GEOMETRY OF THE DUAL BALL
OF THE SPIN FACTOR

Y. FRIEDMAN and B. RUSSO

ABSTRACT

The main result of this paper is a geometric characterization of the unit ball of the dual of a complex spin factor.

THEOREM. *A strongly facially symmetric space of type I_2 in which every proper norm closed face in the unit ball is norm exposed, and which satisfies 'symmetry of transition probabilities', is linearly isometric to the dual of a complex spin factor.*

This result is an important step in the authors' program of showing that the class of all strongly facially symmetric spaces satisfying certain natural and physically significant axioms is equivalent isometrically to the class of all predual spaces of JBW*-triples.

The result can be interpreted as a characterization of the non-ordered state space of 'two state' physical systems.

A new tool for working with concrete spin factors, the so-called facial decomposition, is also developed.

Facially symmetric spaces were introduced and studied in [6] and [7] as a geometrical model for quantum mechanics. They provide an appropriate framework in which to study the problem of characterizing the unit ball of the predual of a JBW*-triple (cf. [5]) in terms of geometric and physically significant properties of a convex set. In this paper we will restrict ourselves to atomic spaces (Definition 2.5). The rank-1 case, together with the axiom (STP) (symmetry of transition probabilities, Definition 2.8) leads easily to a Hilbert space, as will be shown in § 2 (Corollary 2.11). The next significant case, that of rank 2, will be treated in detail in §§ 3 and 4. This case occurs already as the model for the state space of a spin-$\frac{1}{2}$ particle. More generally, this model can occur for any physical system in which each measurement results in at most two distinct outcomes ('two state systems').

A standard algebraic model in this context is the (real or complex) spin factor. Since this is a Jordan algebra (JB-algebra in the real case, and JB*-algebra in the complex case), its state space has been extensively studied. Less well known is the structure of the entire dual ball. In the real case, as will be shown below (see Proposition 1.8), this is the intersection of two cylinders based on real Hilbert balls. These two Hilbert balls are either of the same dimension, or their dimensions differ by 1. In the complex case, the geometry of the spin factor has a much richer structure, an analysis of which is carried out in this paper.

The main result of this paper is a geometric characterization of the predual ball of a complex spin factor. This is the same as the unit ball of the dual since the spin factor is a reflexive Banach space. We show (in Theorem 4.16) that any facially symmetric space of type I_2 (Definition 3.10) satisfying some physically significant geometric axioms ((STP) and (FE), Definition 3.3) is linearly isometric to the predual of a complex spin factor. This result is an important step in the

program whose objective is to show that the dual of any facially symmetric space which satisfies certain natural geometric axioms supports the structure of a JB*-triple. Conversely, it is known that the predual of every JBW*-triple is a neutral strongly facially symmetric space [8, Theorem 3.1] which satisfies (STP) [5, Lemma 2.2] and (FE) [4, Corollary 4.5].

The organization of this paper is as follows. In § 1 we describe the facial structure of the dual ball of the concrete spin factor, using the basis called a spin grid in [3] and [10]. This facial structure was revealed from the study of facially symmetric spaces of type I_2 carried out in §§ 3 and 4. The first named author wishes to thank Professor Itamar Pitowsky of the Hebrew University for discussions of the connections of this facial structure to models in quantum mechanics. These discussions were instrumental in the formulation of the material of § 1. Although the results of § 1 are presented here primarily to illustrate in a concrete form the abstract results of §§ 3 and 4, nevertheless, some of these results are used to simplify the proof of our main theorem. Moreover, the (facial) decomposition introduced here (Proposition 1.11) may provide a new tool for working with spin factors.

In § 2 the definition and some properties of facially symmetric spaces are re-called and some supplementary results from the global theory of facially symmetric spaces are proved. The definition and some properties of atomic spaces are given and the principal geometric axioms are introduced. It is shown that an atomic facially symmetric space satisfying these axioms admits a symmetric sesquilinear form, which leads to a Hilbert space structure in the rank-1 case.

In § 3 we show that a rank-2 face in an atomic neutral strongly facially symmetric space satisfying the axioms (FE) and (STP) is affinely isometrically isomorphic to the unit ball of a real Hilbert space, and moreover that the unit ball of the real linear span of this face is a cylinder with the Hilbert ball as its base. This provides a fundamental computational tool for the main result in § 4.

In § 4 we study the *complex* span of a rank-2 face, which by definition is dense in a facially symmetric space of type I_2. Starting from an arbitrary orthonormal basis in the real Hilbert space whose unit ball is affinely isomorphic to the given rank-2 face, we construct a total subset (called a *dual spin grid*) which resembles the 'dual' of a spin grid in a concrete spin factor. In this construction, the imaginary unit enters naturally from geometric considerations. After a reduction to finite dimensions, it is shown by a induction argument that the linear extension of the natural map between the two dual spin grids (abstract and concrete) is isometric, and therefore extends to an isometry of the dual of the spin factor onto the norm closure of the complex span of the rank-2 face. The induction proceeds by two dimensions at a time, so it is necessary to analyse in complete detail the cases in which the Hilbert spaces arising from the rank-2 face are of dimensions 2 and 3. This leads to characterizations of the spin factors of dimensions 3 and 4, namely, the JBW*-triples $S_2(\mathbb{C})$ of 2 by 2 symmetric complex matrices and $M_2(\mathbb{C})$ of all 2 by 2 complex matrices.

The following symbols will be used in this paper: the closed unit ball of a normed space X will be denoted by X_1, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$.

The duality between a normed space X and its dual X^* will be denoted by $\langle \varphi, x \rangle$ or by $\langle x, \varphi \rangle$ for $(\varphi, x) \in X^* \times X$. The real and imaginary parts of a complex number z will be denoted by $\Re z$ and $\Im z$.

For a convex set K in a normed space Z, ext K denotes the set of extreme points of K. An element $f \in K$ is said to be *norm exposed* in K if $\{f\} = K \cap H$ for a suitable hyperplane H. The set of norm exposed points of K will be denoted by exp K. More generally, $F \subset K$ is a *norm exposed face* of K if $F = K \cap H$. In general, exp $K \subset$ ext K, with equality holding, for example, in the case that $K = U_{*,1}$ is the unit ball of the predual U_* of a JBW*-triple U [5, Proposition 4]. The equality 'exp $Z_1 =$ ext Z_1' for a normed space Z is a physically meaningful assumption and is analogous to one of the pure state properties of Alfsen and Shultz [1] (cf. Definition 2.8 below).

1. *Facial structure of the dual ball of the spin factor*

The spin factor, or 'spinors', occur naturally in different areas of mathematics and physics. Several (equivalent) definitions exist and are used. In this paper, the definition of spin factor will be based on algebraic properties of a natural basis called a spin grid, which is shown in [3] and [10] to generate the spin factor linearly and topologically. It is known (cf. for example [9]) that the complex spin factor can be represented as a norm closed subspace of the bounded operators on a complex Hilbert space which is stable under the operation $a \mapsto aa^*a$ and which generates the Clifford algebra (CAR algebra) as a C*-algebra. The spin factor has the structure of a complex Banach Jordan *-algebra (JB*-algebra) and is thus an example of a JB*-triple (cf. [5]). It is the complexification of the real spin factor, which is a JB-algebra.

We begin by recalling the recent description (see [3]) of the algebraic structure of a *spin factor* or so-called *Cartan factor of type* 4, which we shall refer to as a concrete spin factor. Although the concrete spin factor can be realized as a norm closed subspace of $B(\mathcal{H})$, our approach will be via spin grids. This approach simplifies our calculations and makes possible an elementary description of the facial structure of the unit ball of the dual space.

DEFINITION 1.1. A Banach space U over $\mathbb{C}$ is said to be a *JB*-triple* if it is equipped with a continuous triple product $(a, b, c) \mapsto \{abc\}$ mapping $U \times U \times U$ to U such that

 (i) $\{abc\}$ is linear in a and c and conjugate linear in b;

 (ii) $\{abc\}$ is symmetric in the outer variables, that is, $\{abc\} = \{cba\}$;

 (iii) for any $x \in U$, the operator $\delta(x)$ from U to U defined by $\delta(x)y = \{xxy\}$, $y \in U$, is hermitian (that is, $\exp it\delta$ is an isometry for all real t) with non-negative spectrum;

 (iv) the triple product satisfies the following identity, called the 'main identity':

$$\delta(x)\{abc\} = \{\delta(x)a, b, c\} - \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\}; \qquad (1)$$

 (v) the following norm condition holds:

$$\|\{xxx\}\| = \|x\|^3. \qquad (2)$$

A non-zero element a in a JB*-triple is called a *tripotent* if $a = \{aaa\}$, and we say that a is a *minimal* tripotent if it cannot be written as a sum of two orthogonal tripotents, where elements a and b are said to be *orthogonal* if $\{abx\} = 0$ for every $x \in U$. For example, in the above-mentioned representation

of a spin factor as bounded operators on a Hilbert space $\mathcal{H}$, the triple product is given by $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$ for $x, y, z \in B(\mathcal{H})$, so the tripotents are precisely the partial isometries, and orthogonality of tripotents corresponds to orthogonality of operators. We shall not make use of this representation anywhere in this paper.

We are now going to construct the triple product and norm in a concrete spin factor in an elementary way by using properties of a spin grid. The assumptions that we make in the following definition are known to hold for a spin grid in a spin factor [3].

DEFINITION 1.2. Let I be an index set of arbitrary cardinality. A *basis*, or *spin grid* is a collection $\mathcal{G}$ of linearly independent elements $\{u_i, \bar{u}_i\}_{i \in I}$ or $\{u_0, u_i, \bar{u}_i\}_{i \in I}$. Define a triple product $\{uvw\}$ for elements of the basis by

1. $\{uuu\} = u$ for all $u \in \mathcal{G}$ (the basis will consist of tripotents);
2. for distinct non-zero i, j,

$$\{u_i u_i u_j\} = \{\bar{u}_i \bar{u}_i u_j\} = \tfrac{1}{2}u_j, \quad \{u_j u_j u_i\} = \{\bar{u}_j \bar{u}_j u_i\} = \tfrac{1}{2}u_i,$$

$$\{\bar{u}_i \bar{u}_i \bar{u}_j\} = \tfrac{1}{2}\bar{u}_j, \quad \text{and} \quad \{\bar{u}_j \bar{u}_j \bar{u}_i\} = \tfrac{1}{2}\bar{u}_i$$

(u_i will be *colinear* with u_j and with $\bar{u}_j$, and $\bar{u}_i$ will be *colinear* with $\bar{u}_j$), and

$$\{u_i u_j \bar{u}_i\} = -\tfrac{1}{2}\bar{u}_j, \quad \{u_j \bar{u}_i \bar{u}_j\} = -\tfrac{1}{2}u_i$$

(the quadruple $(u_i, u_j, \bar{u}_i, \bar{u}_j)$ will be an *odd quadrangle*);
3. in the case where u_0 exists, for each $i \neq 0$,

$$\{u_i u_i u_0\} = \{\bar{u}_i \bar{u}_i u_0\} = \tfrac{1}{2}u_0, \quad \{u_0 u_0 u_i\} = u_i, \quad \{u_0 u_0 \bar{u}_i\} = \bar{u}_i$$

(u_0 *governs* u_i and $\bar{u}_i$), and

$$\{u_0 u_i u_0\} = -\bar{u}_i, \quad \{u_0 \bar{u}_i u_0\} = -u_i;$$

4. $\{uvw\} = \{wvu\}$ for all $u, v, w \in \mathcal{G}$;
5. all other products $\{uvw\}$ where u, v, w are from the basis, are zero; in particular, for each $i \neq 0$,

$$\{u_i \bar{u}_i u\} = 0 = \{\bar{u}_i u_i u\} \quad \text{for all } u \in \mathcal{G}$$

($u_i, \bar{u}_i$ will be *orthogonal*).

It follows from these properties that the set of all scalar multiples of basis elements is closed under the triple product $\{\cdot, \cdot, \cdot\}$. Hence, the triple product $\{\cdot, \cdot, \cdot\}$ can be extended to the real or complex span of the basis elements to be linear in the outer variables and (in the complex case) conjugate linear in the middle variable.

Define an inner product on sp $\mathcal{G}$ by

$$\langle a \mid b \rangle = \sum a_i \bar{b}_i + \sum \bar{a}_i \bar{\bar{b}}_i + 2a_0 \bar{b}_0, \tag{3}$$

where $a = \sum a_i u_i + \sum \bar{a}_i \bar{u}_i + a_0 u_0$ and $b = \sum b_i u_i + \sum \bar{b}_i \bar{u}_i + b_0 u_0$ are two elements of sp $\mathcal{G}$.

DEFINITION 1.3. The completion of sp $\mathcal{G}$ with respect to the norm $\|\cdot\|_2$ determined by the inner product (3) is called a (*concrete*) *spin factor*, and will be denoted by $\mathcal{C}$.

If I is finite with n elements, the dimension of $\mathcal{C}$ is $2n$ or $2n + 1$. Otherwise, $\mathcal{C}$ is infinite-dimensional, and (3) is then a convergent sum. The norm on the spin factor $\mathcal{C}$ which will make it into a JB*-triple is not the Hilbert space norm used in the definition, since that norm does not satisfy (2). In order to define the correct norm, which will be equivalent to the Hilbert norm, we introduce the following concepts.

Define a conjugation $^{\#}$ on basis elements by $u_i^{\#} = \bar{u}_i$, $\bar{u}_i^{\#} = u_i$, and $u_0^{\#} = u_0$, and extend this to the linear span in a conjugate linear way.

The connection between the triple product, inner product, and conjugation is given by

$$2\{abc\} = \langle a \mid b \rangle c + \langle c \mid b \rangle a - \langle a \mid c^{\#} \rangle b^{\#}. \tag{4}$$

For each element a of $\mathcal{C}$, the notion of *determinant* is defined by

$$\det a := \sum a_i \bar{a}_i + a_0^2 = \tfrac{1}{2} \langle a \mid a^{\#} \rangle. \tag{5}$$

PROPOSITION 1.4 [3, Proposition 3.3, Lemma 3.4]. *Let $\mathcal{C}$ be a spin factor.*

1. *If $a \in \mathcal{C}$, then a is a scalar multiple of a minimal tripotent if and only if $\det a = 0$ and in this case, from (2) and (4), the norm must be defined by $\|a\| = \langle a \mid a \rangle^{\frac{1}{2}} = \|a\|_2$ for such a.*

2. *Elements a and b in $\mathcal{C}$ with $\det a = \det b = 0$ are scalar multiples of orthogonal tripotents if and only if there is $\lambda \in \mathbb{C}$ such that $b = \lambda a^{\#}$.*

PROPOSITION 1.5 [3, Proposition 3.6]. *For any element a in a spin factor $\mathcal{C}$, with $\det a \neq 0$, there is a unique set of non-negative numbers $\{s_1, s_2\}$ determined by*

$$s_1^2 + s_2^2 = \langle a \mid a \rangle, \quad s_1 s_2 = |\det a|. \tag{6}$$

Also, if $s_1 \neq s_2$, two orthogonal minimal tripotents e, f are determined uniquely by a such that

$$a = s_1 e + s_2 f. \tag{7}$$

Let $\lambda = \det a / |\det a|$. The coordinates of the tripotents e and f with respect to the spin grid are given by

$$e_i = \frac{1}{s_1^2 - s_2^2} (s_1 a_i - \lambda s_2 \bar{a}_i),$$

$$\bar{e}_i = \frac{1}{s_1^2 - s_2^2} (-\lambda s_2 \bar{\bar{a}}_i + s_1 \bar{a}_i), \tag{8}$$

$$e_0 = \frac{1}{s_1^2 - s_2^2} (s_1 a_0 - \lambda s_2 \bar{a}_0),$$

and

$$f_i = \frac{1}{s_1^2 - s_2^2}(-s_2 a_i + \lambda s_1 \bar{a}_i),$$

$$\bar{f}_i = \frac{1}{s_1^2 - s_2^2}(\lambda s_1 \bar{a}_i - s_2 \bar{a}_i), \tag{9}$$

$$f_0 = \frac{1}{s_1^2 - s_2^2}(\lambda s_1 \bar{a}_0 - s_2 a_0).$$

COROLLARY 1.6 [**3**, Corollary 3.7]. *If a has decomposition* (7), *then from* (2) *and* [**5**, *Lemma* 1.3(a)], $\|a\|$ *must be defined as* $\max\{s_1, s_2\}$. *From* (6) *it follows that this norm is equivalent to the Hilbert norm* $\|\cdot\|_2$, *and therefore* $\mathscr{C}$ *is complete and reflexive in this norm.*

Hence every element $a \in \mathscr{C}$ has unique coordinates $\{a_i, \bar{a}_i, a_0\}$ with

$$a = \sum a_i u_i + \sum \bar{a}_i \bar{u}_i + a_0 u_0, \tag{10}$$

where convergence is in $\|\cdot\|$ (u_0 may not exist).

The space $\mathscr{C}$ will now be denoted by U. The Banach space dual U^* (which is the same as the predual U_* since U is reflexive) can be identified with U via the conjugate linear (Riesz) map π: $U^* \ni \varphi \mapsto a_\varphi = \pi(\varphi) \in U$, where

$$\langle \varphi, x \rangle = \langle x \mid a_\varphi \rangle \quad (x \in U, \varphi \in U^*). \tag{11}$$

In the rest of this section, by abuse of notation, we shall sometimes use the same notation for $\varphi \in U^*$ and $a_\varphi \in U$. Thus $\langle b, x \rangle = \langle x \mid b \rangle$ for $b \in U^*, x \in U$, and if a is given by (10), then

$$\pi^{-1}(a) = \sum \bar{a}_i \pi^{-1}(u_i) + \sum \bar{\bar{a}}_i \pi^{-1}(\bar{u}_i) + \bar{a}_0 \pi^{-1}(u_0),$$

$\overline{\det a} = \det(\pi^{-1}(a))$, and the inner product (3) transferred to $\mathscr{C}_*$ satisfies

$$\langle \pi^{-1}(a) \mid \pi^{-1}(b) \rangle = \langle b \mid a \rangle. \tag{12}$$

Since the spin factor U is a JBW*-triple, we have from [**5**, Propositions 4 and 8] a one-to-one correspondence between tripotents and norm exposed faces in the unit ball of the predual U_*, in which minimal tripotents correspond to minimal faces, that is, extreme points of the unit ball. We also have the following corollary.

COROLLARY 1.7 [**8**, Theorem 3.1; **3**, Corollary 3.8]. *The space* $Z := U_*$ *is a neutral strongly facially symmetric space (see § 2 for the definition) with the norm given for any a with $\pi(a)$ decomposed as in* (7) *by*

$$\|a\|_Z = s_1 + s_2 = \sqrt{(\langle a \mid a \rangle + 2 |\det a|)}. \tag{13}$$

We now use (13) to describe the dual ball of the real spin factor.

PROPOSITION 1.8. *If U is a spin factor over $\mathbb{R}$, then the unit ball $U_{*,1}$ of the predual U_* is the intersection of two cylinders with base a real Hilbert ball. The dimensions of these Hilbert spaces differ by at most one. More precisely, $U_{*,1}$ is the unit ball of the l^∞-sum of these two Hilbert spaces.*

Proof. By (13) and (5),

$$\|a\|_Z \leqslant 1 \Leftrightarrow \langle a \mid a \rangle + 2\,|\det a| \leqslant 1 \Leftrightarrow \langle a \mid a \pm a^\# \rangle \leqslant 1.$$

Thus, if a has (real) coordinates a_i, $\bar{a}_i$ and possibly a_0, we have, by (3), $\|a\|_Z \leqslant 1$ if and only if

$$\sum_i a_i(a_i \pm \bar{a}_i) + \sum_i \bar{a}_i(\bar{a}_i \pm a_i) + 2a_0(a_0 \pm a_0) \leqslant 1,$$

or equivalently, with $b_i := \frac{1}{2}(a_i + \bar{a}_i)$ and $\bar{b}_i := \frac{1}{2}(a_i - \bar{a}_i)$,

$$\max\left(\sum_i (2b_i)^2 + 4a_0^2,\; \sum_i (2\bar{b}_i)^2 \right) \leqslant 1.$$

Since by Proposition 1.5 any tripotent v in U is either minimal or a sum of two orthogonal minimal tripotents, each non-trivial norm exposed face of Z_1 is either a single point (an extremal point of Z_1, corresponding to a minimal tripotent) or a (so-called) face of rank 2. For any rank-2 face

$$F_v = \{a \in Z: \ \|a\|_Z = \langle a, v \rangle = 1\}$$

defined by a tripotent v, define an element of F_v called the *centre* of F_v by $\xi = \frac{1}{2}\pi^{-1}(v)$. Note that by our convention, we could write $\xi = \frac{1}{2}v$ and still have $v \in U$ and $\xi \in F_v \subset Z$. We shall usually write F_ξ for F_v, since ξ also determines the face. Later in this section we shall show that F_ξ is affinely isomorphic to the unit ball of a real Hilbert space in such a way that the centre ξ corresponds to the origin. This should now be compared with Theorem 3.8.

DEFINITION 1.9. Let Z be the predual of a concrete spin factor. An element $\xi \in Z$ is said to be *unitary* if $\|\xi\|_Z = 1$ and $\xi^\# = \lambda\xi$ for some $\lambda \in \mathbb{T}$. For $0 \neq a \in Z$ define the *phase* $\zeta(a)$ of a to be

$$\zeta(a) = \begin{cases} \det a/|\det a| & \text{if } \det a \neq 0, \\ 1 & \text{if } \det a = 0. \end{cases}$$

Note that for any unitary element ξ, $\|\xi\|_2^2 = \frac{1}{2}$ and that in the spectral decomposition (7) of ξ, $s_1 = s_2 = \frac{1}{2}$. Indeed, $|\det \xi| = |\frac{1}{2}\langle \xi \mid \bar{\lambda}\xi \rangle| = \frac{1}{2}\|\xi\|_2^2$ and so

$$1 = \|\xi\|_Z = \|\xi\|_2^2 + 2\,|\det \xi| = 2\,\|\xi\|_2^2,$$

and by Proposition 1.5, $(s_1 - s_2)^2 = \langle \xi \mid \xi \rangle - 2\,|\det \xi| = 0$. Hence, any scalar multiple η of a unitary satisfies $\|\eta\|_2 = (1/\sqrt{2})\,\|\eta\|_Z$.

PROPOSITION 1.10. *Let a be any norm-1 element of Z and let $\lambda = \zeta(a)$. Then the centre ξ of a rank-2 face F_v containing a is given by*

$$\xi = \tfrac{1}{2}(a + \lambda a^\#). \tag{14}$$

Moreover, ξ is unitary and $\zeta(\xi) = \lambda$.

Proof. If $\det a = 0$, that is, a is extremal, then a and $a^\#$ are orthogonal extreme points, so that $\langle a, \pi(a) + \pi(a)^\# \rangle = 1$, that is, $a \in F_\xi$. Also $\xi^\# = \xi$ and $\det \xi = \frac{1}{2}\langle \xi \mid \xi \rangle$ which implies $\zeta(\xi) = 1$.

If $\det a \neq 0$, applying (7) to $\pi(a)$, we have $a \in F_v \cap F_{e+f}$ which is therefore a rank-2 face contained in each of F_v and F_{e+f}. Therefore F_v and F_{e+f} both equal this intersection, and hence are equal and $v = e + f$. Now, using (8) and (9) applied to $\pi(a)$, we obtain

$$v_i = e_i + f_i = \frac{1}{s_1^2 - s_2^2}(s_1\bar{a}_i - \bar{\lambda}s_2\bar{a}_i) + \frac{1}{s_1^2 - s_2^2}(-s_2\bar{a}_i + \bar{\lambda}s_1\bar{a}_i)$$

$$= \frac{1}{s_1 + s_2}\bar{a}_i + \frac{\bar{\lambda}}{s_1 + s_2}\bar{a}_i = \bar{a}_i + \bar{\lambda}\bar{a}_i.$$

Similarly,

$$\tilde{v}_i = \tilde{e}_i + \tilde{f}_i = \tilde{\bar{a}}_i + \bar{\lambda}a_i \quad \text{and} \quad v_0 = e_0 + f_0 = \bar{a}_0 + \bar{\lambda}a_0,$$

which yields $v = \pi(a) + \bar{\lambda}(\pi(a))^\#$. Hence, $\xi = \frac{1}{2}\pi^{-1}(v) = \frac{1}{2}(a + \lambda a^\#)$.

The following proposition is the facial decomposition of an arbitrary element of the predual of a spin factor.

PROPOSITION 1.11 (facial decomposition). *Let Z be the predual of a spin factor. For each non-zero $a \in Z$, there are unique elements $\xi, h \in Z$ such that*

(i) *ξ, h are scalar multiples of unitaries and $a = \xi + h$;*

(ii) *$\zeta(\xi) = -\zeta(h) = \zeta(a)$.*

Moreover,

(iii) *$\langle \xi \mid h \rangle = 0$, and hence $\|a\|_2^2 = \|\xi\|_2^2 + \|h\|_2^2$;*

(iv) *$\|a\|_Z = \|\xi\|_Z \geqslant \|h\|_Z$;*

(v) *$|\det a| = \frac{1}{4}(\|\xi\|_Z^2 - \|h\|_Z^2)$, and hence $\|\xi\|_Z = \|h\|_Z$ if and only if $\det a = 0$.*

Proof. Let $\lambda = \zeta(a)$ and set $\xi = \frac{1}{2}(a + \lambda a^\#)$ and $h = \frac{1}{2}(a - \lambda a^\#)$. Then $\xi^\# = \bar{\lambda}\xi$ and $h^\# = -\bar{\lambda}h$. This proves (i).

Since $\det \xi = \frac{1}{2}\langle \xi \mid \xi^\# \rangle = \frac{1}{2}\zeta(a)\|\xi\|_2^2 \neq 0$, $\zeta(\xi) = \zeta(a)$. Similarly, $\det h = -\frac{1}{2}\zeta(a)\|h\|_2^2$ and $\zeta(h) = -\zeta(a)$. This proves (ii).

Suppose that ξ' and h' satisfy (i) and (ii). Then with $\lambda := \zeta(\xi')$ we have $(\xi')^\# = \bar{\lambda}\xi'$ and $(h')^\# = -\bar{\lambda}h'$. Thus, $a = \xi' + h'$ implies that $a^\# = \bar{\lambda}\xi' - \bar{\lambda}h'$ and $\lambda a^\# = \xi' - h'$ so that $\xi' = \frac{1}{2}(a + \lambda a^\#)$ and $h' = \frac{1}{2}(a - \lambda a^\#)$. This proves the uniqueness.

Let $\lambda = \zeta(a)$. Then

$$\langle \xi \mid h \rangle = \frac{1}{4}(\langle a \mid a \rangle + \lambda\langle a^\# \mid a \rangle - \bar{\lambda}\langle a \mid a^\# \rangle - \langle a^\# \mid a^\# \rangle)$$

$$= \frac{1}{4}(\lambda\langle a^\# \mid a \rangle - \bar{\lambda}\langle a \mid a^\# \rangle) = -i\frac{1}{2}\Im(\lambda\langle a^\# \mid a \rangle)$$

$$= -i\Im(\lambda\,\overline{\det a}) = -i\Im|\det a| = 0.$$

This proves (iii).

By Proposition 1.10, $\xi/\|a\|_Z$ is the centre of the face containing $a/\|a\|_Z$. Therefore $\|\xi\|_Z = \|a\|_Z$. Now

$$\|\xi\|_2^2 = \langle \xi \mid \xi \rangle = \frac{1}{2}[\|a\|_2^2 + \Re\langle \lambda a^\# \mid a \rangle] = \frac{1}{2}\|a\|_2^2 + |\det a|,$$

and therefore $\|\xi\|_Z^2 = \|a\|_2^2 + 2\,|\det a|$. Similarly, $\|h\|_Z^2 = \|a\|_2^2 - 2\,|\det a|$ and so $\|h\|_Z^2 \leqslant \|\xi\|_Z^2 = \|a\|_Z^2$, proving (iv) and (v).

In Proposition 1.11, if $\|a\|_Z = 1$, then a and ξ belong to the face F_ξ containing a whereas $h = a - \xi$ is 'parallel' to F. Moreover, by this proposition, the face F_ξ is a real Hilbert ball defined by

$$F_\xi = \{\xi + h: \ h \in Z, \ \|h\|_2 \leqslant 1, \ \langle \xi \mid h \rangle = 0, \ h^\# = -\zeta(\xi)h\}.$$

Note that since $\langle h \mid h' \rangle = \langle h'^\# \mid h^\# \rangle = \langle h' \mid h \rangle$, the inner product

$$\langle \xi + h \mid \xi + h' \rangle = \langle \xi \mid \xi \rangle + \langle h \mid h' \rangle$$

is real.

2. *Atomic facially symmetric spaces*

In this section we shall recall some basic facts about facially symmetric spaces from [7] and prove several new results which supplement those in [7] and are needed in the present paper. We then establish some properties of atomic facially symmetric spaces, including a Hilbert space structure on those atomic facially symmetric spaces which satisfy two geometric axioms.

In this and the next section, Z will be a weakly or strongly facially symmetric space over the real or complex field. In § 4 we shall restrict our study to complex spaces.

Let Z be a real or complex normed space. Elements $f, g \in Z$ are *orthogonal*, notation $f \diamondsuit g$, if $\|f + g\| = \|f - g\| = \|f\| + \|g\|$. A *norm exposed face* of the unit ball Z_1 of Z is a non-empty set (necessarily not equal to Z_1) of the form $F_x = \{f \in Z_1: f(x) = 1\}$, where $x \in Z^*$, $\|x\| = 1$. An element $u \in Z^*$ is called a *projective unit* if $\|u\| = 1$ and $\langle u, F_u^\diamondsuit \rangle = 0$. We use $\mathcal{F}$ and $\mathcal{U}$ to denote the collections of norm exposed faces of Z_1 and projective units in Z^*, respectively.

Motivated by measuring processes in quantum mechanics, we define a *symmetric face* to be a norm exposed face F in Z_1 with the following property: there is a linear isometry S_F of Z onto Z, with $S_F^2 = I$, such that the fixed point set of S_F is $(\overline{\mathrm{sp}}\,F) \oplus F^\diamondsuit$ (topological direct sum). A real or complex normed space Z is said to be *weakly facially symmetric* (WFS) if every norm exposed face in Z_1 is symmetric. For each symmetric face F we define contractive projections $P_k(F)$, with $k = 0, 1, 2$ on Z as follows. First $P_1(F) = \frac{1}{2}(I - S_F)$ is the projection on the -1 eigenspace of S_F. Next we define $P_2(F)$ and $P_0(F)$ as the projections of Z onto $\overline{\mathrm{sp}}\,F$ and $F^\diamondsuit$ respectively, so that $P_2(F) + P_0(F) = \frac{1}{2}(I + S_F)$. These projections were called *generalized Peirce projections* in [6] and [7]. A *generalized tripotent* is a projective unit $u \in \mathcal{U}$ with the property that $F := F_u$ is a symmetric face and $S_F^* u = u$ for some choice of symmetry S_F corresponding to F. In this paper we shall call such u *geometric tripotents*, and the projections $P_k(u)$ will be called *geometric Peirce projections*.

We use $\mathcal{GT}$ and $\mathcal{SF}$ to denote the collections of geometric tripotents and symmetric faces respectively, and the map $\mathcal{GT} \ni u \mapsto F_u \in \mathcal{SF}$ is a bijection [7, Proposition 1.6]. For each geometric tripotent u in the dual of a WFS space Z, we shall denote the geometric Peirce projections by $P_k(u) = P_k(F_u)$, for $k = 0, 1, 2$. Also we let $U := Z^*$, $Z_k(u) = Z_k(F_u) := P_k(u)Z$ and $U_k(u) = U_k(F_u) := P_k(u)^*(U)$, so that $Z = Z_2(u) + Z_1(u) + Z_0(u)$ and $U = U_2(u) + U_1(u) + U_0(u)$. A symmetry corresponding to the symmetric face F_u will sometimes be denoted by S_u.

Elements a and b of U are *orthogonal* if one of them belongs to $U_2(u)$ and the other to $U_0(u)$ for some geometric tripotent u. Two geometric tripotents u and v are said to be *compatible* if their associated geometric Peirce projections commute, that is $[P_k(u), P_j(v)] = 0$ for $k, j \in \{0, 1, 2\}$.

A contractive projection Q on a normed space X is said to be *neutral* if for each $\xi \in X$, $\|Q\xi\| = \|\xi\|$ implies $Q\xi = \xi$. A normed space Z is *neutral* if for every symmetric face F, the projection $P_2(F)$ corresponding to some choice of symmetry S_F, is neutral.

If Y is a closed subspace of a normed space Z, the collections of norm exposed faces and symmetric faces in ∂Y_1, will be denoted by $\mathscr{F}_Y$ and $\mathscr{S}\mathscr{F}_Y$ respectively. We define $\mathscr{U}_Y$, $\mathscr{G}\mathscr{T}_Y$ similarly.

A WFS space Z is *strongly facially symmetric* (SFS) if for every norm exposed face F in Z_1 and every $y \in Z^*$ with $\|y\| = 1$ and $F \subset F_y$, we have $S_F^* y = y$, where S_F denotes a symmetry associated with F. Let Z be a strongly facially symmetric space and suppose that $u, v \in \mathscr{G}\mathscr{T}_Z$. If $F_u \subset F_v$, we write $u \leq v$.

In a neutral strongly facially symmetric space Z, every non-zero element has a polar decomposition [7, Theorem 4.3]: for $0 \neq f \in Z$ there exists a unique geometric tripotent $v = v(f)$ with $f(v) = \|f\|$ and $\langle v, \{f\}^{\diamond} \rangle = 0$. If $f, g \in Z$, then $f \diamond g$ if and only if $v(f) \diamond v(g)$, as follows from [6, Corollary 1.3(b) and Lemma 2.1]. Recall that the geometric tripotent $v = v(f)$ has the following additional properties if f is a unit vector. First,

$$F_v \text{ is the smallest norm exposed face containing } f; \qquad (15)$$

and second, f is positive and faithful on $U_2(v)$ in the following sense:

$$\text{for any } u, w \in \mathscr{G}\mathscr{T} \text{ with } u \leq w \text{ and } f \in F_w, f(u) = \|P_2(u)f\| \; (\geq 0); \qquad (16)$$

$$\text{for any } u \in \mathscr{G}\mathscr{T} \text{ with } u \leq v, f(u) = \|P_2(u)f\| > 0. \qquad (17)$$

Another property of the polar decomposition is that if $f \in Z_k(w)$ for some $w \in \mathscr{G}\mathscr{T}$, then $v(f) \in U_k(w)$. This follows from the uniqueness of the polar decomposition as we now show.

PROPOSITION 2.1. *Let Z be a neutral SFS space and let $v \in \mathscr{G}\mathscr{T}$. Then for $k \in \{0, 1, 2\}$,*

$$\mathscr{G}\mathscr{T}_{Z_k(v)} = \mathscr{G}\mathscr{T} \cap U_k(v).$$

Proof. Let $u \in \mathscr{G}\mathscr{T}_Z \cap U_k(v)$. By [7, Theorem 3.6, Proposition 4.1], $Z_k(v)$ is neutral and SFS. Hence to prove that $u \in \mathscr{G}\mathscr{T}_{Z_k(v)}$ we need only show that

$$\langle u, (F_u \cap Z_k(v))^{\diamond} \cap Z_k(v) \rangle = 0.$$

By [7, Lemma 3.5], this reduces to $\langle u, F_u^{\diamond} \cap Z_k(v) \rangle = 0$, which is obvious since $\langle u, F_u^{\diamond} \rangle = 0$. Thus $\mathscr{G}\mathscr{T}_{Z_k(v)} \supset \mathscr{G}\mathscr{T} \cap U_k(v)$.

Conversely, let $w \in \mathscr{G}\mathscr{T}_{Z_k(v)}$ so that we have $\|w\| = 1$, $w \in U_k(v)$, and $\langle w, (F_w \cap Z_k(v))^{\diamond} \cap Z_k(v) \rangle = 0$. We must prove that $\langle w, F_w^{\diamond} \rangle = 0$. Again, by [7, Lemma 3.5], $\langle w, F_w^{\diamond} \cap Z_k(v) \rangle = 0$. Now let $\rho \in F_w^{\diamond}$. Then $\langle w, \rho \rangle = \langle P_k(v)^* w, \rho \rangle = \langle w, P_k(w)\rho \rangle$ and $P_k(v)\rho \in F_w^{\diamond} \cap Z_k(v)$, since by compatibility of w and v, $P_k(v)\rho = P_k(v)P_0(w)\rho = P_0(w)P_k(v)\rho$. Therefore $\langle w, \rho \rangle = \langle w, P_k(v)\rho \rangle = 0$, that is, $\langle w, F_w^{\diamond} \rangle = 0$. Thus $\mathscr{G}\mathscr{T}_{Z_k(v)} \subset \mathscr{G}\mathscr{T} \cap U_k(v)$.

COROLLARY 2.2. *If $f \in Z_k(v)$, then $v(f) \in U_k(v)$.*

Proof. Since $Z_k(v)$ is a neutral SFS space and $f \in Z_k(v)$, there is a unique $w \in \mathcal{GT}_{Z_k(v)}$ with $f(w) = \|f\|$ and $\langle w, \{f\}^\diamond \cap Z_k(v) \rangle = 0$. Now $v(f) \in \mathcal{GT}_Z$ is uniquely determined by the conditions $f(v(f)) = \|f\|$ and $\langle v(f), \{f\}^\diamond \rangle = 0$. By the proposition, $w \in \mathcal{GT}_Z$ so it suffices to show that $\langle w, \{f\}^\diamond \rangle = 0$ in order to conclude that $v(f) = w \in U_k(v)$.

For $k \in \{0, 1, 2\}$, we have $P_k(v)\{f\}^\diamond \subset \{f\}^\diamond$. Indeed, for $k = 1$, $P_1(v) = \frac{1}{2}(I - S_v)$ and $S_v\{f\}^\diamond = \{S_v f\}^\diamond = \{-f\}^\diamond = \{f\}^\diamond$; and since $P_2(v) + P_0(v) = \frac{1}{2}(I + S_v)$, we have $(P_2(v) + P_0(v))\{f\}^\diamond \subset \{f\}^\diamond$. Then, by [7, Remarks 1.3 and 3.2], for $k = 0$ or 2, $P_k(v)\{f\}^\diamond \subset \{f\}^\diamond$. To complete the proof, note that $\langle w, \{f\}^\diamond \rangle = \langle P_k(v)^* w, \{f\}^\diamond \rangle = \langle w, P_k(v)\{f\}^\diamond \rangle \subset \langle w, \{f\}^\diamond \cap Z_k(v) \rangle = 0$.

The following is part of a joint Peirce decomposition. We believe that in the complex case, equality holds in equations (18) and (19), but we have been unable to prove it (equality does *not* hold in the real case). The inclusions proved in Lemma 2.3 are sufficient for the results proved in the present paper.

LEMMA 2.3. *Let Z be a neutral WFS space, and let w and u be a pair of orthogonal geometric tripotents. Then*

$$F_{w+u} \subset Z_2(w) + Z_2(u) + Z_1(w) \cap Z_1(u), \tag{18}$$

and therefore

$$U_2(w + u) \subset U_2(w) + U_2(u) + U_1(w) \cap U_1(u). \tag{19}$$

Proof. Since S_w^* fixes $U_2(w) + U_0(w)$, we have $S_w^*(u + w) = u + w$. Thus $S_w(F_{u+w}) \subset F_{u+w}$, and therefore $(P_2(w) + P_0(w))F_{u+w} = \frac{1}{2}(I + S_w)F_{u+w} \subset F_{u+w}$.

For $\varphi \in F_{u+w}$,

$$1 = \langle P_2(w)\varphi + P_0(w)\varphi, u + w \rangle = \langle P_2(w)\varphi, w \rangle + \langle P_0(w)\varphi, u \rangle$$

$$\leq |\langle P_2(w)\varphi, w \rangle| + |\langle P_0(w)\varphi, u \rangle| \leq \|P_2(w)\varphi\| + \|P_0(w)\varphi\|$$

$$= \|(P_2(w) + P_0(w))\varphi\| \leq \|\varphi\| = 1.$$

Thus, $\|P_2(u)P_0(w)\varphi\| = \|P_0(w)\varphi\|$, so by neutrality, $P_2(u)P_0(w)\varphi = P_0(w)\varphi$. Therefore, $P_2(u)\varphi = P_2(u)P_0(w)\varphi = P_0(w)\varphi$, and by symmetry $P_2(w)\varphi = P_0(u)\varphi$. Since

$$\varphi = P_2(w)\varphi + P_1(w)\varphi + P_0(w)\varphi = P_2(u)\varphi + P_1(u)\varphi + P_0(u)\varphi,$$

we have $P_1(w)\varphi = P_1(u)\varphi$. Therefore, $\varphi \in Z_2(w) + Z_1(\omega) \cap Z_1(u) + Z_2(u)$. This proves (18), and (19) follows by duality: with

$$P = P_2(w + u), \quad Q = P_2(w) + P_2(u) + P_1(w)P_1(u),$$

we have $PQ = QP = P$, so $P^*Q^* = Q^*P^* = P^*$.

Let $\mathcal{M}$ denote the collection of minimal geometric tripotents of U, that is, $\mathcal{M} = \{v \in \mathcal{GT}: U_2(v)$ is one-dimensional$\}$. In the setting of a JBW*-triple and its predual, there is a one-to-one correspondence between minimal tripotents and extreme points of the unit ball of the predual. Moreover, these extreme points are all norm exposed (cf. [5, Proposition 4]). The following proposition is a generalization of this correspondence.

PROPOSITION 2.4. *If Z is WFS, then there is a bijection η of $\exp Z_1$ onto $\mathcal{M}$. If Z is neutral and SFS, then $\eta(f) = v(f)$, the geometric tripotent occurring in the polar decomposition of f. For $v = \eta(f) \in \mathcal{M}$, with $f \in \exp Z_1$,*

$$P_2(v)^*x = f(x)v \text{ for } x \in U, \quad \text{and} \quad P_2(v)g = g(v)f \text{ for } g \in Z.$$

Proof. If $f \in \exp Z_1$, then $\{f\} = F_v$ for some unique $v \in \mathcal{GT}$, which we denote by $\eta(f)$. Since $\operatorname{sp} F_v$ is one-dimensional, $U_2(v) \cong (\operatorname{sp} F_v)^*$ is one-dimensional, so that $v \in \mathcal{M}$.

Conversely, let $v \in \mathcal{M}$. Then $\operatorname{sp} F_v$ is one-dimensional, so that F_v is a point. Hence, η is surjective.

If $v = \eta(f) = \eta(g)$, with $f, g \in \exp Z_1$, then $\{f\} = F_v = \{g\}$, and η is one-to-one.

To prove the last statement, note that since $P_2(v)^*x = \lambda(x)v$ and $P_2(v)g = \mu(g)f$ for some scalars $\lambda(x), \mu(g)$, we have

$$\lambda(x)g(v) = \langle P_2(v)^*x, g \rangle = \langle x, P_2(v)g \rangle = \langle x, f \rangle \mu(g).$$

Put $x = v$ to get $g(v) = \mu(g)$ and $\lambda(x) = f(x)$.

DEFINITION 2.5. A normed space Z is said to be *atomic* if every symmetric face of Z_1 has an extreme point.

Let $\mathscr{I}$ denote the collection of indecomposable geometric tripotents of U, that is,

$$\mathscr{I} = \{v \in \mathcal{GT} : u \in \mathcal{GT}, u \leq v \Rightarrow u = v\}.$$

In general, $\mathcal{M} \subset \mathscr{I}$, and we shall show in Proposition 2.9 below that under a mild assumption, equality holds.

DEFINITION 2.6. A neutral SFS space Z is said to satisfy the axiom (PE) (for 'point exposed') if $\exp Z_1 = \operatorname{ext} Z_1$.

It is easy to see that being atomic is equivalent, in SFS spaces satisfying (PE), to the assertion that every $u \in \mathcal{GT}$ is greater than or equal to some v in $\mathcal{M}$.

PROPOSITION 2.7. *If Z is an atomic SFS space satisfying (PE), then*

(a) $U = \overline{\operatorname{sp}} \mathcal{M}$ *(weak*-closure)*;

(b) $Z_1 = \overline{\operatorname{co}} \operatorname{ext} Z_1$ *(norm closure)*.

Proof. Observe first that if Z is an atomic SFS space, then the set

$$\{x \in U : \langle x, f \rangle = \|x\| \neq 0 \text{ for some } f \in \operatorname{ext} Z_1\}$$

is norm dense in U. Indeed, by the Bishop–Phelps theorem for the bounded closed convex set Z_1 [2, p. 45], the set of elements $x \in U := Z^*$ which assume their norm on some element $f \in Z_1$, that is, $\langle x, f \rangle = \|x\|$, is norm dense in U. But for all such x, $F_{x/\|x\|} \neq \varnothing$ so there exists $\varphi \in \operatorname{ext} Z_1 \cap F_{x/\|x\|}$ with $\langle x, \varphi \rangle = \|x\|$.

Let $f \in Z$, and suppose $f(\mathcal{M}) = 0$. Then $v(f) \in \mathcal{GT}$ so there exists $u \in \mathcal{M}$ with $u \leq v(f)$. By (17), $f(u) > 0$, a contradiction. This proves (a).

Let K denote $\overline{\operatorname{co}} \operatorname{ext} Z_1$. If $K \neq Z_1$, let $f \in Z_1, f \notin K$. Choose $y \in U$ and real

numbers c, $c + \varepsilon$, such that

$$\Re\langle y, \psi \rangle < c < c + \varepsilon \leqslant \Re\langle y, f \rangle$$

for all $\psi \in K$.

By the above observation, there are $x \in U$, $\varphi \in \text{ext } Z_1$ such that $\|x - y\| < \delta$ and $\langle x, \varphi \rangle = \|x\|$. Thus for sufficiently small δ,

$$\|x\| = \Re\langle x, \varphi \rangle \leqslant c + \tfrac{1}{4}\varepsilon < c + \tfrac{3}{4}\varepsilon \leqslant \Re\langle x, f \rangle \leqslant \|x\|,$$

a contradiction. This proves (b).

We now introduce the principal axioms, which are geometric and physically significant properties of the state space of a physical system.

DEFINITION 2.8. Let f and g be extreme points of the unit ball of a neutral SFS space Z. The *transition probability* of f and g is the number

$$\langle f \,|\, g \rangle := f(v(g)).$$

A neutral SFS space Z is said to satisfy '*symmetry of transition probabilities*' (STP) if for every pair of extreme points $f, g \in \text{ext } Z_1$, we have

$$\overline{\langle f \,|\, g \rangle} = \langle g \,|\, f \rangle,$$

where in the case of complex scalars, the bar denotes conjugation.

For example, in a concrete spin factor, the transition probability coincides with the restriction of the inner product (12) to extremal elements.

PROPOSITION 2.9. *Let Z be an atomic neutral SFS space satisfying (PE). Then $\mathcal{M} = \mathcal{I}$. If furthermore Z satisfies (STP), then the map π: sp ext $Z_1 \to$ sp $\mathcal{M}$ defined by*

$$\pi\left(\sum_{j=1}^{n} z_j \varphi_j\right) = \sum_{j=1}^{n} \bar{z}_j v(\varphi_j)$$

is well-defined and a conjugate-linear bijection.
Denoting this extension also by π, we find that the scalar-valued map

$$\langle \cdot \,|\, \cdot \rangle \colon (f, g) \in Z \times Z \mapsto \langle f, \pi(g) \rangle \tag{20}$$

is a continuous symmetric sesquilinear form.

Proof. Let $v \in \mathcal{I}$ and let φ be an extreme point of F_v. Then φ is an extreme point of Z_1 and so $\{\varphi\} = F_u$ for some $u \in \mathcal{M}$, by Proposition 2.4. Since $\varphi \in F_v$, $F_u \subset F_v$ so $u \leqslant v$ and therefore $u = v$, that is, sp F_v is one-dimensional, and $v \in \mathcal{M}$. Thus $\mathcal{I} = \mathcal{M}$.

By (STP), for all ψ, $\varphi_1, \ldots, \varphi_n \in \text{ext } Z_1$ and scalars $z_1, \ldots, z_n$,

$$\left\langle \sum z_j \varphi_j, v(\psi) \right\rangle = \sum z_j \overline{\psi(v(\varphi_j))} = \overline{\sum \bar{z}_j \psi(v(\varphi_j))} = \overline{\psi\left(\sum \bar{z}_j v(\varphi_j)\right)}.$$

Therefore

$$\sum z_j \varphi_j = 0 \iff \left\langle \psi, \sum \bar{z}_j v(\varphi_j) \right\rangle = 0 \text{ for all } \psi \in \text{ext } Z_1 \iff \sum \bar{z}_j v(\varphi_j) = 0.$$

Finally, by (STP), the map $\langle \cdot \mid \cdot \rangle$ is symmetric, that is, $\overline{\langle f \mid g \rangle} = \langle g \mid f \rangle$, on the linear span of extreme points, and since it is obviously sesquilinear and continuous, the last statement follows.

As a consequence of the sesquilinear form just introduced, we are now able to handle the rank-1 case.

DEFINITION 2.10. A normed space Z is said to be of *rank* 1 if no two non-zero elements of Z are orthogonal.

In an atomic neutral SFS space of rank 1, satisfying (PE), every geometric tripotent is indecomposable, and hence minimal. Thus, every element f in such a space is a multiple of an extreme point of Z_1. It follows that the sesquilinear form $\langle \cdot \mid \cdot \rangle$ is positive definite and moreover, if $f \in Z$ is non-zero, then $f/\|f\|_Z$ is an extreme point and so has norm 1 in the inner product space determined by $\langle \cdot \mid \cdot \rangle$. Therefore, $\|f\|_Z = \|f\|_2$ and Z is a Hilbert space. Furthermore, for $\|f\| = 1$, by Proposition 2.4, $P_2(f)$ is the orthogonal projection onto the span of f in this Hilbert space.

Two geometric tripotents u, w are said to be *colinear* if each of them belongs to the geometric Peirce 1-space of the other, that is, $u \in U_1(w)$ and $w \in U_1(u)$. Thus, in a rank-1 space, since all geometric Peirce 0-projections are zero, colinearity of $v(f)$ and $v(g)$ is equivalent to $P_2(f)g = 0$, that is, orthogonality in the Hilbert space structure. Therefore orthonormal bases in the Hilbert space structure correspond to maximal families of mutually colinear geometric tripotents. Hence we have:

COROLLARY 2.11. *Let Z be an atomic neutral SFS space satisfying (PE) and (STP). Assume that Z is of rank 1. Then Z is linearly isometric with a real or complex Hilbert space. Moreover, there is a one-to-one correspondence between orthonormal bases in this Hilbert space and maximal families of mutually colinear geometric tripotents.*

3. Rank-two faces in facially symmetric spaces

We now begin the study of rank-2 faces, culminating in Theorem 3.8 (the Hilbert ball property). This property will be a basic tool in the following section.

DEFINITION 3.1. Let v and $\tilde{v}$ be orthogonal minimal geometric tripotents in a neutral strongly facially symmetric space Z. The norm exposed face $F_{v+\tilde{v}}$ determined by the geometric tripotent $v + \tilde{v}$ will be called a *face of rank* 2. Let $v = v(f)$ and $\tilde{v} = v(\tilde{f})$ for orthogonal extreme points $f, \tilde{f}$ of Z_1. Then $\xi := \frac{1}{2}(f + \tilde{f})$ is called the centre of $F_{v+\tilde{v}}$ and we shall write $F_{v+\tilde{v}} = F_\xi$.

It will be shown in Theorem 3.8 that $F_{v+\tilde{v}}$ is a Hilbert ball with centre ξ.

The following proposition justifies the terminology 'rank 2', that is, meaning that there are no more than two orthogonal extreme points in such a face. If the face represents the state space of a quantum mechanical system, the proposition implies that for such systems no observable can assume more than two distinct values.

PROPOSITION 3.2. *Let v and $\bar{v}$ be orthogonal minimal geometric tripotents in an atomic neutral strongly facially symmetric space Z, and assume the axioms (PE) and (STP). Let $v = v(f)$ and $\bar{v} = v(\bar{f})$ for orthogonal extreme points $f, \bar{f}$ of Z_1. If ρ and σ are orthogonal elements of $F_{v+\bar{v}}$, then $v(\rho) + v(\sigma) = v + \bar{v}$, ρ and σ are extreme points, and $\rho + \sigma = f + \bar{f}$. Moreover, each norm exposed face of Z_1, which is properly contained in $F_{v+\bar{v}}$, is a point, and if ρ is an extreme point of $F_{v+\bar{v}}$, then $v + \bar{v} - v(\rho)$ is a minimal geometric tripotent.*

Proof. We show first that if ρ and σ are orthogonal extreme points of the rank-2 face $F_{v+\bar{v}}$, then $v(\rho) + v(\sigma) = v + \bar{v}$. Since $\rho \in F_{v+\bar{v}}$, we have by Lemma 2.3,

$$\rho = \rho(v)f + (1 - \rho(v))\bar{f} + h$$

for some $h \in Z_1(v) \cap Z_1(\bar{v})$. Moreover, by (16), $\alpha := \rho(v) \geq 0$. By Lemma 2.3, since $v(\rho) \in U_2(v + \bar{v})$,

$$v(\rho) = av + b\bar{v} + v_1,$$

for some $v_1 \in U_1(v) \cap U_1(\bar{v})$, and scalars a, b.

By (STP), $\alpha = \rho(v) = f(v(\rho)) = a$ and $1 - \alpha = b$, so that

$$v(\rho) = \alpha v + (1 - \alpha)\bar{v} + v_1.$$

Similarly

$$\sigma = \gamma f + (1 - \gamma)\bar{f} + h'$$

and

$$v(\sigma) = \gamma v + (1 - \gamma)\bar{v} + v_1',$$

for some $h' \in Z_1(v) \cap Z_1(\bar{v})$, $v_1' \in U_1(v) \cap U_1(\bar{v})$, and $\gamma \geq 0$.

Since $\rho \lozenge \sigma$, $v(\rho) + v(\sigma) \in \mathcal{GF}$ and therefore has norm 1. Since

$$v(\rho) + v(\sigma) = (\alpha + \gamma)v + v_1 + v_1' + (2 - \alpha - \gamma)\bar{v},$$

and since $P_2(v)^*$ is contractive, we have $\alpha + \gamma \leq 1$ and similarly $2 - \alpha - \gamma \leq 1$. Hence $\alpha + \gamma = 1$ and

$$v(\rho) + v(\sigma) = v + (v_1 + v_1') + \bar{v}.$$

With $x := v + (v_1 + v_1') + \bar{v}$, we have $f \in F_x$, so that $F_v = \{f\} \subset F_x$. Therefore, by strong facial symmetry, $P_1(v)^*x = 0$, that is, $v_1 + v_1' = 0$. This proves that $v(\rho) + v(\sigma) = v + \bar{v}$.

We can now easily complete the proof of the proposition. Since Z is atomic, there exist extreme points g and $\bar{g}$ such that $F_{v(g)} \subset F_{v(\rho)}$ and $F_{v(\bar{g})} \subset F_{v(\sigma)}$. By the previous paragraph, $v + \bar{v} = v(g) + v(\bar{g}) \leq v(\rho) + v(\sigma) \leq v + \bar{v}$, and ρ and σ are extreme points.

If $F_w \subset F_{v+\bar{v}}$ is not a point, then it contains an extreme point φ, which by assumption is norm exposed. Since $v(\varphi) \leq w$, we have $F_{w-v(\varphi)} \lozenge F_{v(\varphi)}$. Thus by the previous paragraph, $F_w = F_{v+\bar{v}}$.

Finally, $v + \bar{v} - v(\rho)$ is the unique minimal geometric tripotent orthogonal to $v(\rho)$. Therefore, $\pi^{-1}(v + \bar{v} - v(\rho)) = f + \bar{f} - \rho$ is the unique extreme point orthogonal to ρ, where π is defined in Proposition 2.9.

To proceed further, we need to replace the property (PE) by an apparently stronger property (FE). It is very likely that (PE) is equivalent to (FE) in facially symmetric spaces, since (FE) holds in the case of the predual of a JBW*-triple [4], but we have been unable to prove this.

DEFINITION 3.3. A neutral SFS space Z is said to satisfy property (FE) if every norm closed face of Z_1 different from Z_1 is a norm exposed face.

An important consequence of this axiom is the following Krein–Milman type result.

PROPOSITION 3.4. *Let Z be an atomic neutral WFS space which satisfies the axiom (FE), and let u be a geometric tripotent. Then*

$$F_u = \overline{co} \text{ ext } F_u \quad (norm \ closure).$$

Proof. Let $K := F_u$ and $K_1 := \overline{co} \text{ ext } F_u \subset K$. If $K_1 \neq K$ then there is an $x \in U_2(u)$ and $c \in \mathbb{R}$ such that

$$\sup \mathfrak{R}\langle x, K_1 \rangle < c < \sup \mathfrak{R}\langle x, K \rangle. \tag{21}$$

By the Bishop–Phelps Theorem for the convex set K [2, p. 45], we have, for any $\varepsilon > 0$, a $\rho \in K$ and $y \in U_2(u)$ such that $\|x - y\| < \varepsilon$ and $r := \sup \mathfrak{R}\langle y, K \rangle = \mathfrak{R}\langle y, \rho \rangle$. By taking ε small enough, we may assume that (21) holds with y in place of x.

The set $G := F_u \cap \{\varphi \in Z_2(u): \mathfrak{R}\langle y, \varphi \rangle = r\}$ is a non-empty norm closed face in F_u, and is therefore norm exposed and symmetric by our assumptions on Z. Since Z is atomic, G has an extreme point, which is then automatically an extreme point of F_u. Because (21) holds with y in place of x, this extreme point is not in K_1, a contradiction.

COROLLARY 3.5. *Let Z be an atomic neutral strongly facially symmetric space satisfying (FE) and (STP), and let F_ξ be a rank-2 face. Then $F_\xi - \xi$ is a symmetric convex set.*

Proof. For any $g \in F_\xi$, define $S(g - \xi) = -(g - \xi)$. From Proposition 3.2, for any extreme point $\rho \in F_\xi$, there exists an extreme point $\bar{\rho} \in F_\xi$, orthogonal to ρ such that $S(\rho - \xi) = \bar{\rho} - \xi$. By the proposition, S is a symmetry of $F_\xi - \xi$.

The following theorem is the main result of this section. It gives the structure of rank-2 faces. We shall use the following lemma in its proof.

LEMMA 3.6. *Let v and $\bar{v}$ be orthogonal minimal geometric tripotents in an atomic neutral strongly facially symmetric space Z, and assume the properties (FE), and (STP). Let $v = v(f)$ and $v(\bar{f})$ for orthogonal extreme points $f, \bar{f}$ of Z_1. If $[\xi, \rho]$ denotes the line segment from ξ to ρ, then*

$$F_{v+\bar{v}} = \bigcup \{[\xi, \rho]: \ \rho \in \text{ext } F_{v+\bar{v}}\}. \tag{22}$$

Proof. By Corollary 3.5, $F_\xi - \xi$ is the closed unit ball Y_1 of a real space $Y \subset Z$ with respect to the norm $\|\cdot\|_Y$ given by the Minkowski functional of $F_\xi - \xi$. Let φ

be an arbitrary point of F_ξ, and let ρ be the point where the ray containing φ and emanating from ξ leaves F_ξ, that is, with $\varphi_t = t\varphi + (1-t)\xi$ for $t \geq 0$, $\rho = \varphi_{t_0}$ for some t_0 (necessarily greater than or equal to 1) with $\varphi_t \in F_\xi$ if and only if $t \leq t_0$. By the definition of the Minkowski functional, $\|\rho - \xi\|_Y = 1$. Let Φ be a continuous linear functional on Y such that, with Y_0 equal to the open unit ball of Y,

$$\Phi(Y_0) < c := \Phi(\rho - \xi), \tag{23}$$

and let J be the hyperplane in Y determined by c, that is, $J = \{\sigma \in Y : \Phi(\sigma) = c\}$. Then $\rho - \xi \in G := J \cap (F_\xi - \xi)$ and G is a closed face of $F_\xi - \xi$. Thus $\rho \in G + \xi$ and $G + \xi$ is a closed face of F_ξ. Therefore, $G + \xi$ is a closed face of Z_1, which by our assumption, is norm exposed. By Proposition 3.2, either $G + \xi = \{\rho\}$ so ρ is an extreme point, or $G + \xi = F_\xi$. In the latter case, $F_\xi - \xi = G = J \cap (F_\xi - \xi) \subset J$, contradicting (23). Thus ρ is an extreme point, and $\varphi \in [\xi, \rho]$.

In the following corollary, we obtain an abstract analogue of the facial decomposition ((i), (iii) and (iv) of Proposition 1.11). In the abstract setting of Lemma 3.6 a centre of any rank-2 face F will be called a *unitary* element of Z. For example, $\frac{1}{2}(f + \tilde{f})$ is a unitary if f and $\tilde{f}$ are any two orthogonal extreme points of F.

COROLLARY 3.7. *Let* $\varphi \in F_{v+\bar{v}}$. *Then there exists an element* $h \in Z$ *which is a multiple of a unitary element such that* $\varphi = \xi + h$, $\langle \xi \mid h \rangle = 0$, *and* $\|h\|_Z \leq 1$. *Moreover,* $\|\varphi - \xi\| \leq 1$, *and* $\varphi \in \text{ext } F_{v+\bar{v}}$ *if and only if* $\|\varphi - \xi\| = 1$.

Proof. Let $F = F_{v+\bar{v}}$ and write, by (22), $\varphi = \xi + t(\rho - \xi)$. Then $\|h\| = t \in [0, 1]$, ξ and $\rho - \xi = \frac{1}{2}(\rho - \bar{\rho})$ are unitary elements, and $\langle \xi \mid h \rangle = \frac{1}{4}\langle \rho + \bar{\rho} \mid \rho - \bar{\rho} \rangle = 0$. Also, $\|\varphi - \xi\| = t \leq 1$. If φ is an extreme point, then $\|\varphi - \xi\| = \|\frac{1}{2}(\varphi - \bar{\varphi})\| = 1$. Finally, if $\varphi \in F$ and $\|\varphi - \xi\| = 1$, we have $1 = \|\varphi - \xi\| = t$ and $\varphi = \rho$ is extreme.

THEOREM 3.8 (Hilbert ball property). *Let* v *and* $\bar{v}$ *be orthogonal minimal geometric tripotents in an atomic neutral strongly facially symmetric space* Z, *and assume the properties* (FE), *and* (STP). *Let* $v = v(f)$ *and* $\bar{v} = v(\tilde{f})$ *for orthogonal extreme points* $f, \tilde{f}$ *of* Z_1. *Then* $F_\xi - \xi$ *is the unit ball of a real subspace* Y *of* Z *on which the sesquilinear form* (20) *is a real inner product whose associated Hilbertian norm is a multiple* $(1/\sqrt{2})$ *of the norm of* Z. *Precisely,*

$$Y = \{t(\rho - \xi) : t \geq 0, \rho \in \text{ext } F_{v+\bar{v}}\}$$

and if $(\cdot \mid \cdot)$ *denotes* $2\langle \cdot \mid \cdot \rangle$, *then*

$$(t(\rho - \xi) \mid s(\tau - \xi)) = ts[2\langle \tau \mid \rho \rangle - 1] \tag{24}$$

for $\tau, \rho \in \text{ext } F_{v+\bar{v}}$, *and* $t, s \geq 0$, *where* $\langle \tau \mid \rho \rangle$ *denotes the transition probability.*

Proof. Let Y be the real normed space in Z whose unit ball is $F_\xi - \xi$. We first show that the sesquilinear form $\langle \cdot \mid \cdot \rangle$ is positive definite on Y.

If $\varphi \in F_\xi$ and $\|\varphi - \xi\|_Y = 1$, then as shown in the proof of Lemma 3.6, φ is an extreme point, and by Corollary 3.7, $\|\varphi - \xi\|_Z = 1$. Therefore, if $h \in Y$ is non-zero, then $h/\|h\|_Y \in F_\xi - \xi$ and, as just shown, $\|h/\|h\|_Y\|_Z = 1$, that is, $\|\cdot\|_Y = \|\cdot\|_Z$ on Y. Moreover, for $h \in Y$, with $h/\|h\|_Y = \varphi - \xi$ for some extreme

point $\varphi \in F_\xi$, we have $\langle h \mid h \rangle = \frac{1}{4}\|h\|_Y^2 \langle \varphi - \tilde{\varphi} \mid \varphi - \tilde{\varphi} \rangle = \frac{1}{2}\|h\|_Y^2$. Thus, on Y, $\langle \cdot \mid \cdot \rangle$ is positive definite, and therefore an inner product with $\|h\|_2 = (1/\sqrt{2})\|h\|_Y = (1/\sqrt{2})\|h\|_Z$.

We complete the proof by computing the inner product, thereby showing it to be real and to satisfy (24). For $h_1, h_2 \in Y$, with $h_j/\|h_j\|_Y = \varphi_j - \xi$ for extreme points $\varphi_j \in F_\xi$, we have

$$(\varphi_1 - \xi \mid \varphi_2 - \xi) = 2\langle \varphi_1 - \xi, \pi(\tfrac{1}{2}(\varphi_2 - \tilde{\varphi}_2))\rangle = \langle \varphi_1 \mid \varphi_2 \rangle - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{2},$$

and therefore

$$(h_1 \mid h_2) = \|h_1\|_Y \|h_2\|_Y (\varphi_1 - \xi \mid \varphi_2 - \xi) = \|h_1\|_Y \|h_2\|_Y [2\langle \varphi_1 \mid \varphi_2 \rangle - 1].$$

This proves (24) and by (15) and (16), the inner product is real. Finally, for $h \in Y$, $(h \mid h) = 2\langle h \mid h \rangle = 2\|h\|_2^2 = \|h\|_Y^2$.

COROLLARY 3.9. *The unit ball (in Z) of the real span of $F_{v+\tilde{v}}$ is a cylinder with base the Hilbert ball Y. Precisely,*

$$(\mathrm{sp}_{\mathbb{R}}\, F_{v+\tilde{v}})_1 = \{\alpha\xi + \beta h: \ \alpha, \beta \in \mathbb{R}, |\alpha|, |\beta| \leq 1, \xi \pm h \in \mathrm{ext}\, F_{v+\tilde{v}} \ (\text{if } \beta \neq 0)\}.$$

Proof. Let F denote $F_{v+\tilde{v}}$. Then $\varphi \in \mathrm{sp}_{\mathbb{R}}\, F$ if and only if for some $a, b \geq 0$ and $\sigma, \tau \in F$,

$$\varphi = a\tau - b\sigma = [a(\tau - \xi) - b(\sigma - \xi)] + (a - b)\xi = \beta h + \alpha\xi,$$

where, with $\psi = a(\tau - \xi) - b(\sigma - \xi)$, we have set $h = \psi/\|\psi\|$, $\beta = \|\psi\|$, and $\alpha = a - b$. Moreover, with $\rho^\pm = \xi \pm h$,

$$\varphi = \tfrac{1}{2}(\alpha + \beta)\rho^+ + \tfrac{1}{2}(\alpha - \beta)\rho^-,$$

so

$$\|\varphi\| = |\tfrac{1}{2}(\alpha + \beta)| + |\tfrac{1}{2}(\alpha - \beta)| = \max(|\alpha|, |\beta|).$$

The following notation will be used in the sequel. The real Hilbert space Y determined in Theorem 3.8 by the face $F_{v+\tilde{v}}$ will be denoted by $\mathcal{H}_{v+\tilde{v}}$ or by $\mathcal{H}_\xi$, since it is uniquely determined by ξ or by $v + \tilde{v}$. The Hilbert ball $F_{v+\tilde{v}} - \xi$, which is the unit ball of this Hilbert space, will be denoted by $B_{v+\tilde{v}}$ or B_ξ. An orthonormal set in $\mathcal{H}_\xi$ will be denoted by $\{e_\alpha\}$ or $\{e_1, e_2, \dots\}$.

DEFINITION 3.10. A neutral strongly facially symmetric space is said to be of *type I_2* if it is of the form $Z_2(F)$, where F is a face of rank 2 (cf. Definition 3.1).

Our main theorem, Theorem 4.16 below, states that an atomic neutral strongly facially symmetric space over $\mathbb{C}$ of *type I_2* which satisfies (FE) and (STP) is linearly isometric to the predual of a spin factor.

The following remark follows from the fact that the inner product above is determined globally by use of the mapping π.

REMARK 3.11. Let F_ξ and F_η be faces of rank 2 so that $\mathcal{H}_\xi$ and $\mathcal{H}_\eta$ are defined. If $a, b \in \mathcal{H}_\xi \cap \mathcal{H}_\eta$, then the inner product $(a \mid b)$ is the same whether computed in $\mathcal{H}_\xi$ or in $\mathcal{H}_\eta$.

REMARK 3.12. Formula (24) has the following physical interpretation. Consider the state space of a spin-$\frac{1}{2}$ particle, where we are concerned only with the directions of spin, which are unit vectors in $\mathbb{R}^3$. Identify the Hilbert space $\mathcal{H}$ with $\mathbb{R}^3$. If spin is measured in the direction $\rho - \xi$ and is positive, then the pure state ρ represents the state of the particle after the measurement. In this case, by (24), the transition probability between states ρ and τ is given by

$$\langle \rho \mid \tau \rangle = \rho(v(\tau)) = \tfrac{1}{2}(1 + (\rho - \xi \mid \tau - \xi)) = \tfrac{1}{2}(1 + \cos \theta) = \cos^2 \tfrac{1}{2}\theta,$$

where θ is the angle between the two directions.

This is consistent with the known behaviour of spin-$\frac{1}{2}$ particles in quantum mechanics.

REMARK 3.13. For rank-2 faces F the Jordan decomposition holds: for each $\varphi \in \mathrm{sp}_{\mathbb{R}} F$, there exist $\sigma, \tau \in \mathbb{R}^+ F$ such that

$$\varphi = \sigma - \tau \quad \text{and} \quad \|\varphi\| = \|\sigma\| + \|\tau\|.$$

(This implies also that $\sigma \Diamond \tau$ and that the decomposition is unique.) Hence $\mathrm{ext}(\mathrm{sp}_{\mathbb{R}} F)_1 = F \cup -F$.

Proof. Let $\varphi \in \mathrm{sp}_{\mathbb{R}} F$, where F denotes $F_{v+\bar{v}}$. As shown in the proof of Corollary 3.9, $\varphi \in \mathrm{sp}_{\mathbb{R}}\{\rho^+, \rho^-\}$, say $\varphi = a\rho^+ + b\rho^-$, where $\rho^{\pm} := \xi \pm \psi/\|\psi\|$ are two orthogonal extreme points of F. Then $\|\varphi\| = |a| + |b| = \|a\rho^+\| + \|b\rho^-\|$.

4. The complex span of a rank-2 face

In order to describe the complex span of a rank-2 face F_ξ, we shall construct, from an orthonormal basis of $\mathcal{H}_\xi$ and the centre ξ, a natural basis analogous to the dual of a spin grid in a concrete spin factor. This construction is entirely similar to the method of obtaining the spin grid in a concrete spin factor. The main difficulty here, however, is to show that the natural correspondence between the concrete and the constructed grids extends to an isometry. This task, which requires most of this section to carry out, is accomplished by a reduction to the finite-dimensional case. Since the odd and even dimensional spinors behave differently, the cases of dimension 2 and 3 both need to be treated first. This is done in Theorems 4.6 and 4.8.

We assume in the remainder of this paper that Z is a linear space over the *complex* field $\mathbb{C}$. We shall make use of the main result of § 3, namely Theorem 3.8. We therefore make the following standing assumption:

$$Z \text{ is an atomic neutral SFS space satisfying (FE) and (STP)}. \qquad (25)$$

We begin by considering an arbitrary pair of orthogonal unit vectors $\{e_1, e_2\}$ in the Hilbert space $\mathcal{H}_\xi$ defined by a rank-2 face F_ξ, and describe the real span

$$Y_0 := \mathrm{sp}_{\mathbb{R}}\{e_1, e_2, \xi\} \subset Z \qquad (26)$$

of these two vectors and the centre ξ of the face.

By using the main result of § 3, in the following proposition we construct a part of the dual spin grid and show that the extreme points of F_ξ lying in Y have the same description as their analogues in the dual of a concrete spin factor.

PROPOSITION 4.1. *Let Z satisfy* (25). *Let $\{e_1, e_2\}$ be an orthonormal set in the Hilbert space $\mathcal{H}_\xi$ defined by a rank-2 face F_ξ. Define $f_1 = e_1 + \xi$, $\bar{f}_1 = \xi - e_1$ and $h = e_2$. Then $f_1, \bar{f}_1$ are orthogonal extreme points of F_ξ and $h \in Z_1(f_1) \cap Z_1(\bar{f}_1)$. Moreover,*

$$F_\xi \cap Y_0 = \{\tfrac{1}{2}(1 + a)f_1 + bh + \tfrac{1}{2}(1 - a)\bar{f}_1 : a, b \in \mathbb{R}, a^2 + b^2 \leqslant 1\}, \tag{27}$$

$$(\text{ext } F_\xi) \cap Y_0 = \{\varphi_\alpha^\pm : \alpha \in [0, 1]\}, \tag{28}$$

where

$$\varphi_\alpha^\pm = \alpha f_1 + (1 - \alpha)\bar{f}_1 \pm 2\sqrt{(\alpha(1 - \alpha))}\, h, \quad \alpha \in [0, 1]. \tag{29}$$

Also

$$\varphi_\alpha^+ \Diamond \varphi_{1-\alpha}^- \quad for \ \alpha \in [0, 1], \tag{30}$$

$$h = \tfrac{1}{2}(\varphi_{1/2}^+ - \varphi_{1/2}^-) = \tfrac{1}{2}((h + \xi) + (h - \xi)), \quad v(h) = v(\varphi_{1/2}^+) - v(\varphi_{1/2}^-), \tag{31}$$

and

$$v(\varphi_\alpha^\pm) = \alpha v(f_1) + (1 - \alpha)v(\bar{f}_1) \pm \sqrt{(\alpha(1 - \alpha))}\, v(h) \quad for \ \alpha \in [0, 1]. \tag{32}$$

Proof. By Corollary 3.7, f_1 and $\bar{f}_1$ are extreme points of the face F_ξ. Since their average is the centre, by Proposition 3.2 there is an element ψ of the face F_ξ such that $\psi \Diamond f_1$ and $f_1 + \psi = f_1 + \bar{f}_1$. Therefore $\psi = \bar{f}_1$, which proves that f_1 and $\bar{f}_1$ are orthogonal.

From Theorem 3.8, we have

$$F_\xi \cap Y_0 = \{\xi + ae_1 + be_2 : a, b \in \mathbb{R}, a^2 + b^2 \leqslant 1\}$$
$$= \{\tfrac{1}{2}(f_1 + \bar{f}_1) + \tfrac{1}{2}a(f_1 - \bar{f}_1) + bh : a, b \in \mathbb{R}, a^2 + b^2 \leqslant 1\}.$$

This proves (27), and (28) follows by putting $\alpha = \tfrac{1}{2}(1 + a)$ and recalling that unit vectors in a Hilbert space are extreme points.

Note next that $\varphi_\alpha^+ + \varphi_{1-\alpha}^- = f_1 + \bar{f}_1$. As above, by Proposition 3.2 there is an element ψ of the face F_ξ such that $\psi \Diamond \varphi_\alpha^+$ and $\varphi_\alpha^+ + \psi = f_1 + \bar{f}_1$. Therefore $\psi = \varphi_{1-\alpha}^-$, which proves (30).

By the definition (29),

$$h = \tfrac{1}{2}(\varphi_{1/2}^+ - \varphi_{1/2}^-), \tag{33}$$

and (30) implies that $\varphi_{1/2}^+ \Diamond \varphi_{1/2}^-$. Therefore, $v(h) = v(\varphi_{1/2}^+) - v(\varphi_{1/2}^-)$, and by Proposition 2.9, $v(h) = 2\pi(h)$. Thus we have

$$v(\varphi_\alpha^\pm) = \pi(\varphi_\alpha^\pm) = \alpha v(f_1) + (1 - \alpha)v(\bar{f}_1) \pm 2\sqrt{(\alpha(1 - \alpha))}\, \tfrac{1}{2}v(h)$$
$$= \alpha v(f_1) + (1 - \alpha)v(\bar{f}_1) \pm \sqrt{(\alpha(1 - \alpha))}\, v(h).$$

Finally,

$$2P_2(f_1)h = 2P_2(f_1)(\tfrac{1}{2}(\varphi_{1/2}^+ - \varphi_{1/2}^-)) = \langle \varphi_{1/2}^+ | f_1 \rangle f_1 - \langle \varphi_{1/2}^- | f_1 \rangle f_1 = \tfrac{1}{2}f_1 - \tfrac{1}{2}f_1 = 0.$$

Similarly, $P_2(\bar{f}_1)h = 0$. Thus by Lemma 2.3, $P_1(f_1)P_1(\bar{f}_1)h = h$.

Since the unit ball of any normed space is built up from its faces, to show that a map is an isometry, it is necessary to obtain information about faces different from the original one. A natural face to consider first is the one with centre h arising in the previous proposition. Note that by (33), h is unitary and hence is the centre of a face. The next lemma gives a partial result about this face. The full description of this face will be given in Proposition 4.4.

LEMMA 4.2. *With the assumption* (25) *and the notation and assumption of Proposition* 4.1, *let* $u = v(h)$, $v = v(f_1)$ *and* $\bar{v}_1 = v(\bar{f}_1)$. *Then*

$$Z_2(v_1 + \bar{v}_1) = Z_2(u).$$

Proof. Since $f_1 + \bar{f}_1 = \varphi_{1/2}^+ + \varphi_{1/2}^- \in \text{sp}_\mathbb{R} \, F_h$, we have $v_1 + \bar{v}_1 \in U_2(u)$ and therefore, by [7, Theorem 2.3], $U_2(v_1 + \bar{v}_1) \subset U_2(u)$ and $F_{v_1 + \bar{v}_1} \subset Z_2(u)$ so that $Z_2(v_1 + \bar{v}_1) \subset Z_2(u)$. Similarly, since $h = \frac{1}{2}(\varphi_{1/2}^+ - \varphi_{1/2}^-) \in \text{sp}_\mathbb{R} \, F_{v_1 + \bar{v}_1} \subset Z_2(v_1 + \bar{v}_1)$, we have $u \in U_2(v_1 + \bar{v}_1)$ and $F_u \subset Z_2(v_1 + \bar{v}_1)$, so that $Z_2(u) \subset Z_2(v_1 + \bar{v}_1)$. ∎

The following lemma is needed in the proof of Proposition 4.4.

LEMMA 4.3. *With the assumption* (25) *and the notation and assumption of Proposition* 4.1, *let* $u = v(h)$, $v_1 = v(f_1)$ *and* $\bar{v}_1 = v(\bar{f}_1)$. *Then for each* $\varphi \in F_u$, *we have* $\Re\langle \varphi, v_1 - \bar{v}_1 \rangle = 0$.

Proof. With $\alpha := \frac{1}{2} + t$ and t real and small in absolute value, we have, from (32),

$$1 = \|v(\varphi_\alpha^+) - v(\varphi_{1-\alpha}^-)\| = \|2t(v_1 - \bar{v}_1) + \surd(1 - 4t^2)\, u\|.$$

Therefore,

$$\|2t(v_1 - \bar{v}_1) + (1 - 2t^2 + \ldots)u\| = 1,$$

and for $\varphi \in F_u$,

$$|\langle \varphi, 2t(v_1 - \bar{v}_1) + u + O(t^2)u \rangle| \leqslant 1,$$

that is,

$$|1 + 2t\langle \varphi, v_1 - \bar{v}_1 \rangle + O(t^2)| \leqslant 1.$$

The lemma follows by taking real parts and letting $t \to 0$. ∎

To describe a face, it suffices to obtain orthonormal sets $\{e_1, e_2\}$ 'parallel' to it. In order to predict what these orthonormal sets should be, we examine the corresponding situation in the concrete spin factor. The 'phase' of the face F_ξ is 1 and thus the conditions of Proposition 1.11 yield

$$\xi^\# = \xi, \quad e_1^\# = -e_1, \quad e_2^\# = -e_2. \tag{34}$$

Moreover, ξ, e_1, e_2 form an orthonormal set. The face F_{e_2} with centre e_2 may contain multiples of e_1 and ξ since the orthogonality condition is not changed. Since $e_2^\# = -e_2$, the 'phase' of F_{e_2} is -1 and the orthonormal set must consist of self-adjoint elements. The element ξ is self-adjoint, but the element e_1 must be multiplied by i to become self-adjoint (cf. (35)).

The difficulty in proving Proposition 4.4 is due to the fact that a complex phenomenon is being obtained from a real (that is, facial) structure.

PROPOSITION 4.4. *Let* Z *satisfy* (25). *As in Proposition* 4.1, *let* $\{e_1, e_2\}$ *be an orthonormal set in the Hilbert space* $\mathcal{H}_\xi$ *defined by a rank-*2 *face* F_ξ, *and define* $f_1 = e_1 + \xi$, $\bar{f}_1 = \xi - e_1$, *and* $h = e_2$. *Let* $u = v(h)$, $v_1 = v(f_1)$ *and* $\bar{v}_1 = v(\bar{f}_1)$. *Then* F_h *is a rank-*2 *face*,

$$\{\xi, ie_1\} \text{ is an orthonormal set in } \mathcal{H}_h \tag{35}$$

and

$$F_h \cap Y = \{\tfrac{1}{2}(\lambda f_1 + \bar{\lambda}\bar{f}_1) + h: \ \lambda \in \mathbb{C}, \ |\lambda| \le 1\}, \tag{36}$$

where $Y = \mathrm{sp}_{\mathbb{C}}\{e_1, e_2, \xi\}$. In particular,

$$(\mathrm{ext}\, F_h) \cap Y = \{\tfrac{1}{2}(\lambda f_1 + \bar{\lambda}\bar{f}_1) + h: \ \lambda \in \mathbb{T}\}. \tag{37}$$

Proof. Let $\varphi \in F_u$ and set, in accordance with Lemma 4.3, $\varphi(v_1) = \alpha + i\beta$, $\varphi(\bar{v}_1) = \alpha + i\bar{\beta}$, with $\alpha, \beta, \bar{\beta} \in \mathbb{R}$, From Lemmas 2.3 and 4.2, we have

$$\varphi = (\alpha + i\beta)f_1 + (\alpha + i\bar{\beta})\bar{f}_1 + h' \quad \text{with } h' \in Z_1(v_1) \cap Z_1(\bar{v}_1). \tag{38}$$

Note that $h'(u) = 1$ and $\|h'\| = \|P_1(v_1)\varphi\| \le \|\varphi\| = 1$ so that $h' \in F_u$ as well.

By (31), $u = v(h) = v(\varphi_{1/2}^+) - v(\varphi_{1/2}^-) := w + \bar{w}$, say. Then by (15) and (16), we have $w \le u$ and

$$\begin{aligned}
0 \le \langle \varphi, w \rangle &= \langle (\alpha + i\beta)f_1 + (\alpha + i\bar{\beta})\bar{f}_1 + h', \tfrac{1}{2}(v_1 + \bar{v}_1 + u) \rangle \\
&= \tfrac{1}{2}(\alpha + i\beta) + \tfrac{1}{2}(\alpha + i\bar{\beta}) + \tfrac{1}{2} \\
&= \alpha + \tfrac{1}{2}i(\beta + \bar{\beta}) + \tfrac{1}{2}.
\end{aligned}$$

Thus $\bar{\beta} = -\beta$ and (38) becomes

$$\varphi = (\alpha + i\beta)f_1 + (\alpha - i\beta)\bar{f}_1 + h'. \tag{39}$$

On the other hand, the fact that $|\varphi(\bar{\mu}v_1 + \mu\bar{v}_1)| \le 1$, with $\mu = \exp[i \arg(\alpha + i\beta)]$, leads immediately to $|2(\alpha + i\beta)| \le 1$. Assuming further that φ lies in Y (the complex span of $f_1, \bar{f}_1, h$), we have $h' = h$ so that $\subset$ holds in (36).

To prove the reverse inclusion, note first that $P_2(w)f_1 = f_1(w)\varphi_{1/2}^+ = \tfrac{1}{2}\varphi_{1/2}^+$. Also

$$\begin{aligned}
P_0(w)f_1 &= P_0(w)P_2(w + \bar{w})f_1 = P_0(w)P_2(u)f_1 \\
&= P_0(w)P_2(v_1 + \bar{v}_1)f_1 = P_2(\bar{w})f_1 \\
&= -\tfrac{1}{2}\varphi_{1/2}^-;
\end{aligned}$$

and thus $S_w f_1 = [2(P_2(w) + P_0(w)) - I]f_1 = 2(\varphi_{1/2}^+ - \varphi_{1/2}^-)/2 - f_1 = \bar{f}_1$.

Since $w \le u$, $S_w(F_u) = F_{S_w^* u} = F_u$. Therefore, for any $\varphi \in F_u \cap Y$, by (39),

$$S_w \varphi = (\alpha - i\beta)f_1 + (\alpha + i\beta)\bar{f}_1 + h \in F_u,$$

and thus

$$\varphi - S_w \varphi = 2\beta i(f_1 - \bar{f}_1). \tag{40}$$

We shall show below that

$$\text{there exists some } \varphi \in F_u \cap \mathrm{sp}_{\mathbb{C}}\{f_1, \bar{f}_1, h\} \text{ with } \beta \ne 0 \text{ in (40).} \tag{41}$$

Assuming (41), we can rewrite (40) as

$$\tfrac{1}{2}i(f_1 - \bar{f}_1) = \frac{\varphi - S_w \varphi}{4\beta} = \frac{(\varphi - h) - (S_w \varphi - h)}{4\beta},$$

which shows that $ie_1 = \tfrac{1}{2}i(f_1 - \bar{f}_1) \in \mathscr{H}_h$. By direct calculation,

$$\begin{aligned}
(\xi \mid ie_1) &= 2\langle \xi + h \mid ie_1 + h \rangle - 1 \\
&= 2\langle \varphi_{1/2}^+ \mid \tfrac{1}{2}i(f_1 - \bar{f}_1) + \tfrac{1}{2}(\varphi_{1/2}^+ - \varphi_{1/2}^-) \rangle - 1 \\
&= 2(0 + \tfrac{1}{2}) - 1 = 0,
\end{aligned}$$

proving (35). Therefore the unit ball B_h of the real Hilbert space $\mathcal{H}_h$ contains the orthonormal set $\{\frac{1}{2}i(f_1 - \bar{f}_1),\ \frac{1}{2}(f_1 + \bar{f}_1)\}$, and it follows that

$$F_u - h \supset \{\tfrac{1}{2}si(f_1 - \bar{f}_1) + \tfrac{1}{2}t(f_1 + \bar{f}_1):\ s, t \in \mathbb{R},\ s^2 + t^2 \leqslant 1\},$$

and

$$F_u \supset \{h + \tfrac{1}{2}si(\bar{f}_1 - f_1) + \tfrac{1}{2}t(f_1 + \bar{f}_1):\ s, t \in \mathbb{R},\ s^2 + t^2 \leqslant 1\}$$
$$= \{\tfrac{1}{2}(\lambda f_1 + \bar{\lambda}\bar{f}_1) + h:\ \lambda \in \mathbb{C},\ |\lambda| \leqslant 1\},$$

that is,

$$F_u \cap \mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, h\} \supset \{\tfrac{1}{2}(\lambda f_1 + \bar{\lambda}\bar{f}_1) + h:\ \lambda \in \mathbb{C},\ |\lambda| \leqslant 1\}.$$

This proves equality in (36). Since in a Hilbert space all unit vectors are extreme, (37) follows.

It remains to prove (41). We first show that there is some $\varphi \in F_u$ with $\beta \neq 0$. By (39),

$$\varphi = (\alpha + i\beta)f_1 + (\alpha - i\beta)\bar{f}_1 + h'$$

with $h' \in Z_1(v) \cap Z_1(\bar{v}) \cap F_u$. Now suppose that $\beta = 0$ for all $\varphi \in F_u$. Then for all such φ,

$$\varphi = \alpha(f_1 + \bar{f}_1) + h' \quad \text{with } h' \in Z_1(v) \cap Z_1(\bar{v}) \cap F_u,$$

and $S_w\varphi = \alpha(\bar{f}_1 + f_1) + S_w h'$, so that $\varphi - S_w\varphi = h' - S_w h'$.

But $S_w h' \in Z_1(v_1)$, since $S_w h' \in F_u \subset Z_2(v_1 + \bar{v}_1)$ implies

$$S_w h' = \langle S_w h',\ v_1 \rangle f_1 + \langle S_w h',\ \bar{v}_1 \rangle \bar{f}_1 + P_1(v_1)P_1(\bar{v}_1)h'$$

and, for example, $\langle S_w h',\ v_1 \rangle = \langle h',\ S_w^* v_1 \rangle = \langle h',\ \bar{v}_1 \rangle = 0$.

Therefore, $P_1(w)h' = P_1(v_1)P_1(w)h'$, and we have

$$P_1(w)\varphi = \tfrac{1}{2}(I - S_w)\varphi = \tfrac{1}{2}(I - S_w)h' = P_1(w)h'$$
$$= P_1(v_1)P_1(w)h' = P_1(v_1)P_1(w)\varphi.$$

Since $\varphi \in F_u$ is arbitrary, this implies $P_1(w)P_2(u) = P_1(v_1)P_1(w)P_2(u)$. Now $f_1 \in Z_2(u)$ by Lemma 4.2, so that $P_1(w)f_1 = P_1(v_1)P_1(w)f_1$, and thus

$$f_1 - \bar{f}_1 = (I - S_w)f_1 = 2P_1(w)f_1 = 2P_1(v_1)P_1(w)f_1 = P_1(v_1)(f_1 - \bar{f}_1) = 0,$$

a contradiction. Thus (41) is true for some $\varphi \in F_u$.

We can now complete the proof of (41). We start with a $\varphi \in F_u$, with $\beta \neq 0$. Writing (again)

$$\varphi = (\alpha + i\beta)f_1 + (\alpha - i\beta)\bar{f}_1 + h' \quad \text{with } h' \in F_u,$$

we have $h' - h \in F_u - h = B_h$. Therefore, $h - h' = -(h' - h) \in B_h = F_u - h$ so that $2h - h' \in F_u$. If we now set $\varphi' = \tfrac{1}{2}(\varphi + (2h - h')) \in F_u$, then

$$\varphi' = \tfrac{1}{2}((\alpha + i\beta)f_1 + (\alpha - i\beta)\bar{f}_1 + 2h)$$
$$= (\tfrac{1}{2}\alpha + i\tfrac{1}{2}\beta)f_1 + (\tfrac{1}{2}\alpha - i\tfrac{1}{2}\beta)\bar{f}_1 + h \in F_u \cap Y$$

and $\tfrac{1}{2}\beta \neq 0$ as required.

The following corollary reduces to Proposition 4.1 in the case where $\mu = 1$. For arbitrary μ, the 'phase' of $F_{(\mu f_1 + \bar{\mu}\bar{f}_1)/2}$ remains equal to 1. The centre $\tfrac{1}{2}(\mu f_1 + \bar{\mu}\bar{f}_1)$

remains in the span of ξ and e_1, and so is orthogonal to e_2. Since $\frac{1}{2}(\mu f_1 - \bar{\mu}\tilde{f}_1)$ is anti-self-adjoint and orthogonal to e_2, it provides a natural candidate for the second vector (cf. (34)).

COROLLARY 4.5. *With the notation and assumptions of Proposition 4.4, for each* $\mu \in \mathbb{T}$,

$$\{\tfrac{1}{2}(\mu f_1 - \bar{\mu}\tilde{f}_1), e_2\} \text{ is an orthonormal set in } \mathcal{H}_{(\mu f_1 + \bar{\mu}\tilde{f}_1)/2} \tag{42}$$

and with $\xi_\mu := \frac{1}{2}(\mu f_1 + \bar{\mu}\tilde{f}_1)$,

$$(\text{ext } F_{\xi_\mu}) \cap Y = \{\alpha\mu f_1 + (1-\alpha)\bar{\mu}\tilde{f}_1 \pm 2\sqrt{(\alpha(1-\alpha))} \, h \colon \alpha \in [0, 1]\}. \tag{43}$$

Proof. The assertion (42) holds if and only if

$$\tfrac{1}{2}(\mu f_1 - \bar{\mu}\tilde{f}_1) + \xi_\mu \text{ and } e_2 + \xi_\mu$$

are extreme points in F_{ξ_μ} with transition probability $\frac{1}{2}$. The first vector is μf_1 which is obviously in ext F_{ξ_μ}. The second vector is $h + \frac{1}{2}(\mu f_1 + \bar{\mu}\tilde{f}_1)$ which is in ext F_h by (37). Since $F_{\xi_\mu} = F_{\bar{\mu}v_1 + \mu\bar{v}_1}$, this vector lies in F_{ξ_μ}. Moreover,

$$\begin{aligned}\langle \mu f_1 \mid \tfrac{1}{2}(\mu f_1 + \bar{\mu}\tilde{f}_1) + h \rangle &= \langle \mu f_1 \mid \tfrac{1}{2}(\mu f_1 + \bar{\mu}\tilde{f}_1) + \tfrac{1}{2}(\varphi^+_{1/2} - \varphi^-_{1/2}) \rangle \\ &= \tfrac{1}{2}(1 + 0 + \tfrac{1}{2}\mu - \tfrac{1}{2}\mu) = \tfrac{1}{2}.\end{aligned}$$

We now apply Proposition 4.1 to the data in (42). By (28) we have (43). $\blacksquare$

The following theorem characterizes the spin factor of dimension 3.

THEOREM 4.6. *Let* Z *be an atomic neutral SFS space satisfying the properties* (FE) *and* (STP). *Let* $\{e_1, e_2\}$ *be an orthonormal set in the Hilbert space* $\mathcal{H}_\xi$ *defined by a rank-2 face* F_ξ. *Define* $f_1 = e_1 + \xi$, $\tilde{f}_1 = \xi - e_1$, *and let* $h = e_2$. *Then* $Y := \mathrm{sp}_\mathbb{C}\{f_1, \tilde{f}_1, h\}$ *is linearly isometric with* $S_2(\mathbb{C})_*$, *the predual of the JBW*-triple of two-by-two symmetric complex matrices. In particular, if* $Z = Z_2(F_\xi)$ *and* $\mathcal{H}_\xi$ *has dimension 2, then* $Z \cong S_2(\mathbb{C})_*$.

Proof. Let $M = S_2(\mathbb{C})$ and define the map $\kappa \colon M_* \to Z$ by

$$\kappa\left(\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}\right) = \alpha f_1 + \beta\tilde{f}_1 + 2\gamma h,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. The map κ is a linear isomorphism of M onto Y. We shall show the following:

$$\kappa(\text{ext } M_{*,1}) \subset \text{ext } Z_1; \tag{44}$$

$$R \diamondsuit S \text{ in ext } M_{*,1} \Rightarrow \kappa(R) \diamondsuit \kappa(S). \tag{45}$$

Suppose that we have proved (44) and (45). Every $R \in M_*$ has the form $R = \lambda_1 R_1 + \lambda_2 R_2$ with R_1 and R_2 orthogonal extreme points in M_* and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then $\kappa(R) = \lambda_1\kappa(R_1) + \lambda_2\kappa(R_2)$ and by (44) and (45), $\|\kappa(R)\| = |\lambda_1| + |\lambda_2| = \|R\|$, proving the theorem.

We proceed to the proofs of (44) and (45). Let $R \in \text{ext } M_{*,1}$, say

$$R = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

where without loss of generality, in order to prove (44), we may assume that $a \neq 0$ and $ac \geq 0$. Let $\lambda = a/|a|$ so that $c = \bar{\lambda}|c|$. Since $\det R = 0$, we have $|b|^2 = |a||c|$. Moreover,

$$1 = \text{tr}(R^*R) = |a|^2 + 2|b|^2 + |c|^2 = (|a| + |c|)^2.$$

Thus $c = \bar{\lambda}|c| = (1 - |a|)\bar{\lambda}$. Once more from $\det R = 0$ we have

$$b = \pm \sqrt{(|a|(1 - |a|))}$$

and $\kappa(R) = |a|\lambda f_1 + (1 - |a|)\bar{\lambda}\bar{f}_1 \pm 2\sqrt{(|a|(1 - |a|))}\, h$, with $|a| \in [0, 1]$. Thus (44) follows from (43).

To prove (45), note first that since $M_0(R)$ (the Peirce 0-space of R) is of dimension 1, S is a multiple of

$$\tilde{S} = \begin{bmatrix} \lambda(1 - |a|) & -b \\ -b & |a|\,\bar{\lambda} \end{bmatrix},$$

which is obviously orthogonal to R (that is, $R(\tilde{S})^* = (\tilde{S})^*R = 0$). Now from (43) and (30), it follows that $\kappa(\tilde{S}) = (1 - |a|)\lambda f_1 + |a|\,\bar{\lambda}\bar{f}_1 - 2bh$ is orthogonal to $\kappa(R)$, proving (45).

We next characterize the spin factor of dimension 4. For this we need to add a second pair $f_2, \tilde{f}_2$ of orthogonal extreme points to the vectors $f_1, \tilde{f}_1$ already obtained in such a way that the four vectors correspond to the standard basis in the predual of the JBW*-triple $M_2(\mathbb{C})$. The construction of these extreme points is provided by the following proposition. In a concrete spin factor, elements of the orthonormal set $\{e_\alpha\}$ all have the same 'phase'. The elements of the grid are obtained from Proposition 1.11 by adding two orthogonal elements of opposite 'phase'. Since multiplication by i changes the 'phase' to its opposite, elements of the form $e_j \pm ie_k$ will be extremal (cf. (49)).

PROPOSITION 4.7. *Let Z be an atomic neutral SFS space satisfying the properties (FE) and (STP). Let $\{e_1, e_2, e_3\}$ be an orthonormal set in the Hilbert space $\mathcal{H}_\xi$ defined by a rank-2 face F_ξ. Then with $h = e_2$,*

$$\{ie_1, \xi, ie_3\} \text{ is an orthonormal set in } \mathcal{H}_h \tag{46}$$

and there exist orthogonal extreme points $f_2, -\tilde{f}_2 \in F_h$ such that

$$e_2 = \tfrac{1}{2}(f_2 - \tilde{f}_2), \quad e_3 = \tfrac{1}{2}i(f_2 + \tilde{f}_2). \tag{47}$$

With $f_1 = e_1 + \xi$, $\tilde{f}_1 = \xi - e_1$ as before, we have

$$\text{sp}_\mathbb{C}\{e_1, e_2, e_3, \xi\} = \text{sp}_\mathbb{C}\{f_1, \tilde{f}_1, f_2, \tilde{f}_2\}. \tag{48}$$

Proof. Apply Proposition 4.4 to the orthonormal pairs $\{e_1, e_2\}$ and $\{e_3, e_2\}$ in $\mathcal{H}_\xi$ to obtain orthonormal pairs $\{\xi, ie_1\}$ and $\{\xi, ie_3\}$ in the Hilbert space $\mathcal{H}_{e_2}$. Now ie_1, ie_3 are orthogonal in this Hilbert space if and only if $ie_1 + h$ and $ie_3 + h$ are extreme points of F_h with transition probability $\tfrac{1}{2}$. That these are extreme points follows by setting $\lambda = i$ in Proposition 4.4. The transition probability can be computed using the easily verified formula

$$\langle e_i + \xi \mid e_j \pm \xi \rangle = \pm \tfrac{1}{2} \quad \text{for } i \neq j.$$

This proves (46).

Now apply Proposition 4.1 to the orthonormal set $\{\xi, ie_3\}$ in the Hilbert space $\mathcal{H}_{e_2}$. Then, with

$$f_2 := ie_3 + e_2 \quad \text{and} \quad \bar{f}_2 := -ie_3 - e_2, \tag{49}$$

f_2 and $-\bar{f}_2$ are orthogonal extreme points of F_{e_2}, and (47) is satisfied. From this (48) is immediate.

The following example illustrates Proposition 4.7 and will be used in Lemma 4.9. The -1 in the definition of $\bar{f}_2$ is due to the fact that spin grids contain *odd* quadrangles [3, Corollary, p. 313].

EXAMPLE. Let $Z = M_2(\mathbb{C})_*$, let E_{ij} be the standard matrix unit in $M_2(\mathbb{C})_*$, and let

$$f_1 = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{f}_1 = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad h = \tfrac{1}{2}(E_{12} + E_{21}) = \begin{bmatrix} 0 & \tfrac{1}{2} \\ \tfrac{1}{2} & 0 \end{bmatrix},$$

$$f_2 = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{f}_2 = -E_{21} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \xi = \tfrac{1}{2}(f_1 + \bar{f}_1) = \begin{bmatrix} \tfrac{1}{2} & 0 \\ 0 & \tfrac{1}{2} \end{bmatrix}.$$

Then F_ξ is the state space $\mathcal{S}$ of the C*-algebra $M_2(\mathbb{C})$. We claim that $\{e_1, e_2, e_3\}$ is an orthonormal set in $\mathcal{H}_\xi$, where $e_1 = \tfrac{1}{2}(f_1 - \bar{f}_1)$, $e_2 = h$, $e_3 = \tfrac{1}{2}i(f_2 + \bar{f}_2)$. For this it is required to show that $e_1 + \xi$, $e_2 + \xi$, $e_3 + \xi$ all belong to F_ξ and each pair in this family has transition probability $\tfrac{1}{2}$. Now

$$e_1 + \xi = f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 + \xi = h + \tfrac{1}{2}(f_1 + \bar{f}_1) = \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix},$$

and

$$e_3 + \xi = \tfrac{1}{2}i(f_2 + \bar{f}_2) + \tfrac{1}{2}(f_1 + \bar{f}_1) = \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2}i \\ -\tfrac{1}{2}i & \tfrac{1}{2} \end{bmatrix}.$$

Therefore each of these vectors belongs to F_ξ, and moreover,

$$\langle e_1 + \xi \mid e_2 + \xi \rangle = \mathrm{Tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix} \right) = \tfrac{1}{2},$$

and similarly for the others.

We now recall the description of the state space $\mathcal{S}$ of the C*-algebra $M_2(\mathbb{C})$. Let C denote the centre ξ of the face $\mathcal{S}$. Since $M_2(\mathbb{C})_*$ is the predual of a JBW*-triple, it is an atomic neutral strongly facially symmetric space satisfying (FE) and (STP), so by Proposition 4.7,

$$\mathcal{S} - C = \{\tfrac{1}{2}r(E_{11} - E_{22}) + \tfrac{1}{2}s(E_{12} + E_{21}) + \tfrac{1}{2}ti(E_{12} - E_{21}):$$
$$r^2 + s^2 + t^2 \leqslant 1, \, r, s, t \in \mathbb{R}\},$$

and therefore

$$\mathcal{S} = \{C + \tfrac{1}{2}r(E_{11} - E_{22}) + \tfrac{1}{2}s(E_{12} + E_{21}) + \tfrac{1}{2}ti(E_{12} - E_{21}):$$
$$r^2 + s^2 + t^2 \leqslant 1, \, r, s, t \in \mathbb{R}\},$$

which, with $\lambda = s + it$ and

$$\rho_{\lambda, r} = \tfrac{1}{2}(1 + r)E_{11} + \tfrac{1}{2}(1 - r)E_{22} + \tfrac{1}{2}\lambda E_{12} + \tfrac{1}{2}\bar{\lambda}E_{21} \tag{50}$$

implies

$$\mathscr{S} = \{\rho_{\lambda,r}: \ r \in \mathbb{R}, \ \lambda \in \mathbb{C}, \ r^2 + |\lambda|^2 \leqslant 1\}, \tag{51}$$

and

$$\text{ext } \mathscr{S} = \{\rho_{\lambda,r}: \ r \in \mathbb{R}, \ \lambda \in \mathbb{C}, \ r^2 + |\lambda|^2 = 1\}. \tag{52}$$

Finally, since $\rho_{\lambda,r} + \rho_{-\lambda,-r} = C$, we have

$$\rho_{\lambda,r} \diamondsuit \rho_{-\lambda,-r}. \tag{53}$$

We now obtain the characterization of the complex spin factor of dimension 4.

THEOREM 4.8. *Let Z be an atomic neutral SFS space satisfying (FE) and (STP). Let $\{e_1, e_2, e_3\}$ be an orthonormal set in the Hilbert space $\mathscr{H}_\xi$ defined by a rank-2 face F_ξ. Then with $f_1 = e_1 + \xi$, $\bar{f}_1 = \xi - e_1$, $h = e_2$, and $f_2, \bar{f}_2$ given by Proposition 4.7, we have that $Y' := \text{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_2, \bar{f}_2\}$ is linearly isometric with $M_2(\mathbb{C})_*$, the predual of the JBW*-triple of two-by-two complex matrices. More precisely, letting E_{ij} denote the canonical matrix units in $M_2(\mathbb{C})_*$, we see that the map*

$$E_{11} \mapsto f_1, \quad E_{22} \mapsto \bar{f}_1, \quad E_{12} \mapsto f_2, \quad E_{21} \mapsto -\bar{f}_2,$$

extends linearly to an isometry v of $M_2(\mathbb{C})_*$ onto Y', and

$$v(\text{ext } M_2(\mathbb{C})_{*,1}) \subset \text{ext } Z_1.$$

In particular, if $Z = Z_2(F_\xi)$ and $\mathscr{H}_\xi$ has dimension 3, then $Z \cong M_2(\mathbb{C})_*$.

Proof. The map v is obviously a linear isomorphism of $M_2(\mathbb{C})_*$ onto Y'. As in the proof of Theorem 4.6, we shall show the following:

$$v(\text{ext } M_2(\mathbb{C})_{*,1}) \subset \text{ext } Z_1; \tag{54}$$

$$R \diamondsuit S \text{ in ext } M_2(\mathbb{C})_{*,1} \Rightarrow v(R) \diamondsuit v(S). \tag{55}$$

Suppose that we have proved (54) and (55). Every $R \in M_2(\mathbb{C})_*$ has the form $R = \lambda_1 R_1 + \lambda_2 R_2$ with R_1 and R_2 orthogonal extreme points in $M_2(\mathbb{C})_*$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then $v(R) = \lambda_1 v(R_1) + \lambda_2 v(R_2)$ and by (54) and (55),

$$\|v(R)\| = |\lambda_1| + |\lambda_2| = \|R\|,$$

proving that v is an isometry.

The statements (54) and (55) follow from the next two lemmas, as shown below after the proofs of the two lemmas. In Lemma 4.9, the notation is that of Theorem 4.8. The mapping π_Z is defined in Proposition 2.9 above.

LEMMA 4.9. *For any two orthogonal non-zero projections $P, Q \in M_2(\mathbb{C})$, $p := \pi_Z \circ v \circ \pi_{M_2(\mathbb{C})_*}^{-1}(P)$ and $q := \pi_Z \circ v \circ \pi_{M_2(\mathbb{C})_*}^{-1}(Q)$ are orthogonal minimal geometric tripotents of Z. Moreover, for any $\mu \in \mathbb{T}$, v maps $F_{\mu P + \bar{\mu} Q}$ onto $F_{\mu p + \bar{\mu} q} \cap Y'$, where $Y' = \text{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_2, \bar{f}_2\}$,*

$$v(\text{ext } F_{\mu P + \bar{\mu} Q}) \subset \text{ext } Z_1,$$

and v maps orthogonal elements of $F_{\mu P + \bar{\mu} Q}$ into orthogonal elements of Z.

Proof. By the above example, the state space $\mathscr{S}$ of the C*-algebra $M_2(\mathbb{C})$ is a norm exposed face F_C with centre $C = \frac{1}{2}(E_{11} + E_{22})$ and $\mathscr{S} - C$ is a Hilbert ball with orthonormal basis

$$\tfrac{1}{2}(E_{11} - E_{22}), \ \tfrac{1}{2}(E_{12} + E_{21}), \ \tfrac{1}{2}i(E_{12} - E_{21}).$$

The map v sends the centre C into $\frac{1}{2}(f_1 + \bar{f}_1)$ and the orthonormal basis into

$$\tfrac{1}{2}(f_1 - \bar{f}_1), \ \tfrac{1}{2}(f_2 - \bar{f}_2), \ \tfrac{1}{2}i(f_2 + \bar{f}_2),$$

which is an orthonormal set, say $\{e_1, e_2, e_3\}$ in the Hilbert space $\mathscr{H}_\xi$ determined by the given rank-2 face F_ξ, with centre $\xi = \frac{1}{2}(f_1 + \bar{f}_1)$. Therefore (using Theorem 3.8) we have

$$v(\mathscr{S}) = F_\xi \cap Y', \tag{56}$$

$v(\text{ext } \mathscr{S}) \subset \text{ext } F_\xi$ and (by Theorem 3.8 and Proposition 4.1) v maps orthogonal elements of $\mathscr{S}$ into orthogonal elements of Z. Thus p, q are orthogonal minimal geometric tripotents.

Let $\hat{P} = \pi^{-1}_{M_2(\mathbb{C})_*}(P)$, $\hat{Q} = \pi^{-1}_{M_2(\mathbb{C})_*}(Q)$ be the extreme points in $M_2(\mathbb{C})_{*,1}$ corresponding to the elements P, Q respectively. Then $\frac{1}{2}(\hat{P} + \hat{Q})$ is the centre of the norm exposed face $F_{P+Q} = \mathscr{S}$ and the Hilbert ball $F_{P+Q} - \frac{1}{2}(\hat{P} + \hat{Q}) = \mathscr{S} - C$ contains an orthonormal basis of the form

$$\{\tfrac{1}{2}(\hat{P} - \hat{Q}), E_2', E_3'\},$$

where E_2', E_3' are elements of $\mathscr{H}_C$. By (56), the map v takes this centre and basis into the centre $\frac{1}{2}(\hat{p} + \hat{q})$ of F_ξ and an orthonormal set in $\mathscr{H}_\xi$ of the form

$$\{\tfrac{1}{2}(\hat{p} - \hat{q}), e_2', e_3'\},$$

where e_2', e_3' are elements of $\mathscr{H}_\xi$. For any $\mu \in \mathbb{T}$, by Corollary 4.5 and Remark 3.11, v takes the orthonormal basis

$$\{\tfrac{1}{2}(\mu\hat{P} - \bar{\mu}\hat{Q}), E_2', E_3'\} \quad \text{in } \mathscr{H}_{(\mu\hat{P} + \bar{\mu}\hat{Q})/2}$$

into the orthonormal set

$$\{\tfrac{1}{2}(\mu\hat{p} - \bar{\mu}\hat{q}), e_2', e_3'\} \quad \text{in } \mathscr{H}_{(\mu\hat{p} + \bar{\mu}\hat{q})/2}.$$

Therefore, as above, v maps $F_{\mu P + \bar{\mu}Q}$ onto $F_{\mu p + \bar{\mu}q} \cap Y'$ in such a way that extremality and orthogonality are preserved.

A special case of the following lemma, which concerns a concrete spin factor, is needed in the proof of Theorem 4.8.

LEMMA 4.10. *Let F_ξ be a rank-2 face with $\xi^\# = \xi$ which spans the predual Z of a concrete spin factor. For any extreme point $\varphi \in Z_1$, there exist $\mu \in \mathbb{T}$ and orthogonal extreme points $f, \tilde{f}$ in F_ξ such that $\varphi \in F_{(\mu f + \bar{\mu}\tilde{f})/2}$.*

Proof. Since φ is an extreme point, $\eta := \frac{1}{2}(\varphi + \varphi^\#)$ is the centre of a rank-2 face containing φ. Now η and ξ are vectors in the Hilbert space Z and the vector $\gamma := \eta - 2\langle \eta \mid \xi \rangle \xi$ is orthogonal to ξ. If $\gamma = 0$, then $|2\langle \eta \mid \xi \rangle| = 1$. But $2\langle \eta \mid \xi \rangle = 2\Re\langle \varphi \mid \xi \rangle$ and therefore $\eta = \pm\xi$. By definition of face, $\varphi \in F_{\pm\xi}$, so the lemma follows with $\mu = \pm 1$.

We may now assume that $\gamma \neq 0$, and therefore $\|2\langle \eta \mid \xi \rangle \xi\|_2 < \|\xi\|_2$, that is, $2|\langle \eta \mid \xi \rangle| < 1$. Set $\mu = \Re 2\langle \varphi \mid \xi \rangle + i(1 - (\Re 2\langle \varphi \mid \xi \rangle)^2)^{\frac{1}{2}}$. If $\Im \mu = 0$, then $\langle \eta \mid \xi \rangle = \frac{1}{2}$, a contradiction. Thus $\Im \mu \neq 0$ and we can solve for the vector e_1 in the equation

$$\eta = (\Re \mu)\xi + i(\Im \mu)e_1.$$

We find that $e_1^{\#} = -e_1$, $\langle e_1 \mid \xi \rangle = 0$, and $\|e_1\|_2^2 = \frac{1}{2}$, the latter since $\frac{1}{2} = \|\eta\|_2^2 = (\Re \mu)^2 \frac{1}{2} + (\Im \mu)^2 \|e_1\|_2^2$. Therefore, by Proposition 1.11, $\xi \pm e_1$ are extreme points in F_ξ and $\eta = \frac{1}{2}[\mu(\xi + e_1) + \bar{\mu}(\xi - e_1)]$.

If we apply Lemma 4.10 to $Z = M_2(\mathbb{C})_*$ which is the span of the state space $\mathscr{S}$ of the C*-algebra $M_2(\mathbb{C})$, we obtain the following corollary.

COROLLARY 4.11. *Every element* $A \in \text{ext } M_2(\mathbb{C})_{*,1}$ *belongs to a face of the form* $F_{\mu P + \bar{\mu}Q}$ *for some orthogonal non-zero projections* $P, Q \in M_2(\mathbb{C})$ *and some* $\mu \in \mathbb{T}$.

We can now complete the proof of Theorem 4.8. To prove (54), note that by Corollary 4.11, an extreme point A in $M_2(\mathbb{C})_*$ belongs to a face $F_{\mu P + \bar{\mu}Q}$ and by Lemma 4.9 this face is mapped by v onto $F_{\mu p + \bar{\mu}q} \cap Y'$ in such a way that $v(A)$ is an extreme point of Z_1. Moreover, since the space orthogonal to A in $M_2(\mathbb{C})_*$ is one-dimensional and there is an element $\bar{A}$ in $F_{\mu P + \bar{\mu}Q}$ orthogonal to A which is mapped by v to an element $v(\bar{A})$ in $F_{\mu p + \bar{\mu}q}$ orthogonal to $v(A)$, v satisfies (55). This completes the proof of Theorem 4.8.

In a JBW*-triple, the basic operator $D(x, x): y \mapsto \{xxy\}$ is hermitian, that is, $\exp itD(x, x)$ is an isometry for all $t \in \mathbb{R}$. If $x = u$ is a tripotent, this says that $S_\mu(u) := \mu P_2(u) + P_1(u) + \bar{\mu}P_0(u)$, where $P_k(u)$ denotes the Peirce projections associated with u, is an isometry for each $\mu \in \mathbb{T}$. It is an open question whether the corresponding result holds in a facially symmetric space, that is, whether for each geometric tripotent u in a facially symmetric space, the map $S_\mu(u) :=$ $\mu P_2(u) + P_1(u) + \bar{\mu}P_0(u)$, where $P_k(u)$ denotes the *geometric* Peirce projections associated with u, is an isometry for each $\mu \in \mathbb{T}$. However, we do have the following corollary to Theorem 4.8, which answers this question affirmatively in the case of a *minimal* geometric tripotent in a facially symmetric space of type I_2. This corollary is needed in the proof of Theorem 4.16. It is convenient to note the following remark first.

REMARK 4.12. *If* F_ξ *is a rank-2 face and if* $\{e_j\}_{j \in I}$ *is an orthonormal basis for the real Hilbert space* $\mathscr{H}_\xi$, *then*

$$\text{sp}_\mathbb{C}\{e_j, \xi\}_{j \in I} \subset \text{sp}_\mathbb{C} F_\xi \subset \overline{\text{sp}}_\mathbb{C} F_\xi = \overline{\text{sp}}_\mathbb{C}\{e_j, \xi\}_{j \in I} = Z_2(F_\xi).$$

COROLLARY 4.13. *Let* f *be an extreme point of a rank-2 face* F_ξ *and assume that the underlying SFS space is neutral and satisfies* (FE) *and* (STP). *For* $\mu \in \mathbb{T}$, *define* $S_\mu(f) := \mu P_2(f) + P_1(f) + \bar{\mu}P_0(f)$. *Then* $S_\mu(f)$ *is a bounded operator on* Z *with inverse* $S_{\bar{\mu}(f)}$, *and* $S_\mu(f)$ *is an isometry of* $Z_2(F_\xi)$ *onto* $Z_2(F_\xi)$ *(recall that* $P_k(f)$ *denotes* $P_k(v(f))$*).*

Proof. Obviously $\|S_\mu(f)\| \leq 3$ and since $S_{\bar{\mu}}(f)S_\mu(f) = I$, $S_\mu(f)$ is invertible. Since $f \in F_\xi$, $v(f) \in U_2(F_\xi)$ and by [7, Theorem 3.3], $S_\mu(f)(Z_2(F_\xi)) = Z_2(F_\xi)$.

To show that $S_\mu(f)$ is an isometry on $Z_2(F_\xi)$, let $\bar{f}$ be an extreme point of F_ξ orthogonal to f such that $\xi = \frac{1}{2}(f + \bar{f})$, and pick an orthonormal basis $\{e_j\}_{j \in I}$ for $\mathcal{H}_\xi$ containing $e_1 = \frac{1}{2}(f - \bar{f})$. For $\varphi \in \mathrm{sp}_\mathbb{C}\{e_j, \xi\}_{j \in I}$ we have

$$\varphi = zf + w\bar{f} + \sum_{j \neq 1}(\alpha_j + i\beta_j)e_j$$

$$= zf + w\bar{f} + \left(\sum \alpha_j e_j\right) + i\left(\sum \beta_j e_j\right),$$

with $z, w \in \mathbb{C}$ and $\alpha_j, \beta_j \in \mathbb{R}$.

Therefore $\varphi \in \mathrm{sp}_\mathbb{C}\{f, \bar{f}, e_2', e_3'\}$ where $e_2', e_3' \in \mathcal{H}_\xi$ are chosen so that $\{e_1, e_2', e_3'\}$ is an orthonormal set in $\mathcal{H}_\xi$ and

$$\mathrm{sp}_\mathbb{R}\left\{\sum \alpha_j e_j, \sum \beta_j e_j\right\} \subset \mathrm{sp}_\mathbb{R}\{e_2', e_3'\}.$$

For notation's sake, set $f_1 = f$, $\bar{f}_1 = \bar{f}$. By Proposition 4.7, there exist orthogonal extreme points $f_2, -\bar{f}_2 \in F_{e_2'}$ such that $e_2' = \frac{1}{2}(f_2 - \bar{f}_2)$ and $e_3' = \frac{1}{2}i(f_2 + \bar{f}_2)$. Thus $\mathrm{sp}_\mathbb{C}\{e_1, e_2', e_3', \xi\} = \mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_2, \bar{f}_2\}$, and since $\mathrm{sp}_\mathbb{C}\{e_1, \xi\} = \mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1\}$, we have

$$\varphi \in \mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, e_2', e_3'\}$$

$$= \mathrm{sp}_\mathbb{C}\{e_1, \xi, e_2', e_3'\}$$

$$= \mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_2, \bar{f}_2\} \cong M_2(\mathbb{C})_*$$

via

$$\{f_1, \bar{f}_1, f_2, \bar{f}_2\} \mapsto \{E_{11}, E_{22}, E_{12}, -E_{21}\}$$

by Theorem 4.8. Now $e_2', e_3', f_2, \bar{f}_2 \in Z_1(f)$ since e_2' and e_3' are orthogonal in $\mathcal{H}_\xi$ to e_1. Therefore if

$$\varphi = a_{11}f_1 + a_{22}\bar{f}_1 + a_{12}f_2 + a_{21}\bar{f}_2 \quad \text{with } a_{ij} \in \mathbb{C},$$

we have

$$S_\mu(f)\varphi = \mu a_{11}f_1 + \bar{\mu}a_{22}\bar{f}_1 + a_{12}f_2 + a_{21}\bar{f}_2,$$

and

$$\|S_\mu(f)\varphi\|_Z = \left\|\begin{pmatrix} \mu a_{11} & a_{12} \\ -a_{21} & \bar{\mu}a_{22} \end{pmatrix}\right\|_{M_2(\mathbb{C})_*} = \left\|\begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix}\right\|_{M_2(\mathbb{C})_*} = \|\varphi\|_Z.$$

We shall now construct a generating family for a space of type I_2.

DEFINITION 4.14. A *dual spin grid* in a facially symmetric space $Z = Z_2(F_\xi)$ of type I_2 is a family $\{f_j, \bar{f}_j\}_{j \in I \cup \{1\}}$, or $\{f_j, \bar{f}_j\}_{j \in I \cup \{1\}} \cup \{f_0\}$, where I is an index set not containing 0 or 1, and for each $j \in I$, $f_j, \bar{f}_j$ are a pair of orthogonal extreme points of Z_1 such that $\xi = \frac{1}{2}(f_1 + \bar{f}_1)$, and with $e_1 = \frac{1}{2}(f_1 - \bar{f}_1)$, $e_j = \frac{1}{2}(f_j - \bar{f}_j)$ and $e_j' = \frac{1}{2}i(f_j + \bar{f}_j)$ $(j \in I)$, the collection $\{e_1, e_j, e_j'\}_{j \in I}$, or $\{e_1, e_j, e_j'\}_{j \in I} \cup \{f_0\}$, is an orthonormal basis in the Hilbert space $\mathcal{H}_\xi$.

Note that this definition is invariant under surjective isometries (the 'isomorphisms' in the category of facially symmetric spaces). Also, the dual basis of a spin grid in a concrete spin factor (cf. [3]) is an example of a dual spin grid.

The following simple lemma will be used several times in the main theorem which follows it.

LEMMA 4.15. *In a neutral SFS space Z of type I_2, let $\{f_j, \bar{f}_j\}_{j \in I \cup \{1,2\}}$, or $\{f_j, \bar{f}_j\}_{j \in I \cup \{1,2\}} \cup \{f_0\}$, be a dual spin grid, where I is an index set not containing any of the elements $0, 1, 2$. Let $\{f'_1, \bar{f}'_1, f'_2, \bar{f}'_2\}$ be any dual spin grid in $\mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_2, \bar{f}_2\}$ (which exists by Theorem 4.8) and suppose that $f'_1 + \bar{f}'_1 = f_1 + \bar{f}_1$. Then*

$$\{f'_1, \bar{f}'_1, f'_2, \bar{f}'_2\} \cup \{f_j, \bar{f}_j\}_{j \in I}$$

or

$$\{f'_1, \bar{f}'_1, f'_2, \bar{f}'_2\} \cup \{f_j, \bar{f}_j\}_{j \in I} \cup \{f_0\}$$

is also a dual spin grid for Z.

Proof. Let $\xi' = \frac{1}{2}(f'_1 + \bar{f}'_1)$. Then $\xi' = \xi$, $\mathcal{H}_{\xi'} = \mathcal{H}_\xi$, and

$$\{\tfrac{1}{2}(f'_1 - \bar{f}'_1), \tfrac{1}{2}(f'_2 - \bar{f}'_2), \tfrac{1}{2}i(f'_2 + \bar{f}'_2)\}$$

is an orthonormal set which has the same real span as $\{e_1, e_2, e'_2\}$. Therefore,

$$\{\tfrac{1}{2}(f'_1 - \bar{f}'_1), \tfrac{1}{2}(f'_2 - \bar{f}'_2), \tfrac{1}{2}i(f'_2 + \bar{f}'_2)\} \cup \{e_j, e'_j\}_{j \in I}$$

is an orthonormal basis for $\mathcal{H}_\xi$.

The following is the main result of this paper.

THEOREM 4.16. *Let Z be an atomic neutral SFS space of type I_2 over $\mathbb{C}$ and assume the properties (FE) and (STP). Then Z is linearly isometric to the dual of a concrete spin factor.*

Proof. Let F_ξ be a rank-2 face with $Z = Z_2(F_\xi)$. Construct a dual spin grid for Z by applying Proposition 4.7 as follows. First, let $\{e_1\} \cup \{e_j, e'_j\}_{j \in I}$ (if $\mathcal{H}_\xi$ is infinite-dimensional or of odd finite dimension) or $\{e_1\} \cup \{e_j, e'_j\}_{j \in I} \cup \{e_0\}$ (if $\mathcal{H}_\xi$ is of even finite dimension) be any orthonormal basis for $\mathcal{H}_\xi$, where I is an index set not containing either 0 or 1, and let $f_1 = e_1 + \xi$, $\bar{f}_1 = -\xi - e_1$. For each $j \in I$, by Proposition 4.7 applied to the orthonormal set $\{e_1, e_j, e'_j\}$, we obtain $f_j, -\bar{f}_j \in \mathrm{ext}\, F_{e_j}$ such that

$$e_j = \tfrac{1}{2}(f_j - \bar{f}_j), \quad e'_j = \tfrac{1}{2}i(f_j + \bar{f}_j),$$

and for each $j \in I$,

$$\mathrm{sp}_\mathbb{C}\{e_1, e_j, e'_j, \xi\} = \mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_j, \bar{f}_j\}.$$

Moreover, by Theorem 4.8, $\mathrm{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_j, \bar{f}_j\} \cong M_2(\mathbb{C})_*$.

Let $\{g_j, \bar{g}_j\}_{j \in I}$, or $\{g_j, \bar{g}_j\}_{j \in I} \cup \{g_0\}$, be a dual spin grid for the dual of a concrete spin factor $\mathscr{C}$ of the appropriate dimension corresponding to the index set I. This means that $\{\pi_\mathscr{C}(g_j), \pi_\mathscr{C}(\bar{g}_j)\}_{j \in I}$, or this set together with $\pi_\mathscr{C}(g_0)$, is a spin grid for a Cartan factor of type 4 (complex spin factor), as described in [3, Corollary, p. 313] and §1. We shall assume in the rest of this proof, for convenience, that $\mathcal{H}_\xi$ is either infinite-dimensional or of odd finite dimension.

Trivial modifications complete the proof in the other case. Now

$$\text{sp}_\mathbb{C}\{f_1, \bar{f}_1, f_j, \bar{f}_j\}_{j \in I} = \text{sp}_\mathbb{C}\{e_1, e_j, e'_j, \xi\}_{j \in I},$$

which is norm dense in $Z = Z_2(F_\xi)$ by Remark 4.12. Therefore, the map T defined by $Tg_j = f_j$, $T\bar{g}_j = \bar{f}_j$ extends linearly to a map from a dense subspace of $\mathscr{C}_*$ to a dense subspace of Z. We shall show that T is an isometry.

For notation's sake let $\mathscr{C}$ and $\mathscr{C}'$ denote the dual spin grids chosen above for Z and $\mathscr{C}_*$ respectively. Let $g \in \text{sp}_\mathbb{C}\mathscr{C}'$. We shall show that $\|Tg\|_Z = \|g\|_{\mathscr{C}_*}$ which will prove the theorem. First of all, if g is a linear combination of g_j and $\bar{g}_j$ for a fixed j, then by orthogonality of g_j, $\bar{g}_j$ and of f_j, $\bar{f}_j$, it is clear that $\|Tg\|_Z = \|g\|_{\mathscr{C}_*}$. To handle the other cases, we shall prove, by induction on n, the following assertion:

If $n \geqslant 2$ and if $\{g_i, \bar{g}_i\}_{i \in J}$ is *any* dual spin grid for $\mathscr{C}$, then T is an isometry from $\text{sp}_\mathbb{C}\{g_1, \bar{g}_1, \ldots, g_n, \bar{g}_n\}$ into Z.

For $n = 2$ this is proved in Theorem 4.8. Suppose that $n > 2$ and let g belong to $\text{sp}_\mathbb{C}\{g_1, \bar{g}_1, \ldots, g_n, \bar{g}_n\}$. Write $g = h + k$ where $h = a_1 g_1 + \bar{a}_1 \bar{g}_1 + a_2 g_2 + \bar{a}_2 \bar{g}_2$ and $k = \sum_{j=3}^n (a_j g_j + \bar{a}_j \bar{g}_j)$. Since $\text{sp}_\mathbb{C}\{g_1, \bar{g}_1, g_2, \bar{g}_2\}$ is isometric to $M_2(\mathbb{C})_*$, h has a spectral decomposition (cf. Proposition 1.5) $h = t_1 \rho_1 + t_2 \bar{\rho}_1$ where $t_1, t_2 \in \mathbb{C}$ and by Lemma 4.9 (with $Z = \mathscr{C}_*$) and Corollary 4.11, $\rho_1 \in \text{ext}\, F_{\mu P + \bar{\mu} Q}$ for some orthogonal minimal partial isometries P, Q in $\mathscr{C}$ (corresponding to orthogonal minimal projections in $M_2(\mathbb{C})$) and some $\mu \in \mathbb{T}$.

In the following let $\mathscr{C} = \{f_1, \bar{f}_1, f_2, \bar{f}_2\} \cup \mathscr{C}_0 \cup \mathscr{C}_1$, where $\mathscr{C}_0 = \{f_3, \bar{f}_3, \ldots, f_n, \bar{f}_n\}$ and $\mathscr{C}_1 = \mathscr{C} \backslash (\mathscr{C}_0 \cup \{f_1, \bar{f}_1, f_2, \bar{f}_2\})$. Similarly, let $\mathscr{C}' = \{g_1, \bar{g}_1, g_2, \bar{g}_2\} \cup \mathscr{C}'_0 \cup \mathscr{C}'_1$, where $\mathscr{C}'_0 = \{g_3, \bar{g}_3, \ldots, g_n, \bar{g}_n\}$ and $\mathscr{C}'_1 = \mathscr{C}' \backslash (\mathscr{C}'_0 \cup \{g_1, \bar{g}_1, g_2, \bar{g}_2\})$, and refer to the diagram.

$$
\begin{array}{ccccc}
\text{grid} & & & & \text{grid} \\
\mathscr{C}' & \text{sp}_\mathbb{C}\,\mathscr{C}' & \xrightarrow{\ T\ } & \text{sp}_\mathbb{C}\,\mathscr{C} & \mathscr{C} \\
& I \downarrow & & \downarrow I & \\
\{\hat{P}, \hat{Q}, h_1, \bar{h}_1\} & \text{sp}_\mathbb{C}\,\mathscr{C}' & \xrightarrow{\ T\ } & \text{sp}_\mathbb{C}\,\mathscr{C} & \{T\hat{P}, T\hat{Q}, Th_1, T\bar{h}_1\} \\
\cup\, \mathscr{C}'_0 \cup \mathscr{C}'_1 & S_{\bar{\mu}}(\hat{P}) \downarrow & & \downarrow S_\mu(T\hat{P}) & \cup\, \mathscr{C}_0 \cup \mathscr{C}_1 \\
\{\bar{\mu}\hat{P}, \mu\hat{Q}, h_1, \bar{h}_1\} & \text{sp}_\mathbb{C}\,\mathscr{C}' & \xrightarrow{\ T\ } & \text{sp}_\mathbb{C}\,\mathscr{C} & \{\mu T\hat{P}, \bar{\mu} T\hat{Q}, Th_1, T\bar{h}_1\} \\
\cup\, \mathscr{C}'_0 \cup \mathscr{C}'_1 & I \downarrow & & \downarrow I & \cup\, \mathscr{C}_0 \cup \mathscr{C}_1 \\
\{\rho_1, \bar{\rho}_1, \rho_2, \bar{\rho}_2\} & \text{sp}_\mathbb{C}\,\mathscr{C}' & \xrightarrow{\ T\ } & \text{sp}_\mathbb{C}\,\mathscr{C} & \{T\rho_1, T\bar{\rho}_1, T\rho_2, T\bar{\rho}_2\} \\
\cup\, \mathscr{C}'_0 \cup \mathscr{C}'_1 & & & & \cup\, \mathscr{C}_0 \cup \mathscr{C}_1
\end{array}
$$

With $Y'' := \text{sp}_\mathbb{C}\{g_1, \bar{g}_1, g_2, \bar{g}_2\}$ $(\subset \mathscr{C}_*)$, choose $h_1, \bar{h}_1$ such that $\{\hat{P}, \hat{Q}, h_1, \bar{h}_1\}$ is a dual spin grid for Y''. By Lemma 4.15, $\{\hat{P}, \hat{Q}, h_1, \bar{h}_1\} \cup \mathscr{C}'_0 \cup \mathscr{C}'_1$ is a dual spin grid in $\mathscr{C}_*$. Since $T|_{Y''}$ is an isometry (by Theorem 4.8), $\{T\hat{P}, T\hat{Q}, Th_1, T\bar{h}_1\}$ is a dual spin grid for TY'' $(\subset Z)$. By Lemma 4.15, $\{T\hat{P}, T\hat{Q}, Th_1, T\bar{h}_1\} \cup \mathscr{C}_0 \cup \mathscr{C}_1$ is a dual spin grid for Z.

Now $S_{\bar{\mu}}(\hat{P})$ and $S_\mu(T\hat{P})$ are isometries by Corollary 4.13. Therefore a dual spin grid in $\mathscr{C}_*$ is given by $\{\bar{\mu}\hat{P}, \mu\hat{Q}, h_1, \bar{h}_1\} \cup \mathscr{C}'_0 \cup \mathscr{C}'_1$ and one in Z is given by $\{\mu T\hat{P}, \bar{\mu} T\hat{Q}, Th_1, T\bar{h}_1\} \cup \mathscr{C}_0 \cup \mathscr{C}_1$. Next, it is easily checked that, on a dual spin grid, $S_\mu(T\hat{P}) T S_{\bar{\mu}} = T$, and hence the middle part of the diagram commutes.

Choose $\rho_2, \bar{\rho}_2$ such that $\{\rho_1, \bar{\rho}_1, \rho_2, \bar{\rho}_2\}$ form a dual spin grid of Y''. Since $\rho_1, \bar{\rho}_1 \in F_{\mu P + \bar{\mu} Q}$, by Lemma 4.15, $\{\rho_1, \bar{\rho}_1, \rho_2, \bar{\rho}_2\} \cup \mathscr{C}'_0 \cup \mathscr{C}'_1$ is a dual spin grid of

$\mathscr{C}_*$. As above, T maps this dual spin grid onto a dual spin grid of Z. Note that, by induction, $T|_X$ is an isometry, where $X := \mathrm{sp}_{\mathbb{C}}(\{\rho_1, \bar{\rho}_1\} \cup \mathscr{C}'_0)$. Since $g \in X$, $\|Tg\|_Z = \|g\|_{\mathscr{C}_*}$.

References

1. E. ALFSEN and F. SHULTZ, 'State spaces of Jordan algebras', *Acta Math.* 140 (1978) 155–190.
2. R. D. BOURGIN, *Geometric aspects of convex sets with the Radon–Nicodym property*, Lecture Notes in Mathematics 993 (Springer, Berlin, 1983).
3. T. DANG and Y. FRIEDMAN, 'Classification of JBW*-triple factors and applications', *Math. Scand.* 61 (1987) 292–330.
4. C. M. EDWARDS and G. T. RÜTTIMANN, 'On the facial structure of the unit balls in a JBW*-triple and its predual', *J. London Math. Soc.* (2) 38 (1988) 317–332.
5. Y. FRIEDMAN and B. RUSSO, 'Structure of the predual of a JBW*-triple', *J. Reine Angew. Math.* 356 (1985) 67–89.
6. Y. FRIEDMAN and B. RUSSO, 'A geometric spectral theorem', *Quart. J. Math. Oxford* (2) 37 (1986) 263–277.
7. Y. FRIEDMAN and B. RUSSO, 'Affine structure of facially symmetric spaces', *Math. Proc. Cambridge Philos. Soc.* 106 (1989) 107–124.
8. Y. FRIEDMAN and B. RUSSO, 'Some affine geometric aspects of operator algebras', *Pacific J. Math.* 137 (1989) 123–144.
9. L. A. HARRIS, 'Bounded symmetric homogeneous domains in infinite-dimensional spaces', *Infinite-dimensional holomorphy* (eds T. L. Hayden and T. J. Suffridge), Lecture Notes in Mathematics 364 (Springer, Berlin, 1974), pp. 13–40.
10. E. NEHER, *Jordan triple systems by the grid approach*, Lecture Notes in Mathematics 1280 (Springer, Berlin, 1987).

Department of Mathematics
Jerusalem College of Technology
P.O. Box 16031
Jerusalem 91-160
Israel

Department of Mathematics
University of California
Irvine
California 92717
U.S.A.