*-triple with predual $M_*=P'(U')$. The next proposition gives the connection between the polar decompositions of an element $f \in P'(U')$ with respect to M and U''. Together with Lemma 2.1 it will clarify the relationship between orthogonality in M and in U''.

Proposition 2.2. *Let P be a contractive projection on a JB^*-triple U, let $f \in P'(U')$, and let $v=e(f) \in U''$. Then $\tilde{v}:=P''v$ is the tripotent occurring in the polar decomposition of f with respect to the JBW^*-triple $M:=P''(U'')$.*

Proof. We show first that $\tilde{v}$ is a tripotent in $P''(U'')$. By (0.2), we have $\tilde{v}=v+b$ where $b \in P_0(v)U''$. By (2.1) applied to P'',

$$\{\tilde{v}\tilde{v}\tilde{v}\}_M=P''\{\tilde{v}\tilde{v}\tilde{v}\}=P''\{\tilde{v}, v, \tilde{v}\}=P''\{v+b, v, v+b\}=P''\{vvv\}=P''v=\tilde{v}.$$

Therefore $\tilde{v}$ is a tripotent.

We show next that if u is a tripotent in $P''(U'')$ with $f(u)=\|f\|$, then $u \geqq \tilde{v}$ in M. Since $f(u)=\|f\|$, we have $u=v+c$ with $c \in P_0(v)U''$ (by (0.3)). Thus $u=P''u=\tilde{v}+P''c$ and $\tilde{c}:=P''c \in P_0(v)U''$ by Theorem 1. It remains to show that $\tilde{v}$ is orthogonal to $\tilde{c}$ in M. By (2.1), we have

$$\{\tilde{v}\,\tilde{v}\,\tilde{c}\}_M = P''\{\tilde{v}\,\tilde{v}\,\tilde{c}\} = P''\{\tilde{v}\,v\,\tilde{c}\} = P''\{v+b, v, \tilde{c}\} = P''\{v, v, \tilde{c}\} = P''0 = 0. \quad \square$$

To obtain the connection between orthogonality in M and U'' we need the following

Lemma 2.3. *Let f and g be normal functionals on a JBW^*-triple U. Then $f \perp g$ if and only if $\|f \pm g\| = \|f\| + \|g\|$.*

Proof. We may assume $\|f\| = \|g\| = 1$. Let $u = e(g)$, $v = e(f)$.

Suppose first that $f \perp g$. Then $e_\pm := u \pm v$ has norm one and $\langle f \pm g, e_\pm \rangle = 2$. Therefore $2 \leqq \|f \pm g\| \leqq \|f\| + \|g\| = 2$.

Suppose now that $\|f \pm g\| = \|f\| + \|g\|$, and set $w := e(f+g)$. Then

$$2 = \langle f+g, w \rangle = \langle f, w \rangle + \langle g, w \rangle \leqq |\langle f, w \rangle| + |\langle g, w \rangle| \leqq \|f\| + \|g\| = 2.$$

Thus $\langle f, w \rangle = \langle g, w \rangle = 1$ and so $f, g \in P_2(w)\,U_*$ by [11: Prop. 1]. By the Jordan decomposition in JBW-algebras [13: Appendix] f and g are orthogonal states on the JBW^*-algebra $U_2(w)$, and hence $f \perp g$. $\square$

Corollary 2.4. *For a contractive projection P on a JB^*-triple U and $f, g \in P'(U')$, let $v = e(f)$, $u = e(g)$, $\tilde{v} = P''v$, $\tilde{u} = P''u$. Then $u \perp v$ in U'' if and only if $\tilde{u} \perp \tilde{v}$ in $M = P''(U'')$.*

Proof. For any $h \in P'(U') \subseteq U'$, $\|h\|_{M_*} = \|h\|_{U'}$. By Proposition 2.2 and Lemma 2.3,

$$\begin{aligned} u \perp v \text{ in } U'' &\Leftrightarrow \|f \pm g\|_{U'} = \|f\|_{U'} + \|g\|_{U'} \Leftrightarrow \|f \pm g\|_{M_*} \\ &= \|f\|_{M_*} + \|g\|_{M_*} \Leftrightarrow \tilde{u} \perp \tilde{v} \text{ in } M. \quad \square \end{aligned}$$

To extend this corollary to arbitrary tripotents we need:

Lemma 2.5. *Let v be a tripotent in a JBW^*-triple U. Then there is a family of mutually orthogonal functionals $(f_\alpha) \subseteq U_*$ with $v = \sum_\alpha e(f_\alpha)$ (w^*-convergence).*

Proof. Let $A = U_2(v)$ which is a JBW^*-algebra with unit v. Since the normal states of A are separating [13], $v = \sup\{e_\phi \colon \phi \in A_*^+\}$ where e_ϕ is the support projection of ϕ in A. By Zorn there is an orthogonal family $\{e_{\phi_\alpha}\}$ such that $v = \sum e_{\phi_\alpha}$ in A. Then (e_{ϕ_α}) are pairwise orthogonal tripotents in U and $e_{\phi_\alpha} = e(f_\alpha)$ where $f_\alpha = \phi_\alpha P_2(v)$. $\square$

The following lemma gives a correspondence between tripotents in M and in U'' which is needed in order to describe the fine structure of the range of a contractive projection. We first establish some notation.

If P is a contractive projection on a JB^*-triple U, we let

$$\mathscr{C} := \text{the } w^*\text{-closure of span}\{e(f) \colon f \in P'(U')\} \subseteq U'',$$

$$\mathscr{O} := \bigcap_{g \in P'(U')} P_0(g)\,U''.$$

It is obvious that $\mathscr{O}$ is a JB^*-subtriple of U'' and that $\mathscr{C} \perp \mathscr{O}$. Therefore the sum $\mathscr{C} + \mathscr{O}$ is direct and the projection Q of $\mathscr{C} + \mathscr{O}$ onto $\mathscr{C}$ is contractive.

Lemma 2.6. *Let P be a contractive projection on a JB^*-triple U and let w be a tripotent of the JBW^*-triple $M=P''(U'')$. Then Qw is a tripotent of U''. Moreover, if w_1 and w_2 and orthogonal tripotents of M, then Qw_1 and Qw_2 are orthogonal in U''.*

Proof. Let us apply Lemma 2.5 to a tripotent w of the JBW^*-triple $M:=P''(U'')$. We obtain orthogonal elements $(f_\alpha)\subseteq P'(U')$ and by Proposition 2.2 $w=\sum_\alpha (e(f_\alpha)+b_\alpha)$ (w^*-convergence in M) where, by Lemma 2.1, $b_\alpha\in\mathcal{O}$. By Corollary 2.4 the $e(f_\alpha)$ are orthogonal in U'' so $w=\sum_\alpha e(f_\alpha)+\sum_\alpha b_\alpha$, $\sum_\alpha e(f_\alpha)$ w^*-converges to a tripotent $Qw\in\mathcal{C}$ of U'' and $b:=\sum_\alpha b_\alpha\in\mathcal{O}$ exists in U'' as a w^*-limit. The second statement follows from Corollary 2.4 and the formula $Qw=\sum_\alpha e(f_\alpha)$. □

By Lemma 2.6 each tripotent of $M=P''(U'')$ lies in $\mathcal{C}+\mathcal{O}$. By the spectral theorem [11: Rk. 1.9] $M\subset\mathcal{C}+\mathcal{O}$, and Q is a homomorphism of M into U'', i.e., $Q(\{a\,b\,c\}_M)=\{Qa\,Qb\,Qc\}$. On the other hand, for $x\in P(U)\subseteq M$, and $f\in P'(U')$, $f(x)=f(Qx)$, and

$$\|x\|=\sup\{|\langle f, Qx\rangle|:\ f\in P'(U'),\ \|f\|\leqq 1\}\leqq\|Qx\|\leqq\|x\|.$$

Thus, by restricting Q to $P(U)$, we have:

Theorem 2. *Let P be a contractive projection on a JB^*-triple U. Then the JB^*-triple $P(U)$ is isometrically isomorphic to a closed subtriple of U''.*

Since it is elementary that the second dual of a J^*-algebra is a J^*-algebra, we obtain as a consequence of (2.2) and Theorem 2, the main result of [10]:

Corollary 2.7. *Let P be a contractive projection on a J^*-algebra. Then the range of P is a JB^*-triple which is isometrically isomorphic to a J^*-algebra.*

Our next result is a conditional expectation formula analogous to (2.1). Recall that the triple product $\{a\,b\,c\}$ is symmetric and linear in a and c and conjugate linear in b. The formula (2.1) was proved in [15] using holomorphic methods. Holomorphic methods are unavailable for the proof of Theorem 3 because of the conjugate linearity in b. Both formulas had been proved in [9] for J^*-algebras.

Theorem 3. *Let P be a contractive projection on a JB^*-triple U. Then*

$$P\{Pa\,Pb\,c\}=P\{Pa\,Pb\,Pc\}\quad \text{for } a,b,c\in U. \tag{2.3}$$

Proof. There is no loss of generality in assuming that $a=b$. By approximating Pa by finite linear combinations of orthogonal tripotents of $M=P''(U'')$, it suffices to prove that

$$P''\{w_1\,w_2\,x\}=P''\{w_1\,w_2\,P''x\} \tag{2.4}$$

whenever w_1, w_2 are tripotents of M and $x\in U''$. We only need to consider two cases, namely $w_1=w_2$ and $w_1\perp w_2$.

Note first that by the Peirce rules, for any $a_1, a_2 \in \mathcal{O}$,

$$z \in U'' \quad \text{and} \quad g \in P'(U'), \quad \text{we have } \langle g, \{a_1 a_2 z\}\rangle = 0.$$

Therefore $P''\{a_1 a_2 z\} = 0$.

If $w_1 \perp w_2$, then writing $w_i = Q w_i + b_i = v_i + b_i$ as in Lemma 2.6 implies

$$\{w_1 w_2 x\} = \{v_1 v_2 x\} + \{b_1 b_2 x\} = \{b_1 b_2 x\}.$$

Therefore $P''\{w_1 w_2 x\} = 0 = P''\{w_1 w_2 P'' x\}$.

If $w_1 = w_2 = w$ say, write $w = Q w + b = v + b$ as in Lemma 2.6. Then $P''\{w w x\} = P''\{v v x\} + P''\{b b x\} = P'' D(v, v) x$ and since $D(v, v) = P_2(v) + \frac{1}{2} P_1(v)$, Theorem 1 implies

$$P''\{w w x\} = P''(P_2(v) + \tfrac{1}{2} P_1(v)) x = P''(P_2(v) + \tfrac{1}{2} P_1(v)) P'' x = P''\{w w P'' x\}. \quad \square$$

3. Structure of a Bicontractive Projection

In the previous section, we showed that the JB^*-triple $P(U)$ is isomorphic to a subtriple of U''. Here of course P is any contractive projection on a JB^*-triple U. In this section we make the assumption that P is bicontractive, i.e., $I - P$ is also contractive.

It turns out that in this case, the isomorphism Q of $P(U)$ into U'' is the identity, which results in:

Proposition 3.1. *Let P be a bicontractive projection on a JB^*-triple U. Then $P(U)$ is a JB^*-subtriple of U.*

Proof. Since $M := P''(U'')$ is a JBW^*-triple, it suffices to prove that $Q w = w$ for each tripotent w in M. By Lemma 2.5 and Lemma 2.6 it suffices to prove that $P'' v = v$ whenever $v = e(f)$ for some $f \in P'(U')$.

Let $b = P'' v - v$. Then, by (2.1),

$$P''\{b b b\} = P''\{v + b, b, v + b\} = P''\{P'' v, b, P'' v\} = P''\{P'' v, P'' b, P'' v\} = 0.$$

By induction $P''(B) = 0$ where B is the JB^*-triple generated by b. It follows that $I - P''$ restricts to a bicontractive projection on $C v \oplus B$ with range B. Then, for any $z \in B$ with $\|z\| \leqq 1$, we have $\|v + z\| \leqq 1$ so that $\|(I - P'')(v + z)\| \leqq 1$, i.e., $\|-b + z\| \leqq 1$. This forces $b = 0$. $\square$

By Proposition 3.1, if P is bicontractive, the conditional expectation formulas (2.1) and (2.3) take on a neater form:

$$P\{a b x\} = \{a b P x\} \quad \text{and} \quad P\{a x b\} = \{a P x b\}$$

for $a, b \in P(U)$ and $x \in U$.

Finally, let P be a bicontractive projection on a JB^*-triple U, and set $\theta = 2\mathrm{P} - \mathrm{I}$. By the argument in [9: p. 355], the two conditional expectation formulae (2.1), (2.3) and Proposition 3.1 imply that θ is a homomorphism of U.

Since a homomorphism is always contractive (as follows from $\|\{zzz\}\| = \|z\|^3$) we obtain

Theorem 4. *Let P be a bicontractive projection on a JB*-triple U. Then there is a surjective isometry θ on U of order 2 such that $P = \frac{1}{2}(I + \theta)$. Thus all JB*-triples, their duals, and all pre-duals of JBW*-triples belong to the class $\mathscr{S}$.*

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