2-LOCAL TRIPLE DERIVATIONS ON VON NEUMANN ALGEBRAS

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ABSTRACT. We prove that every (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra M is a triple derivation, equivalently, the set $\mathrm{Der}_t(M)$, of all triple derivations on M, is algebraically 2-reflexive in the set $\mathcal{M}(M) = M^M$ of all mappings from M into M.

1. INTRODUCTION

Let X and Y be Banach spaces. According to the terminology employed in the literature (see, for example, [4]), a subset $\mathcal{D}$ of the Banach space $B(X,Y)$, of all bounded linear operators from X into Y, is called *algebraically reflexive* in $B(X,Y)$ when it satisfies the property:

$$T \in B(X,Y) \text{ with } T(x) \in \mathcal{D}(x),\ \forall x \in X \Rightarrow T \in \mathcal{D}. \tag{1.1}$$

Algebraic reflexivity of $\mathcal{D}$ in the space $L(X,Y)$, of all linear mappings from X into Y, a stronger version of the above property not requiring continuity of T, is defined by:

$$T \in L(X,Y) \text{ with } T(x) \in \mathcal{D}(x),\ \forall x \in X \Rightarrow T \in \mathcal{D}. \tag{1.2}$$

In 1990, Kadison proved that (1.1) holds if $\mathcal{D}$ is the set $\mathrm{Der}(M,X)$ of all (associative) derivations on a von Neumann algebra M into a dual M-bimodule X [18]. Johnson extended Kadison's result by establishing that the set $\mathcal{D} = \mathrm{Der}(A,X)$, of all (associative) derivations from a C*-algebra A into a Banach A-bimodule X satisfies (1.2) [17].

Algebraic reflexivity of the set of local triple derivations on a C*-algebra and on a JB*-triple have been studied in [24, 9, 12] and [14]. More precisely, Mackey proves in [24] that the set $\mathcal{D} = \mathrm{Der}_t(M)$, of all triple derivations on a JBW*-triple M satisfies (1.1). The result has been supplemented in [12], where Burgos, Fernández-Polo and the third author of this note prove that

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for each JB*-triple E, the set $\mathcal{D} = \mathrm{Der}_t(E)$ of all triple derivations on E satisfies (1.2).

Hereafter, *algebraic reflexivity* will refer to the stronger version (1.2) which does not assume the continuity of T.

In [6], Brešar and Šemrl proved that the set of all (algebra) automorphisms of $B(H)$ is algebraically reflexive whenever H is a separable, infinite-dimensional Hilbert space. Given a Banach space X. A linear mapping $T : X \to X$ satisfying the hypothesis at (1.2) for $\mathcal{D} = \mathrm{Aut}(X)$, the set of automorphisms on X, is called a *local automorphism.* Larson and Sourour showed in [22] that for every infinite dimensional Banach space X, every surjective local automorphism T on the Banach algebra $B(X)$, of all bounded linear operators on X, is an automorphism.

Motivated by the results of Šemrl in [31], references witness a growing interest in a subtle version of algebraic reflexivity called *algebraic 2-reflexivity* (cf. [1, 2, 10, 11, 21, 23, 25, 26] and [29]). A subset $\mathcal{D}$ of the set $\mathcal{M}(X, Y) = Y^X$, of all mappings from X into Y, is called *algebraically 2-reflexive* when the following property holds: for each mapping T in $\mathcal{M}(X, Y)$ such that for each $a, b \in X$, there exists $S = S_{a,b} \in \mathcal{D}$ (depending on a and b), with $T(a) = S_{a,b}(a)$ and $T(b) = S_{a,b}(b)$, then T lies in $\mathcal{D}$. A mapping $T : X \to Y$ satisfying that for each $a, b \in X$, there exists $S = S_{a,b} \in \mathcal{D}$ (depending on a and b), with $T(a) = S_{a,b}(a)$ and $T(b) = S_{a,b}(b)$ will be called a 2-local $\mathcal{D}$-mapping. If we assume that every mapping $s \in \mathcal{D}$ is r- homogeneous (that is, $S(ta) = t^r S(a)$ for every $t \in \mathbb{R}$ or $\mathbb{C}$) with $0 < r$, then every 2-local $\mathcal{D}$-mapping $T : X \to Y$ is r-homogeneous. Indeed, for each $a \in X$, $t \in \mathbb{C}$ take $S_{a,ta} \in \mathcal{D}$ satisfying $T(ta) = S_{a,ta}(ta) = t^r S_{a,ta}(a) = t^r T(a)$.

Šemrl establishes in [31] that for every infinite-dimensional separable Hilbert space H, the sets $\mathrm{Aut}(B(H))$ and $\mathrm{Der}(B(H))$, of all (algebra) automorphisms and associative derivations on $B(H)$, respectively, are algebraically 2-reflexive in $\mathcal{M}(B(H)) = \mathcal{M}(B(H), B(H))$. Ayupov and the first author of this note proved in [1] that the same statement remains true for general Hilbert spaces (see [20] for the finite dimensional case). Actually, the set $\mathrm{Hom}(A)$, of all homomorphisms on a general C*-algebra A, is algebraically 2-reflexive in the Banach algebra $B(A)$, of all bounded linear operators on A, and the set *-$\mathrm{Hom}(A)$, of all *-homomorphisms on A, is algebraically 2-reflexive in the space $L(A)$, of all linear operators on A (cf. [27]).

In recent contributions, Burgos, Fernández-Polo and the third author of this note prove that the set $\mathrm{Hom}(M)$ (respectively, $\mathrm{Hom}_t(M)$), of all homomorphisms (respectively, triple homomorphisms) on a von Neumann algebra (respectively, on a JBW*-triple) M, is an algebraically 2-reflexive subset of $\mathcal{M}(M)$ (cf. [10], [11], respectively), while Ayupov and the first author of this note establish that set $\mathrm{Der}(M)$ of all derivations on M is algebraically 2-reflexive in $\mathcal{M}(M)$ (see [2]).

In this paper, we consider the set $\mathrm{Der}_t(A)$ of all triple derivations on a C*-algebra A. We recall that every C*-algebra A can be equipped with a ternary product of the form

$$\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a).$$

When A is equipped with this product it becomes a JB*-triple in the sense of [19]. A linear mapping $\delta : A \to A$ is said to be a *triple derivation* when it satisfies the (triple) Leibnitz rule:

$$\delta\{a,b,c\} = \{\delta(a),b,c\} + \{a,\delta(b),c\} + \{a,b,\delta(c)\}.$$

It is known that every triple derivation is automatically continuous (cf. [3]). We refer to [3, 15] and [28] for the basic references on triple derivations. According to the standard notation, 2-local $\mathrm{Der}_t(A)$-mappings from A into A are called *2-local triple derivations*.

The goal of this note is to explore the algebraic 2-reflexivity of $\mathrm{Der}_t(A)$ in $\mathcal{M}(A)$. Our main result proves that every (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra M is a triple derivation (hence linear and continuous) (see Theorem 2.14), equivalently, $\mathrm{Der}_t(M)$ is algebraically 2-reflexive in $\mathcal{M}(M)$.

2. 2-local triple derivations on von Neumann algebras

We start by recalling some generalities on triple derivations. Let A be a C*-algebra. For each $b \in A$, we shall denote by M_b the Jordan multiplication mapping by the element b, that is $M_b(x) = b \circ x = \frac{1}{2}(bx + xb)$. Following standard notation, given elements a, b in A, we denote by $L(a,b)$ the operator on A defined by $L(a,b)(x) = \{a,b,x\} = \frac{1}{2}(ab^*x + xb^*a)$. It is known that he mapping $\delta(a,b) : A \to A$, given by

$$\delta(a,b)(x) = L(a,b)(x) - L(b,a)(x),$$

is a triple derivation on A (cf. [3, 15]), called an inner triple derivation.

Let $\delta : A \to A$ be a triple derivation on a unital C*-algebra. By [15, Lemmas 1 and 2], $\delta(\mathbf{1})^* = -\delta(\mathbf{1})$, and $M_{\delta(\mathbf{1})} = \delta(\frac{1}{2}\delta(\mathbf{1}), \mathbf{1})$ is an inner triple derivation on A and the difference $D = \delta - \delta(\frac{1}{2}\delta(\mathbf{1}), \mathbf{1})$ is a Jordan *-derivation on A, more concretely,

$$D(x \circ y) = D(x) \circ y + x \circ D(y), \text{ and } D(x^*) = D(x)^*,$$

for every $x, y \in A$. By [3, Corollary 2.2], δ (and hence D) is a continuous operator. A widely known result, due to B.E. Johnson, states that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation (cf. [16]). Therefore, D is an associative *-derivation in the usual sense. When $A = M$ is a von Neumann algebra, we can guarantee that D is an inner derivation, that is there exists $\widetilde{a} \in A$ satisfying $D(x) = [\widetilde{a}, x] = \widetilde{a}x - x\widetilde{a}$, for every $x \in A$ (cf. [30, Theorem 4.1.6]). Further, from the condition $D(x^*) = D(x)^*$, for every $x \in A$, we deduce that $(\widetilde{a}^* + \widetilde{a})x = x(\widetilde{a}^* + \widetilde{a})$. Thus, taking $a = \frac{1}{2}(\widetilde{a} - \widetilde{a}^*)$, it follows that $[a, x] = [\widetilde{a}, x]$, for every $x \in M$. We have therefore shown that for every triple derivation δ on a von Neumann algebra M, there exist skew-hermitian elements $a, b \in M$ satisfying

$$\delta(x) = [a, x] + b \circ x,$$

for every $x \in M$.

Our first lemma is a direct consequence of the above arguments (see [15, Lemmas 1 and 2]).

Lemma 2.1. *Let $T : A \to A$ be a (not necessarily linear nor continuous) 2-local triple derivation on a unital C*-algebra. Then*
(a) $T(\mathbf{1})^* = -T(\mathbf{1})$;
(b) $M_{T(\mathbf{1})} = \delta\left(\frac{1}{2}T(\mathbf{1}), \mathbf{1}\right)$ is an inner triple derivation on A;
(c) $\widehat{T} = T - \delta\left(\frac{1}{2}T(\mathbf{1}), \mathbf{1}\right)$ is a 2-local triple derivation on A with $\widehat{T}(\mathbf{1}) = 0$.
□

In what follows, we denote by A_{sa} the hermitian elements of the C*-algebra A.

Lemma 2.2. *Let $T : A \to A$ be a (not necessarily linear nor continuous) 2-local triple derivation on a unital C*-algebra satisfying $T(\mathbf{1}) = 0$. Then $T(x) = T(x)^*$ for all $x \in A_{sa}$.*

Proof. Let $x \in A_{sa}$. By assumptions,

$$T(x)^* = \{\mathbf{1}, T(x), \mathbf{1}\} = \{\mathbf{1}, \delta_{x,\mathbf{1}}(x), \mathbf{1}\} = \delta_{x,\mathbf{1}}\{\mathbf{1}, x, \mathbf{1}\} - 2\{\delta_{x,\mathbf{1}}(\mathbf{1}), x, \mathbf{1}\}$$
$$= \delta_{x,\mathbf{1}}(x^*) - 2\{T(\mathbf{1}), x, \mathbf{1}\} = \delta_{x,\mathbf{1}}(x) = T(x).$$

The proof is complete. □

Lemma 2.3. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra satisfying $T(\mathbf{1}) = 0$. Then for every $x, y \in M_{sa}$ there exists a skew-hermitian element $a_{x,y} \in M$ such that*

$$T(x) = [a_{x,y}, x], \text{ and, } T(y) = [a_{x,y}, y].$$

Proof. For every $x, y \in M_{sa}$ we can find skew-hermitian elements $a_{x,y}, b_{x,y} \in M$ such that

$$T(x) = [a_{x,y}, x] + b_{x,y} \circ x, \text{ and, } T(y) = [a_{x,y}, y] + b_{x,y} \circ y.$$

Taking into account that $T(x) = T(x)^*$ (see Lemma 2.2) we obtain

$$[a_{x,y}, x] + b_{x,y} \circ x = T(x) = T(x)^* = [a_{x,y}, x]^* + (b_{x,y} \circ x)^*$$
$$= [x, a_{x,y}^*] + x \circ b_{x,y}^* = [x, -a_{x,y}] - x \circ b_{x,y} = [a_{x,y}, x] - b_{x,y} \circ x,$$

i.e. $b_{x,y} \circ x = 0$, and similarly $b_{x,y} \circ y = 0$. Therefore $T(x) = [a_{x,y}, x]$, $T(y) = [a_{x,y}, y]$, and the proof is complete. □

We state now an observation, which plays an useful role in our study.

Lemma 2.4. *Let a and b be skew-hermitian elements in a C^*-algebra A. Suppose $x \in A$ is self-adjoint with $ax - xa + bx + xb = 0$. Then $ax = xa$, and $bx = -xb$.*

Proof. Since $0 = ax - xa + bx + xb$. Passing to the adjoint, we obtain $ax - xa - (bx + xb) = 0$. Conclude the proof by adding and subtracting these two equalities. The proof is complete. □

Let M be a von Neumann algebra. If $x \in M_{sa}$, we denote by $s(x)$ the support projection of x – that is, the projection onto $(\ker(x))^{\perp} = \overline{\operatorname{ran}(x)}$. We say that x *has full support* if $s(x) = 1$ (equivalently, $\ker(x) = \{0\}$).

Lemma 2.5. *Let M be a von Neumann algebra. Suppose $u \in M_+$ has full support, $c \in M$ is self-adjoint, and $\sigma(c^2u) \cap (0, \infty) = \emptyset$. Then $c = 0$. Consequently, if u and c are as above, and $uc + cu = 0$ (or $c^2u = -cuc \leq 0$), then $c = 0$.*

Proof. For the fist statement of the lemma, suppose $\sigma(c^2u) \cap (0, \infty) = \emptyset$. Note that

$$(-\infty, 0] \supseteq \sigma(c^2u) \cup \{0\} = \sigma(c \cdot cu) \supseteq \sigma(cuc).$$

However, cuc is positive, hence $\sigma(cuc) \subset [0, \|cuc\|]$, with $\max_{\lambda \in \sigma(cuc)} = \|cuc\|$. Thus, $cu^{1/2}u^{1/2}c = cuc = 0$, which means that $cu^{1/2} = u^{1/2}c = 0$ and hence $s(c) \subset 1 - (u^{1/2}) = 1 - s(u) = 0$, which leads to $c = 0$.

To prove the second part, we have $c^2u = -cuc \leq 0$, hence in particular, $\sigma(c^2u) \subset (-\infty, 0]$. The proof is complete. □

In [2, Lemma 2.2], Ayupov and the first author of this note prove that for every (not necessarily linear nor continuous) 2-local derivation on a von Neumann algebra $\Delta : M \to M$, and every self-adjoint element $z \in M$, there exists $a \in M$ satisfying

$$\Delta(x) = [a, x],$$

for every $x \in \mathcal{W}^*(z)$, where $\mathcal{W}^*(z) = \{z\}''$ denotes the abelian von Neumann subalgebra of M generated by the element z, and the unit element and $\{z\}''$ denotes the bicommutant of the set $\{z\}$. We prove next a ternary version of this result.

Lemma 2.6. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. Let $z \in M$ be a self-adjoint element and let $\mathcal{W}^*(z) = \{z\}''$ be the abelian von Neumann subalgebra of M generated by the element z and the unit element. Then there exist skew-hermitian elements $a_z, b_z \in M$, depending on z, such that*

$$T(x) = [a_z, x] + b_z \circ x = a_z x - x a_z + \frac{1}{2}(b_z x + x b_z)$$

for all $x \in \mathcal{W}^*(z)$. In particular, T is linear on $\mathcal{W}^*(z)$.

Proof. We can assume that $z \neq 0$. Note that the abelian von Neumann subalgebras generated by $\mathbf{1}$ and z and by $\mathbf{1}$ and $\mathbf{1} + \frac{z}{2\|z\|}$ coincide. So, replacing z with $\mathbf{1} + \frac{z}{2\|z\|}$ we can assume that z is an invertible positive element.

By definition, there exist skew-hermitian elements $a_z, b_z \in M$ (depending on z) such that

$$T(z) = [a_z, z] + b_z \circ z.$$

Define a mapping $T_0 : M \to M$ given by $T_0(x) = T(x) - ([a_z, z] + b_z \circ z)$, $x \in M$. Clearly, T_0 is a 2-local triple derivation on M. We shall show that $T_0 \equiv 0$ on $\mathcal{W}^*(z)$. Let $x \in \mathcal{W}^*(z)$ be an arbitrary element. By assumptions, there exist skew-hermitian elements $c_{z,x}, d_{z,x} \in M$ such that

$$T_0(z) = [c_{z,x}, z] + d_{z,x} \circ z, \text{ and, } T_0(x) = [c_{z,x}, x] + d_{z,x} \circ x.$$

Since

$$0 = T_0(z) = [c_{z,x}, z] + d_{z,x} \circ z,$$

we get

$$[c_{z,x}, z] + d_{z,x} \circ z = 0.$$

Taking into account that z is a hermitian element and Lemma 2.4 we get $c_{z,x}z = zc_{z,x}$ and $d_{z,x}z = -zd_{z,x}$.

Since z has a full support, and $d_{z,x}^2 z = -d_{z,x} z d_{z,x}$, Lemma 2.5 implies that $d_{z,x} = 0$. Further

$$c_{z,x} \in \{z\}' = \{z\}''' = \mathcal{W}^*(z)',$$

i.e. $c_{z,x}$ commutes with any element in $\mathcal{W}^*(z)$. Therefore

$$T_0(x) = [c_{z,x}, x] + d_{z,x} \circ x = 0$$

for all $x \in \mathcal{W}^*(z)$. The proof is complete. □

2.1. **Complete additivity of 2-local derivations and 2-local triple derivations on von Neumann algebras.**

Let $\mathcal{P}(M)$ denote the lattice of projections in a von Neumann algebra M. Let X be a Banach space. A mapping $\mu : \mathcal{P}(M) \to X$ is said to be *finitely additive* when

$$\mu\left(\sum_{i=1}^{n} p_i\right) = \sum_{i=1}^{n} \mu(p_i),$$

for every family $p_1, \ldots, p_n$ of mutually orthogonal projections in M. A mapping $\mu : \mathcal{P}(M) \to X$ is said to be *bounded* when the set

$$\{\|\mu(p)\| : p \in \mathcal{P}(M)\}$$

is bounded.

The celebrated Bunce-Wright-Mackey-Gleason theorem ([7, 8]) states that if M has no summand of type I_2, then every bounded finitely additive mapping $\mu : \mathcal{P}(M) \to X$ extends to a bounded linear operator from M to X.

According to the terminology employed in [32] and [13], a completely additive mapping $\mu : \mathcal{P}(M) \to \mathbb{C}$ is called a *charge*. The Dorofeev–Sherstnev theorem ([32, Theorem 29.5] or [13, Theorem 2]) states that any charge on a von Neumann algebra with no summands of type I_n is bounded.

We shall use the Dorofeev-Shertsnev theorem in Corollary 2.8 in order to be able to apply the Bunce-Wright-Mackey-Gleason theorem in Proposition 2.9. To this end, we need Proposition 2.7, which is implicitly applied in [2, proof of Lemma 2.3] for 2-local associative derivations. A proof is included here for completeness reasons.

First, we recall some facts about the strong*-topology. For each normal positive functional φ in the predual of a von Neumann algebra M, the mapping

$$x \mapsto \|x\|_\varphi = \left(\varphi(\frac{xx^* + x^*x}{2})\right)^{\frac{1}{2}} \quad (x \in M)$$

defines a prehilbertian seminorm on M. The *strong* topology* of M is the locally convex topology on M defined by all the seminorms $\|.\|_\varphi$, where φ runs in the set of all positive functionals in M_* (cf. [30, Definition 1.8.7]). It is known that the strong* topology of M is compatible with the duality (M, M_*), that is a functional $\psi : M \to \mathbb{C}$ is strong* continuous if and only if it is weak* continuous (see [30, Corollary 1.8.10]). We also recall that the product of every von Neumann algebra is jointly strong* continuous on bounded sets (see [30, Proposition 1.8.12]).

Suppose $X = W$ is another von Neumann algebra, and let τ denote the norm-, the weak*- or the strong*-topology of W. The mapping μ is said to be *τ-completely additive* (respectively, *countably or sequentially τ-additive*) when

$$\mu\left(\sum_{i\in I} e_i\right) = \tau\text{-}\sum_{i\in I} \mu(e_i) \tag{2.1}$$

for every family (respectively, sequence) $\{e_i\}_{i\in I}$ of mutually orthogonal projections in M.

It is known that every family $(p_i)_{i\in I}$ of mutually orthogonal projections in a von Neumann algebra M is summable with respect to the weak*-topology of M and $p = \text{weak}^*\text{-}\sum_{i\in I} p_i$ is a projection in M (cf. [30, Definition 1.13.4]). Further, for each normal positive functional ϕ in M_* and every finite set $F \subset I$, we have

$$\left\| p - \sum_{i\in F} p_i \right\|_\phi^2 = \phi\left(p - \sum_{i\in F} p_i\right),$$

which implies that the family $(p_i)_{i\in I}$ is summable with respect to the strong*-topology of M with the same limit, that is, $p = \text{strong}^*\text{-}\sum_{i\in I} p_i$.

Proposition 2.7. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. Then the following statements hold:*

(*a*) *The restriction $T|_{\mathcal{P}(M)}$ is sequentially strong*-additive, and consequently sequentially weak*-additive;*

(*b*) *$T|_{\mathcal{P}(M)}$ is weak*-completely additive, i.e.,*

$$T\left(weak^*\text{-}\sum_{i\in I} p_i\right) = weak^*\text{-}\sum_{i\in I} T(p_i) \tag{2.2}$$

for every family $(p_i)_{i\in I}$ of mutually orthogonal projections in M.

Proof. (*a*) Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of mutually orthogonal projections in M. Let us consider the element $z = \sum\limits_{n\in I} \frac{1}{n} p_n$. By Lemma 2.6 there exist skew-hermitian elements $a_z, b_z \in M$ such that $T(x) = [a_z, x] + b_z \circ x$ for all $x \in \mathcal{W}^*(z)$. Since $\sum\limits_{n=1}^{\infty} p_n, p_m \in \mathcal{W}^*(z)$, for all $m \in \mathbb{N}$, and the product of M is jointly strong*-continuous, we obtain that

$$T\left(\sum_{n=1}^{\infty} p_n\right) = \left[a_z, \sum_{n=1}^{\infty} p_n\right] + b_z \circ \left(\sum_{n=1}^{\infty} p_n\right)$$

$$= \sum_{n=1}^{\infty} [a_z, p_n] + \sum_{n=1}^{\infty} b_z \circ p_n = \sum_{n=1}^{\infty} T(p_n),$$

i.e. $T|_{\mathcal{P}(M)}$ is a countably or sequentially strong*-additive mapping.

(*b*) Let φ be a positive normal functional in M_*, and let $\|.\|_\varphi$ denote the prehilbertian seminorm given by $\|z\|_\varphi^2 = \frac{1}{2}\varphi(zz^* + z^*z)$ $(z \in M)$. Let $\{p_i\}_{i\in I}$ be an arbitrary family of mutually orthogonal projections in M. For every $n \in \mathbb{N}$ define

$$I_n = \{i \in I : \|T(p_i)\|_\varphi \geq 1/n\}.$$

We claim, that I_n is a finite set for every natural n. Otherwise, passing to a subset if necessary, we can assume that there exists a natural k such that I_k is infinite and countable. In this case the series $\sum\limits_{i\in I_k} T(p_i)$ does not converge with respect to the semi-norm $\|.\|_\varphi$. On the other hand, since I_k is a countable set, by (*a*), we have

$$T\left(\sum_{i\in I_k} p_i\right) = \text{strong}^*\text{-}\sum_{i\in I_k} T(p_i),$$

which is impossible. This proves the claim.

We have shown that the set

$$I_0 = \left\{i \in I : \|T(p_i)\|_\varphi \neq 0\right\} = \bigcup_{n\in\mathbb{N}} I_n$$

is a countable set, and $\|T(p_i)\|_\varphi = 0$, for every $i \in I\backslash I_0$.

Set $p = \sum\limits_{i\in I\backslash I_0} p_i \in M$. We shall show that $\varphi(T(p)) = 0$. Let q denote the support projection of φ in M. Having in mind that $\|T(p_i)\|^2_\varphi = 0$, for every $i \in I\backslash I_0$, we deduce that $T(p_i) \perp q$ for every $i \in I\backslash I_0$.

Replacing T with $\widehat{T} = T - \delta(\frac{1}{2}T(\mathbf{1}), \mathbf{1})$ we can assume that $T(\mathbf{1}) = 0$ (cf. Lemma 2.1) and $T(x) = T(x)^*$, for every $x \in M_{sa}$ (cf. Lemma 2.2). By Lemma 2.3, for every $i \in I \setminus I_0$ there exists a skew-hermitian element $a_i = a_{p,p_i} \in M$ such that

$$T(p) = a_i p - p a_i, \text{ and, } T(p_i) = a_i p_i - p_i a_i.$$

Since $T(p_i) \perp q$ we get $(a_i p_i - p_i a_i)q = q(a_i p_i - p_i a_i) = 0$, for all $i \in I \setminus I_0$. Thus, since $p a_i p_i q = p_i a_i q$,

$$\begin{aligned}(T(p)p_i)q &= (a_i p - p a_i)p_i q = a_i p_i q - p a_i p_i q\\ &= a_i p_i q - p_i a_i q = (a_i p_i - p_i a_i)q = 0,\end{aligned}$$

and similarly

$$q(p_i T(p)) = 0,$$

for every $i \in I \setminus I_0$. Consequently,

$$(T(p)p)q = T(p)\left(\sum_{i\in I\backslash I_0} p_i\right) q = 0 = q\left(\sum_{i\in I\backslash I_0} p_i\right) T(p) = q(pT(p)). \tag{2.3}$$

Therefore,

$$\begin{aligned}T(p) = \delta_{p,\mathbf{1}}(p) = \delta_{p,\mathbf{1}}\{p,p,p\} &= 2\{\delta_{p,\mathbf{1}}(p), p, p\} + \{p, \delta_{p,\mathbf{1}}(p), p\}\\ = 2\{T(p),p,p\} + \{p,T(p),p\} &= pT(p) + T(p)p + pT(p)^*p\\ &= pT(p) + T(p)p + pT(p)p,\end{aligned}$$

which implies that

$$\begin{aligned}\varphi(T(p)) &= \varphi(pT(p) + T(p)p + pT(p)p)\\ = \varphi(qpT(p)q) + \varphi(qT(p)pq) &+ \varphi(qpT(p)pq) = (\text{by } (2.3)) = 0.\end{aligned}$$

Finally, by (a) we have

$$T\left(\sum_{i\in I_0} p_i\right) = \|.\|_\varphi\text{-}\sum_{i\in I_0} T(p_i).$$

Two more applications of (a) give:

$$\varphi\left(T\left(\sum_{i\in I} p_i\right)\right) = \varphi\left(T\left(p + \sum_{i\in I_0} p_i\right)\right) = \varphi\left(T(p) + T\left(\sum_{i\in I_0} p_i\right)\right)$$

$$= \varphi\left(T(p)\right) + \varphi\left(T\left(\sum_{i\in I_0} p_i\right)\right) = \sum_{i\in I_0} \varphi\left(T\left(p_i\right)\right).$$

By the Cauchy-Schwarz inequality, $0 \leq |\varphi T(p_i)|^2 \leq \|T(p_i)\|_\varphi^2 = 0$, for every $i \in I\backslash I_0$, and hence $\sum_{i\in I_0} \varphi\left(T\left(p_i\right)\right) = \sum_{i\in I} \varphi\left(T\left(p_i\right)\right)$. The arbitrariness of φ shows that $T\left(\text{weak}^*\text{-}\sum_{i\in I} p_i\right) = \text{weak}^*\text{-}\sum_{i\in I} T(p_i)$. □

Let ϕ be a normal functional in the predual of a von Neumann algebra M. Our previous Proposition 2.7 assures that for every (not necessarily linear nor continuous) 2-local triple derivation $T : M \to M$ the mapping $\phi \circ T|_{\mathcal{P}(M)} : \mathcal{P}(M) \to \mathbb{C}$ is a completely additive mapping or a charge on M. Under the additional hypothesis of M being a continuous von Neumann algebra or, more generally, a von Neumann algebra with no Type I_n-factors $(1 < n < \infty)$ direct summands (i.e. without direct summand isomorphic to a matrix algebra $M_n(\mathbb{C})$, $1 < n < \infty$), the Dorofeev–Sherstnev theorem ([32, Theorem 29.5] or [13, Theorem 2]) imply that $\phi \circ T|_{\mathcal{P}(M)}$ is a bounded charge, that is, the set $\{|\phi \circ T(p)| : p \in \mathcal{P}(M)\}$ is bounded. The uniform boundedness principle gives:

Corollary 2.8. *Let M be a von Neumann algebra with no Type I_n-factor direct summands $(1 < n < \infty)$ and let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then the restriction $T|_{\mathcal{P}(M)}$ is a bounded weak*-completely additive mapping.* □

2.2. Additivity of 2-local triple derivations on hermitian parts of von Neumann algebras.

Suppose now that M is a von Neumann algebra with no Type I_n-factor direct summands $(1 < n < \infty)$, and $T : M \to M$ is a (not necessarily linear nor continuous) 2-local triple derivation. By Corollary 2.8 combined with the Bunce-Wright-Mackey-Gleason theorem [7, 8], there exits a bounded linear operator $G : M \to M$ satisfying that $G(p) = T(p)$, for every projection $p \in M$.

Let z be a self-adjoint element in M. By Lemma 2.6, there exist skew-hermitian elements $a_z, b_z \in M$ such that $T(x) = [a_z, x] + b_z \circ x$, for every $x \in \mathcal{W}^*(z)$. Since $G|_{\mathcal{W}^*(z)}, T|_{\mathcal{W}^*(z)} : \mathcal{W}^*(z) \to M$ are bounded linear operators, which coincide on the set of projections of $\mathcal{W}^*(z)$, and every self-adjoint element in $\mathcal{W}^*(z)$ can be approximated in norm by finite linear combinations of mutually orthogonal projections in $\mathcal{W}^*(z)$, it follows that $T(x) = G(x)$ for every $x \in \mathcal{W}^*(z)$, and hence

$$T(a) = G(a), \text{ for every } a \in M_{sa},$$

in particular, T is additive on M_{sa}.

The above arguments materialize in the following result.

Proposition 2.9. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra with no Type I_n-factor direct summands ($1 < n < \infty$). Then the restriction $T|_{M_{sa}}$ is additive.* □

Corollary 2.10. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a properly infinite von Neumann algebra. Then the restriction $T|_{M_{sa}}$ is additive.*

Next we shall show that the conclusion of the above corollary is also true for a finite von Neumann algebra.

First we show that every 2-local triple derivation on a von Neumann algebra "intertwines" central projections.

Lemma 2.11. *If T is a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra M, and p is a central projection in M, then $T(Mp) \subset Mp$. In particular, $T(px) = pT(x)$ for every $x \in M$.*

Proof. Consider $x \in Mp$, then $x = pxp = \{x, p, p\}$. T coincides with a triple derivation $\delta_{x,p}$ on the set $\{x, p\}$, hence

$$T(x) = \delta_{x,p}(x) = \delta_{x,p}\{x, p, p\} = \{\delta_{x,p}(x), p, p\}+\{x, \delta_{x,p}(p), p\}+\{x, p, \delta_{x,p}(p)\}$$

lies in Mp.

For the final statement, fix $x \in M$, and consider skew-hermitian elements $a_{x,xp}$, $b_{x,xp} \in M$ satisfying

$$T(x) = [a_{x,xp}, x] + b_{x,xp} \circ x, \text{ and } T(xp) = [a_{x,xp}, xp] + b_{x,xp} \circ (xp).$$

The assumption p being central implies that $pT(x) = T(px)$. □

Proposition 2.12. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a finite von Neumann algebra. Then the restriction $T|_{M_{sa}}$ is additive.*

Proof. Since M is finite there exists a faithful normal semi-finite trace τ on M. We shall consider the following two cases.

Case 1. Suppose τ is a finite trace. Replacing T with $\widehat{T} = T-\delta(\frac{1}{2}T(\mathbf{1}), \mathbf{1})$ we can assume that $T(\mathbf{1}) = 0$ (cf. Lemma 2.1) and $T(x) = T(x)^*$, for every $x \in M_{sa}$ (cf. Lemma 2.2). By Lemma 2.3, for every $x, y \in M_{sa}$ there exists a skew-hermitian element $a_{x,y} \in M$ such that $T(x) = [a_{x,y}, x]$ and $T(y) = [a_{x,y}, y]$. Then

$$T(x)y + xT(y) = [a_{x,y}, x]y + x[a_{x,y}, y] = [a_{x,y}, xy],$$

that is,

$$[a_{x,y}, xy] = T(x)y + xT(y).$$

Further

$$0 = \tau([a_{x,y}, xy]) = \tau\left(T(x)y + xT(y)\right),$$

i.e. $\tau(T(x)y) = -\tau(xT(y))$, for every $x, y \in M_{sa}$. For arbitrary $u, v, w \in M_{sa}$, set $x = u + v$, and $y = w$. The above identity implies

$$\tau\left(T(u+v)w\right) = -\tau\left((u+v)T(w)\right) =$$

$$= -\tau\left(uT(w)\right) - \tau\left(vT(w)\right) = \tau\left(T(u)w\right) + \tau\left(T(v)w\right) = \tau\left((T(u) + T(v))w\right),$$

and so

$$\tau\left((T(u+v) - T(u) - T(v))w\right) = 0$$

for all $u, v, w \in M_{sa}$. Take $w = T(u+v) - T(u) - T(v)$. Then $\tau(ww^*) = 0$. Since the trace τ is faithful it follows that $ww^* = 0$, and hence $w = 0$. Therefore

$$T(u+v) = T(u) + T(v).$$

Case 2. As in *Case 1*, we may assume $T(\mathbf{1}) = 0$. Suppose now that τ is a semi-finite trace. Since M is finite there exists a family of mutually orthogonal central projections $\{z_i\}$ in M such that z_i has finite trace for all i and $\bigvee z_i = \mathbf{1}$ (cf. [30, §2.2 or Corollary 2.4.7]). By Lemma 2.11, for each i, T maps z_iM into itself. From Case 1, $T|_{z_iM} : z_iM \to z_iM$ is additive. Furthermore,

$$z_iT(x+y) = T|_{z_iM}(z_ix + z_iy) = T|_{z_iM}(z_ix) + T|_{z_iM}(z_iy) = z_iT(x) + z_iT(y),$$

for every $x, y \in M$ and every i. Therefore

$$T(x+y) = \left(\sum_i z_i\right) T(x+y) = \sum_i z_iT(x+y) = \sum_i \left(z_iT(x) + z_iT(y)\right)$$

$$= \left(\sum_i z_i\right) T(x) + \left(\sum_i z_i\right) T(y) = T(x) + T(y),$$

for every $x, y \in M$. The proof is complete. □

Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra. In this case there exist orthogonal central projections $z_1, z_2 \in M$ with $z_1 + z_2 = \mathbf{1}$ such that:

(−) z_1M is a finite von Neumann algebra;
(−) z_2M is a properly infinite von Neumann algebra,

(cf. [30, §2.2]).

By Lemma 2.11, for each $k = 1, 2$, z_kT maps z_kM into itself. By Corollary 2.10 and Proposition 2.12 both z_1T and z_2T are additive on M_{sa}. So $T = z_1T + z_2T$ also is additive on M_{sa}.

We have thus proved the following result:

Proposition 2.13. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on an arbitrary von Neumann algebra. Then the restriction $T|_{M_{sa}}$ is additive.* □

2.3. **Main result.**

We can state now the main result of this paper.

Theorem 2.14. *Let M be an arbitrary von Neumann algebra and let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation. Then T is a triple derivation (hence linear and continuous). Equivalently, the set $Der_t(M)$, of all triple derivations on M, is algebraically 2-reflexive in the set $\mathcal{M}(M) = M^M$ of all mappings from M into M.*

We need the following two Lemmata.

Lemma 2.15. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra with $T(\mathbf{1}) = 0$. Then there exists a skew-hermitian element $a \in M$ such that $T(x) = [a, x]$, for all $x \in M_{sa}$.*

Proof. Let $x \in M_{sa}$. By Lemma 2.3 there exist a skew-hermitian element $a_{x,x^2} \in M$ such that

$$T(x) = [a_{x,x^2}, x],\ T(x^2) = [a_{x,x^2}, x^2].$$

Thus

$$T(x^2) = [a_{x,x^2}, x^2] = [a_{x,x^2}, x]x + x[a_{x,x^2}, x] = T(x)x + xT(x),$$

i.e.

$$T(x^2) = T(x)x + xT(x), \tag{2.4}$$

for every $x \in M_{sa}$.

By Proposition 2.13 and Lemma 2.2, $T|_{M_{sa}} : M_{sa} \to M_{sa}$ is a real linear mapping. Now, we consider the linear extension $\hat{T}$ of $T|_{M_{sa}}$ to M defined by

$$\hat{T}(x_1 + ix_2) = T(x_1) + iT(x_2),\ x_1, x_2 \in M_{sa}.$$

Taking into account the homogeneity of T, Proposition 2.13 and the identity (2.4) we obtain that $\hat{T}$ is a Jordan derivation on M. By [5, Theorem 1] any Jordan derivation on a semi-prime algebra is a derivation. Since M is von Neumann algebra, $\hat{T}$ is a derivation on M (see also [33] and [16]). Therefore there exists an element $a \in M$ such that $\hat{T}(x) = [a, x]$ for all $x \in M$. In particular, $T(x) = [a, x]$ for all $x \in M_{sa}$. Since $T(M_{sa}) \subseteq M_{sa}$, we can assume that $a^* = -a$, which completes the proof. □

Lemma 2.16. *Let $T : M \to M$ be a (not necessarily linear nor continuous) 2-local triple derivation on a von Neumann algebra. If $T|_{M_{sa}} \equiv 0$, then $T \equiv 0$.*

Proof. Let $x \in M$ be an arbitrary element and let $x = x_1 + ix_2$, where $x_1, x_2 \in M_{sa}$. Since T is homogeneous, if necessary, passing to the element $(1 + \|x_2\|)^{-1}x$, we can suppose that $\|x_2\| < 1$. In this case the element $y =$

$\mathbf{1}+x_2$ is positive and invertible. Take skew-hermitian elements $a_{x,y}, b_{x,y} \in M$ such that

$$T(x) = [a_{x,y}, x] + b_{x,y} \circ x,$$
$$T(y) = [a_{x,y}, y] + b_{x,y} \circ y.$$

Since $T(y) = 0$, we get $[a_{x,y}, y] + b_{x,y} \circ y = 0$. By Lemma 2.4 we obtain that $[a_{x,y}, y] = 0$ and $ib_{x,y} \circ y = 0$. Taking into account that $ib_{x,y}$ is hermitian, y is positive and invertible, Lemma 2.5 implies that $b_{x,y} = 0$.

We further note that

$$0 = [a_{x,y}, y] = [a_{x,y}, \mathbf{1} + x_2] = [a_{x,y}, x_2],$$

i.e.

$$[a_{x,y}, x_2] = 0.$$

Now,

$$T(x) = [a_{x,y}, x] + b_{x,y} \circ x = [a_{x,y}, x_1 + ix_2] = [a_{x,y}, x_1],$$

i.e.

$$T(x) = [a_{x,y}, x_1].$$

Therefore,

$$T(x)^* = [a_{x,y}, x_1]^* = [x_1, a_{x,y}^*] = [x_1, -a_{x,y}] = [a_{x,y}, x_1] = T(x).$$

So

$$T(x)^* = T(x). \tag{2.5}$$

Now replacing x by ix on (2.5) we obtain from the homogeneity of T that

$$T(x)^* = -T(x). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain that $T(x) = 0$, which finishes the proof. □

Proof of Theorem 2.14. Let us define $\widehat{T} = T - \delta\left(\frac{1}{2}T(\mathbf{1}), \mathbf{1}\right)$. Then $\widehat{T}$ is a 2-local triple derivation on M with $\widehat{T}(\mathbf{1}) = 0$ (cf. Lemma 2.1) and $\widehat{T}(x) = \widehat{T}(x)^*$, for every $x \in M_{sa}$ (cf. Lemma 2.2). By Lemma 2.15 there exists an element $a \in M$ such that $\widehat{T}(x) = [a, x]$ for all $x \in M_{sa}$. Consider the 2-local triple derivation $\widehat{T} - [a, \cdot]$. Since $(\widehat{T} - [a, \cdot])|_{M_{sa}} \equiv 0$, Lemma 2.16 implies that $\widehat{T} = [a, \cdot]$, and hence $T = [a, \cdot] + \delta\left(\frac{1}{2}T(\mathbf{1}), \mathbf{1}\right)$, witnessing the desired statement. □

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