THE DUNFORD–PETTIS PROPERTY FOR SOME FUNCTION ALGEBRAS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT

The Dunford–Pettis property is shown to hold for the uniform algebra $A(\Omega)$ and its dual for some standard domains Ω, including strongly pseudoconvex bounded domains in $\mathbb{C}^n$, pseudoconvex bounded domains of finite type in $\mathbb{C}^2$, and bounded domains in $\mathbb{C}$. Previously the result was known for the unit ball and unit polydisc in $\mathbb{C}^n$. Techniques used involve Bourgain algebras, Hankel operators, properties of the Bergman kernel, quasi-metrics on the boundary, and $\bar{\partial}$-theory.

1. *Introduction*

A Banach space X over the complex field is said to have the *Dunford–Pettis property* if for each Banach space Y, every weakly compact linear operator $T: X \to Y$ is completely continuous. There are several conditions equivalent to this which have been discussed in [**12**]. Here is one of them which is often used as a definition of the Dunford–Pettis property.

THEOREM 1.1 [**15**]. *Let X be a Banach space. Then X has the Dunford–Pettis property if and only if for every weakly null sequence (x_n) in X and weakly null sequence (x_n^*) in X^*, we have* $\lim_{n\to\infty}\langle x_n^*, x_n\rangle = 0$.

No infinite dimensional reflexive Banach space X has the Dunford–Pettis property since the identity operator $I: X \to X$ is a weakly compact operator, but the unit ball of X is not norm compact. The basic result proved by Dunford and Pettis in [**13**] is that $L^1(\mu)$ has the Dunford–Pettis property. Since the canonical map of X into X^{**} is weakly continuous, Theorem 1.1 implies that X must have the Dunford–Pettis property if X^* has it. In particular, the space $C_0(K)$ of all continuous functions which vanish at infinity on a locally compact Hausdorff space K has the Dunford–Pettis property.

Work has been done by several authors to determine which Banach spaces have the Dunford–Pettis property. The C^*-algebras and their duals and preduals which have the Dunford–Pettis property have been completely analysed by Chu and Iochum in [**6**] (see also [**7**]). In [**3**], Bourgain proved that H^∞ of the unit disk has the Dunford–Pettis property. In [**4**], Bourgain proved that the Ball algebra $A(B_n)$ and the polydisc algebra $A(D^n)$ (and their dual spaces) have the Dunford–Pettis property, thereby generalizing in two directions the result of J. Chaumat for the disc algebra (see [**22**, Section 8]). Recall that $A(\Omega)$ is the subspace of $C(\bar{\Omega})$ of functions which are holomorphic on Ω.

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Now the unit ball and unit polydisc in $\mathbb{C}^n$ are examples of bounded symmetric domains, and the unit ball is a special case of a strongly pseudoconvex domain. The purpose of the present paper is to prove that both $A(\Omega)$ and its dual space $A(\Omega)^*$ have the Dunford–Pettis property for a quite large class of domains in $\mathbb{C}^n$ including the unit ball, strongly pseudoconvex domains, and finite type domains in $\mathbb{C}^2$. Our purpose is facilitated by exploiting the connection between the Dunford–Pettis property holding for $A(\Omega)$ and the boundedness and compactness of certain Hankel operators H_ϕ on $A(\Omega)$.

The paper is organized as follows. In Section 2, we give some relations between $A(\Omega)$ or $A(\Omega)^*$ having the Dunford–Pettis property and the complete continuity of some Hankel operators on $A(\Omega)$. In Section 3, we consider more general integral operators which are modelled on the Bergman projection, and we prove that they satisfy the conditions used in Section 2. In Section 4, several examples of standard domains which satisfy the conditions, and hence for which $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property, are given. Finally, in Section 5, we show, as an application of $\bar\partial$-theory, that $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property when Ω is a bounded domain in the complex plane with C^1 boundary.

Bourgain's methods in [**4**] are very useful for finding subspaces of $C(K)$ (where K is compact) having the Dunford–Pettis property. This theory has been developed by Cima and Timoney in [**9**], where they define the so-called *Bourgain algebra* of a subspace of $C(K)$. Here we describe briefly their result. Let $\phi \in C(K)$ and let M_ϕ denote the multiplication operator on $C(K)$. Use ϕx^{**} to denote $M_\phi^{**}(x^{**})$ for all $x^{**} \in C(K)^{**}$ and let X be any closed subspace of $C(K)$. Then we let X_B denote the space of all functions $\phi \in C(K)$ satisfying the following condition: if (x_n^{**}) is a weakly null sequence in $X^{**} \subset C(K)^{**}$, then $\lim_{n\to\infty} \operatorname{dist}(\phi x_n^{**}, X^{**}) = 0$. Here

$$\operatorname{dist}(\phi x_n^{**}, X^{**}) = \inf\{\|\phi x_n^{**} - x^{**}\|_{C(K)^{**}} : x^{**} \in X^{**}\} = \|\phi x_n^{**} + X^{**}\|_{C(K)^{**}/X^{**}}.$$

We let X_b be the space of all functions $\phi \in C(K)$ satisfying: if $(x_n) \subset X$ is a weakly null sequence, then $\lim_{n\to\infty} \operatorname{dist}(\phi x_n, X) = 0$. It has been proved in [**9**] that X_B and X_b are algebras, which are the so-called Bourgain algebras of X. From [**4**, Proposition 2] and the proof of [**4**, Theorem 1], Cima and Timoney in [**9**] formulated and proved the following theorem.

THEOREM 1.2 [**9**]. *Let K be a compact Hausdorff space. Let X be a closed subspace of $C(K)$. Then we have*

(i) *if $X_B = C(K)$, then both X and X^* have the Dunford–Pettis property;*
(ii) *if $X_b = C(K)$, then X has the Dunford–Pettis property.*

2. *Sufficient Condition on the Bergman Kernel*

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. We let dV denote Lebesgue volume measure over Ω. For $0 < p \leqslant \infty$, we let $L^p(\Omega)$ be the usual Lebesgue space with respect to the measure dV. We let $A^p(\Omega)$ be the subspace of $L^p(\Omega)$ consisting of holomorphic functions. It is well known and easy to show that $A^p(\Omega)$ is a closed subspace of $L^p(\Omega)$. In particular, $A^2(\Omega)$ is a Hilbert space. By subharmonicity of $|f|^p$ for $f \in A^p(\Omega)$, it follows easily that the evaluation functional $e_z(f) = f(z)$ is bounded on $A^p(\Omega)$ for every point $z \in \Omega$. By the Riesz representation theorem, there is $K_z \in A^2(\Omega)$ so that

$$f(z) = \int_\Omega f(w)\overline{K_z}(w)\,dV(w)$$

for all $f \in A^2(\Omega)$. Let $K(z, w) = \overline{K_z}(w)$. It is well known that $K(z, w)$ is a reproducing kernel of $A^2(\Omega)$, called the *Bergman kernel*. Let $P: L^2(\Omega) \to A^2(\Omega)$ be the orthogonal projection, the so-called *Bergman projection*. Then we have

$$P(f)(z) = \int_\Omega f(w)\,K(z, w)\,dV(w)$$

for all $f \in L^2(\Omega)$. For $\phi \in L^2(\Omega)$, we define the multiplication operator formally by $M_\phi(f) = \phi f$. The commutator of M_ϕ and P is defined by $[M_\phi, P] = M_\phi P - PM_\phi$, and we define the Hankel operator H_ϕ as follows: $H_\phi = [M_\phi, P]P$. From these definitions, one can easily see that $H_\phi = [M_\phi, P]$ on $A^2(\Omega)$. In other words, we can also look at $[M_\phi, P]$ as an integral operator with kernel function:

$$K_\phi(z, w) = (\phi(z) - \phi(w))\,K(z, w) \quad \text{for } z, w \in \Omega.$$

Let $A(\Omega) = A^2(\Omega) \cap C(\bar{\Omega})$. Then $A(\Omega)$ is a Banach algebra, and it is easy to show that $A(\Omega)$ is a closed subspace of $C(\bar{\Omega})$. The first relation between the Dunford–Pettis property of the Banach space $A(\Omega)$ and the boundedness and complete continuity of the Hankel operators H_ϕ on $A(\Omega)$ is given by the following simple proposition.

PROPOSITION 2.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. Assume that for all $\phi \in C^1(\bar{\Omega})$, the Hankel operator H_ϕ maps $A(\Omega)$ into $C(\bar{\Omega})$ and is completely continuous. Then $A(\Omega)_b = C(\bar{\Omega})$. Hence, by Theorem* 1.2, *$A(\Omega)$ has the Dunford–Pettis property.*

Proof. It is required to prove that for each weakly null sequence $(f_n)_{n=1}^\infty \subset A(\Omega)$, and all $\phi \in C(\bar{\Omega})$, we have

$$\lim_{n\to\infty} \operatorname{dist}(\phi f_n, A(\Omega)) = 0. \tag{2.1}$$

Since $C^1(\bar{\Omega})$ is dense in $C(\bar{\Omega})$, it suffices to prove (2.1) for all $\phi \in C^1(\bar{\Omega})$.

Let $\phi \in C^1(\bar{\Omega})$. Since $H_\phi(A(\Omega)) \subset C(\bar{\Omega})$, we have $P(\phi f_n) \in C(\bar{\Omega}) \cap A^2(\Omega) = A(\Omega)$ for all n. Since H_ϕ is completely continuous from $A(\Omega)$ to $C(\bar{\Omega})$, we have

$$\lim_{n\to\infty} \|H_\phi(f_n)\|_\infty = 0.$$

However,

$$\operatorname{dist}(\phi f_n, A(\Omega)) \leqslant \|\phi f_n - P(\phi f_n)\|_\infty = \|H_\phi(f_n)\|_\infty.$$

This implies that (2.1) holds for all $\phi \in C^1(\bar{\Omega})$, and completes the proof.

From the assumption of Proposition 2.1, one realizes that in order to prove that $A(\Omega)$ has the Dunford–Pettis property, it is important to understand when Hankel operators carry $A(\Omega)$ into $C(\bar{\Omega})$ and are completely continuous from $A(\Omega)$ to $C(\bar{\Omega})$. Conditions for the boundedness and compactness of the Hankel operators on Bergman spaces or Hardy spaces have been studied by many authors (see for example [**2**, **1**] and references therein).

In the next two theorems, we first give a sufficient condition for the complete continuity of $H_\phi : A(\Omega) \to C(\bar{\Omega})$ and then we show that this condition implies the Dunford–Pettis property for $A(\Omega)^*$ and $A(\Omega)$. In the following two sections, we shall show that certain standard domains satisfy this condition.

THEOREM 2.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. Let $\phi \in C(\bar{\Omega})$. Then the following statements hold.*

(i) *If $\|K_\phi(z,\cdot)\|_1 \leqslant C_\phi$ for all $z \in \Omega$, then $H_\phi : A^p(\Omega) \to L^p(\Omega)$ is bounded for all $1 \leqslant p \leqslant \infty$. Here we use the notation $A^\infty = H^\infty$, and C_ϕ is a constant depending only on ϕ and Ω.*

(ii) *If the set $\{K_\phi(z,\cdot) : z \in \Omega\}$ is relatively compact in $L^1(\Omega)$, then $H_\phi : A(\Omega) \to L^\infty(\Omega)$ is completely continuous.*

(iii) *If the set $\{K_\phi(z,\cdot)\, z \in \Omega\}$ is relatively compact in $L^1(\Omega)$ and if $K(z,w) \in C(\bar{\Omega} \times \Omega_\varepsilon)$ for all $0 < \varepsilon \ll 1$, where $\Omega_\varepsilon = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) \geqslant \varepsilon\}$, then $H_\phi : A(\Omega) \to C(\bar{\Omega})$ is completely continuous.*

Proof. Since the Bergman kernel is conjugate symmetric, we have $|K_\phi(z,w)| = |K_\phi(w,z)|$ for all $z, w \in \Omega$. Applying Schur's lemma [25] with the assumption in (i), the integral operator with kernel function K_ϕ is bounded on $L^p(\Omega)$ for all $1 \leqslant p \leqslant \infty$, so $H_\phi : A^p(\Omega) \to L^p(\Omega)$ is bounded.

Now we prove (ii). By (i), $H_\phi : A^p(\Omega) \to L^p(\Omega)$ is bounded for all $1 \leqslant p \leqslant \infty$. In particular, $H_\phi : A(\Omega) \to L^\infty(\Omega)$ is bounded. Let $\{f_n\} \subset A(\Omega)$ be a weakly null sequence. We shall show that $\{H_\phi(f_n)\}$ is strongly null in $L^\infty(\Omega)$. Suppose not. Then there is a positive number ε_0 and a subsequence $z_{n_k} \in \Omega$ such that

$$|H_\phi(f_{n_k})(z_{n_k})| \geqslant \varepsilon_0 \quad \text{for } k = 1, 2, \ldots. \tag{2.2}$$

Since $\{K_\phi(z,\cdot) : z \in \Omega\}$ is relatively compact in $L^1(\Omega)$, there is a further subsequence (we use the same notation) $\{K_\phi(z_{n_k},\cdot)\}$ converging strongly to some $g \in L^1(\Omega)$. Since $\{f_n\}$ is weakly null, it is bounded in $L^\infty(\Omega)$. Thus

$$\begin{aligned}|H_\phi(f_{n_k})\,(z_{n_k})| &\leqslant \left|\int_\Omega f_{n_k}(w) g(w)\, dV(w)\right| + \int_\Omega |K_\phi(z_{n_k}, w) - g(w)|\,|f_{n_k}(w)|\, dV(w) \\ &\leqslant \left|\int_\Omega f_{n_k}(w) g(w)\, dV(w)\right| + \|f_{n_k}\|_\infty \,\|K_\phi(z_{n_k},\cdot) - g(\cdot)\|_1 \longrightarrow 0\end{aligned}$$

as $k \to \infty$. This contradicts (2.2). Therefore $\|H_\phi(f_n)\|_\infty \to 0$ as $n \to \infty$.

We prove (iii) now; we note that, by (ii), it suffices to prove that $H_\phi(f) \in C(\bar{\Omega})$ for all $f \in A(\Omega)$. Let $f \in A(\Omega)$. Since $H_\phi(f) \in C(\Omega)$, it suffices to prove that $H_\phi(f)$ is uniformly continuous on Ω. If not, there are sequences $(z_k^1)_k$ and $(z_k^2)_k$ in Ω and a positive number ε_0 such that $\lim_{k\to\infty} |z_k^1 - z_k^2| = 0$, and

$$|H_\phi(f)(z_k^1) - H_\phi(f)(z_k^2)| \geqslant \varepsilon_0 \quad \text{for } k = 1, 2, \ldots. \tag{2.3}$$

Since $\{K_\phi(z,\cdot) : z \in \Omega\}$ is relatively compact in $L^1(\Omega)$, without loss of generality, we may assume that there are functions $g_1, g_2 \in L^1(\Omega)$ such that $\|K_\phi(z_k^1,\cdot) - g_1(\cdot)\|_1 \to 0$ and $\|K_\phi(z_k^2,\cdot) - g_2(\cdot)\|_1 \to 0$ as $k \to \infty$. Let

$$\delta(\varepsilon) = \max_{j=1,2} \left\{ \int_{\Omega \setminus \Omega_\varepsilon} |g_j(w)|\, dV(w) \right\},$$

so that $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$. Since $K(z,w)$ is uniformly continuous on $\bar{\Omega} \times \Omega_\varepsilon$, for each $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for all $z^1, z^2 \in \bar{\Omega}$ and all $w \in \bar{\Omega}_\varepsilon$ we have

$$|K_\phi(z^1, w) - K_\phi(z^2, w)| \leqslant \delta(\varepsilon) \quad \text{if } |z^1 - z^2| \leqslant \eta(\varepsilon).$$

Thus

$$\begin{aligned}|H_\phi(f)(z_k^1)-H_\phi(f)(z_k^2)| &\leqslant \int_\Omega |K_\phi(z_k^1,w)-K_\phi(z_k^2,w)|\,|f(w)|\,dV(w)\\ &\leqslant \|f\|_\infty \int_{\Omega_\varepsilon} |K_\phi(z_k^1,w)-K_\phi(z_k^2,w)|\,dV(w)\\ &\quad+\|f\|_\infty \int_{\Omega-\Omega_\varepsilon} |K_\phi(z_k^1,w)-K_\phi(z_k^2,w)|\,dV(w)\\ &\leqslant \|f\|_\infty \sum_{j=1}^{2}\int_{\Omega-\Omega_\varepsilon} |K_\phi(z_k^j,w)-g_j(w)|\,dV(w)\\ &\quad+\|f\|_\infty \sum_{j=1}^{2}\int_{\Omega-\Omega_\varepsilon} |g_j(w)|\,dV(w)\\ &\quad+\|f\|_\infty \int_{\Omega_\varepsilon} |K_\phi(z_k^1,w)-K_\phi(z_k^2,w)|\,dV(w)\\ &\leqslant C\delta(\varepsilon)\quad \text{as } k\to\infty.\end{aligned}$$

The above inequality contradicts (2.3) as $\varepsilon\to 0$. Therefore, (iii) follows.

THEOREM 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. Suppose that for all $\phi\in C^1(\bar\Omega)$ the set $\{K_\phi(z,\cdot):z\in\Omega\}$ is relatively compact in $L^1(\Omega)$, and $K(z,w)\in C(\bar\Omega\times\Omega_\varepsilon)$ for all $0<\varepsilon\ll 1$, where $\Omega_\varepsilon=\{z\in\Omega: dist(z,\partial\Omega)\geqslant\varepsilon\}$. Then $A(\Omega)_B=C(\bar\Omega)$, and hence by Theorem 1.2, $A(\Omega)^*$ and $A(\Omega)$ have the Dunford–Pettis property.*

In order to prove Theorem 2.3, we need the following well-known elementary proposition.

PROPOSITION 2.4. *Let ϕ be the embedding map of a Banach space X into X^{**}. Let τ be the weak topology of X, and σ be the weak*-topology of X^{**}. Then*

(a) *ϕ is a homeomorphism from (X,τ) onto a dense subspace of (X^{**},σ);*

(b) *if B is the closed unit ball of X, then $\phi(B)$ is σ-dense in B^{**}, the closed unit ball of X^{**}.*

For the proof of Theorem 2.3, it is required to prove the following.

CLAIM. *If (x_n^{**}) is weakly null in $A(\Omega)^{**}$ and $\phi\in C(\bar\Omega)$, then*

$$\lim_{n\to\infty} \operatorname{dist}(\phi x_n^{**}, A(\Omega)^{**})=0.$$

Proof of Claim. Since (x_n^{**}) is weakly null in $A(\Omega)^{**}$, it is bounded. Since $C^1(\bar\Omega)$ is dense in $C(\bar\Omega)$, without loss of generality, we may assume that $\phi\in C^1(\bar\Omega)$. We first show that for each $\varepsilon>0$, there is $N_\varepsilon>0$ such that

$$\sup_{z\in\Omega}|\langle\overline{K_\phi}(z,\cdot), i^{**}(x_n^{**})\rangle|<\tfrac{1}{2}\varepsilon \quad \text{for all } n\geqslant N_\varepsilon, \tag{2.4}$$

where $i: A(\Omega)\to C(\bar\Omega)$ is the inclusion map.

Suppose that (2.4) is not true. Then there are a sequence $\{z_k\}$ of points in Ω, a subsequence $\{x_{n_k}^{**}\} \subset A(\Omega)^{**}$ and a positive number ε_0 so that

$$|\langle \overline{K_\phi}(z_k, \cdot), i^{**}(x_{n_k}^{**})\rangle| \geqslant \tfrac{1}{2}\varepsilon_0 \quad \text{for } k = 1, 2, \ldots. \tag{2.5}$$

Since $\{\overline{K_\phi}(z_k, \cdot)\}$ is relatively compact in $L^1(\Omega)$, there is a convergent subsequence, still denoted by $\{\overline{K_\phi}(z_k, \cdot)\}$, and a function $g \in L^1(\Omega)$ so that

$$\|\overline{K_\phi}(z_k, \cdot) - g\|_1 \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Since $i^*: C(\bar{\Omega})^* = \mathscr{M}(\bar{\Omega}) \to A(\Omega)^*$ is a bounded operator with norm 1, we have

$$\|i^*(\overline{K_\phi}(z_k, \cdot)) - i^*(g)\|_{A(\Omega)^*} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Thus

$$\begin{aligned} |\langle i^*(\overline{K_\phi}(z_k, \cdot)), x_{n_k}^{**}\rangle| &\leqslant |\langle i^*(g), x_{n_k}^{**}\rangle| + |\langle i^*(\overline{K_\phi}(z_k, \cdot)) - i^*(g), x_{n_k}^{**}\rangle| \\ &\leqslant |\langle i^*(g), x_{n_k}^{**}\rangle| + \|\overline{K_\phi}(z_k, \cdot) - g\|_1 \, \|x_{n_k}^{**}\|_{A(\Omega)^{**}} \longrightarrow 0 \end{aligned}$$

as $k \to \infty$. This contradicts (2.5). Therefore, we have completed the proof of (2.4).

By Proposition 2.4 with $X = A(\Omega)$, for each n there is a net (x_α) in $A(\Omega)$ so that

(a) $\|x_\alpha\|_\infty \leqslant \|x_n^{**}\|_{A(\Omega)^{**}}$;
(b) $x_n^{**} = \sigma(A^{**}, A^*)\text{-}\lim_\alpha x_\alpha = \sigma(C(\bar{\Omega})^{**}, C(\bar{\Omega})^*)\text{-}\lim_\alpha x_\alpha$.

Since $\{K_\phi(z, \cdot)\}$ is relatively compact in $L^1(\Omega)$, an argument similar to the proof of (2.4) above shows that

$$\sup_{z \in \Omega} |\langle x_\alpha, \overline{K_\phi}(z, \cdot)\rangle| \leqslant \sup_{z \in \Omega} |\langle x_n^{**}, \overline{K_\phi}(z, \cdot)\rangle| + \tfrac{1}{2}\varepsilon \leqslant \varepsilon \tag{2.6}$$

when α is sufficiently large. We shall show that

$$\operatorname{dist}(\phi x_n^{**}, A(\Omega)^{**}) \leqslant \varepsilon \quad \text{for } n \geqslant N_\varepsilon.$$

In the first place, with

$$(A(\Omega)^{**})_\perp = \{x^* \in C(\bar{\Omega})^* : \langle x^*, A^{**}\rangle = 0\} \quad \text{and} \quad A(\Omega)^\perp = \{x^* \in C(\bar{\Omega})^* : \langle x^*, A(\Omega)\rangle = 0\},$$

so that $(A(\Omega)^{**})_\perp \subset A(\Omega)^\perp$, we have

$$\begin{aligned} \operatorname{dist}(\phi x_n^{**}, A(\Omega)^{**}) &= \|\phi x_n^{**} + A(\Omega)^{**}\|_{C(\bar{\Omega})^{**}/A(\Omega)^{**}} \\ &= \sup_{\|x^*\|=1} \{|\langle \phi x_n^{**}, x^*\rangle| : x^* \in (A(\Omega)^{**})_\perp\} \\ &= \sup_{\|x^*\|=1} \{\lim_\alpha |\langle \phi x_\alpha, x^*\rangle| : x^* \in (A(\Omega)^{**})_\perp\} \\ &\leqslant \limsup_\alpha \sup_{\|x^*\|=1} \{|\langle \phi x_\alpha, x^*\rangle| : x^* \in (A(\Omega)^{**})_\perp\} \\ &\leqslant \limsup_\alpha \sup_{\|x^*\|=1} \{|\langle \phi x_\alpha, x^*\rangle| : x \in (A(\Omega))^\perp\} \\ &= \limsup_\alpha \|\phi x_\alpha + A(\Omega)\|_{C(\bar{\Omega})/A(\Omega)} \\ &= \limsup_\alpha \operatorname{dist}_{C(\bar{\Omega})}(\phi x_\alpha, A(\Omega)). \end{aligned} \tag{2.7}$$

Therefore we need only to show that dist $(\phi x_\alpha, A(\Omega)) < \varepsilon$. We have

$$|\phi(z)\, x_\alpha(z) - P(\phi x_\alpha)(z)| = |H_\phi(x_\alpha)(z)| = \left| \int_\Omega K_\phi(z,w)\, x_\alpha(w)\, dV(w) \right|$$
$$= |\langle x_\alpha, \overline{K_\phi}(z,\cdot)\rangle| < \varepsilon \tag{2.8}$$

by (2.6) for all $z \in \Omega$. By Theorem 2.2(iii), $P(\phi x_\alpha) \in A(\Omega)$. The proof of Theorem 2.3 is complete.

3. *Sufficient condition on the domain*

In this section, we shall study the boundedness and compactness of Hankel operators on $L^\infty(\Omega)$ and $A(\Omega)$. In fact we shall consider more general operators which are modelled on Hankel operators (see **[1]** for details). Let X be a Hausdorff space. A *quasi-metric* d on X is a continuous function $d: X \times X \to R^+$ which satisfies the requirements for a topological metric except that the triangle inequality is replaced by

$$d(x,z) \leqslant C(d(x,y) + d(y,z)) \quad \text{for } x, y, z \in X. \tag{3.1}$$

(See **[10]** or **[11**, Section 2] for the structure of spaces of homogeneous type, upon which this is based.)

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Let $r(z)$ denote the distance from z to $\partial\Omega$ and let $\pi(z)$ be the projection of $z \in \Omega$ onto $\partial\Omega$ (see **[18]**). It is clear that this projection exists and is unique when $z \in \Omega$ is near $\partial\Omega$, say $r(z) \leqslant \varepsilon_0$. Let d be a quasi-metric on $\partial\Omega$ as defined above. We define balls $B(P,\delta)$, for $P \in \partial\Omega$ and $\delta > 0$, with respect to this quasi-metric. Then we define a Carleson region to be

$$\mathscr{C}(w,\delta) = \{z \in \Omega : r(z) < \delta \text{ and } \pi(z) \in B(\pi(w),\delta)\}$$

when $w \in \bar{\Omega}$ and $r(w) \leqslant \delta < \varepsilon_0$.

Let $\psi(z,\delta) = \delta\sigma(B(\pi(z),\delta))$, where σ denotes surface measure on $\partial\Omega$. For $z, w \in \Omega$, we let

$$r(z,w) := \inf\{\rho \geqslant r(z) : \mathscr{C}(\pi(z), r(z)) \subset \mathscr{C}(w,\rho)\}.$$

It has been proved in **[1]** that $r(z,w)$ is quasi-symmetric, that is, $r(z,w) \approx r(w,z)$.

A complex-valued measurable function K on $\Omega \times \Omega$ is a *homogeneous kernel* if it is locally bounded on $\Omega \times \Omega$ and satisfies

$$|K(z,w)| \leqslant C\psi(z, r(z,w))^{-1} \quad \text{for } z, w \in \Omega \quad \text{and} \quad r(z), r(w) < \varepsilon_0, \tag{3.2}$$

where C is some constant. If K is a homogeneous kernel, we denote the associated integral operator by I_K, which is defined formally by

$$I_K f(z) = \int_\Omega f(w)\, K(z,w)\, dV(w). \tag{3.3}$$

Note that if K is the Bergman kernel over Ω, then I_K is the Bergman projection P.

We shall now prove the following theorem.

THEOREM 3.1. *Let Ω be a bounded domain with C^2 boundary in $\mathbb{C}^n$ and let d be a quasi-metric on $\partial\Omega$ satisfying the inequality:*

$$|z-w| \leqslant \left(\frac{C}{1 + |\log r(z,w)|}\right)^{1+\varepsilon} \tag{3.4}$$

for all $z, w \in \bar{\Omega}$ and some $\varepsilon > 0$. Let $K(\cdot,\cdot)$ be a homogeneous kernel on Ω. Then

(a) *if* $\phi \in C^1(\bar{\Omega})$, *then* $H_\phi^K = [M_\phi, I_K] I_K : L^\infty(\Omega) \to L^\infty(\Omega)$ *is bounded;*

(b) *if* $K(\cdot,\cdot) \in C(\bar{\Omega} \times \Omega_\varepsilon)$ *for all* $0 < \varepsilon \leqslant 1$, *then* $\{K_\phi(z,\cdot) : z \in \Omega\}$ *is relatively compact in* $L^1(\Omega)$.

Proof. Part (a) follows from the fact that $|z - \cdot|\, K(z,\cdot) \in L^1(\Omega)$ uniformly for $z \in \Omega$, which we now show.

Let $N = N(z)$ be the smallest integer such that $\partial\Omega \subset B(\pi(z), 2^N r(z))$. Then

$$
\begin{aligned}
&\int_\Omega |z-w|\, |K(z,w)|\, dV(w) \\
&\leqslant C \int_\Omega |K(z,w)| \left(\frac{1}{1+|\log r(z,w)|}\right)^{1+\varepsilon} dV(w) \\
&\leqslant C \sum_{k=1}^{N} \int_{2^k \mathscr{C}(z,r(z)) - 2^{k-1}\mathscr{C}(z,r(z))} \frac{1}{|2^k \mathscr{C}(z,r(z))|} \left(\frac{1}{1+|\log r(z,w)|}\right)^{1+\varepsilon} dV(w) \\
&\leqslant C\left(1 + \sum_{k=1}^{N} \left(\frac{1}{1+|\log r(z,w)|}\right)^{1+\varepsilon}\right) \\
&\leqslant C\left(1 + \sum_{k=1}^{\infty} \left(\frac{1}{1+\log 2^{-k}|}\right)^{1+\varepsilon}\right) \\
&\leqslant C\left(1 + \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{1+\varepsilon}\right) \leqslant C,
\end{aligned}
$$

where $c\mathscr{C}(z,\delta)$ is shorthand for $\mathscr{C}(x, c\delta)$.

Next we establish (b). Let $\{K_\phi(z_k,\cdot)\}$ be any sequence with $z_k \in \Omega$ (for $k = 1, 2, \ldots$). We show that it has a convergent subsequence in $L^1(\Omega)$. Since $\bar{\Omega}$ is compact, there is a subsequence $\{z_{k_l}\}$ and an element $z_0 \in \bar{\Omega}$ so that $\lim_{l\to\infty} z_{k_l} = z_0$. We shall show that $K_\phi(z_{k_l},\cdot)$ is a convergent sequence in $L^1(\Omega)$. It suffices to show it is a Cauchy sequence in $L^1(\Omega)$. For any $0 < \delta \leqslant 1$, since $K_\phi(\cdot,\cdot)$ is uniformly continuous on $\bar{\Omega} \times \Omega_\delta$, there is a positive integer N_δ such that if $l, m > N_\delta$, then

$$|K_\phi(z_{k_l}, w) - K_\phi(z_{k_m}, w)| \leqslant \delta$$

for all $w \in \Omega_\delta$. Thus for $l, m > N_\delta$, we have

$$
\begin{aligned}
&\|K_\phi(z_{k_l},\cdot) - K_\phi(z_{k_m},\cdot)\|_1 \\
&\leqslant \|(K_\phi(z_{k_l},\cdot) - K_\phi(z_{k_m},\cdot))\mathscr{X}_{\Omega_\delta}\|_1 + \|(K_\phi(z_{k_l},\cdot) - K_\phi(z_{k_m},\cdot))(1-\mathscr{X}_{\Omega_\delta})\|_1 \\
&\leqslant \|(K_\phi(z_{k_l},\cdot)(1-\mathscr{X}_{\Omega_{2\delta}})\|_1 + \|K_\phi(z_{k_m},\cdot))(1-\mathscr{X}_{\Omega_{2\delta}})\|_1 + \|(K_\phi(z_{k_l},\cdot) - K_\phi(z_{k_m},\cdot))\mathscr{X}_{\Omega_\delta}\|_1 \\
&\leqslant C \int_0^\delta \int_{\partial\Omega} (|K_\phi(z_{k_l}, w + tv(w))| + |K_\phi(z_{k_m}, w + tv(w))|)\, d\sigma(w)\, dt + C|\Omega|\,\delta \\
&\leqslant C \int_0^\delta \frac{1}{t(\log(1/t))^{1+\varepsilon}}\, dt + C|\Omega|\,\delta \\
&\leqslant C_\varepsilon \frac{1}{|\log \delta|^\varepsilon} + C|\Omega|\,\delta,
\end{aligned}
$$

where $v(w)$ is the unit inner normal vector to $\partial\Omega$ at w. Here we use the fact that $|\phi(z) - \phi(w)| \leqslant C|z-w| \leqslant C/(1+|\log r(z,w)|)$. Now (b) follows by letting $\delta \to 0$, and so the proof of Theorem 3.1 is complete.

Applying Theorem 3.1 and Theorem 2.3, we have the following result.

COROLLARY 3.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary and quasi-metric d on $\partial\Omega$ which satisfies (3.4). If K is a homogeneous kernel on Ω such that $K \in C(\bar{\Omega} \times \Omega_\varepsilon)$ for all $\varepsilon > 0$, then both $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.*

4. *Some standard domains satisfying the sufficient condition*

In this section, we shall give several examples of domains Ω in $\mathbb{C}^n$ which satisfy the condition of Corollary 3.2, and hence for which $A(\Omega)$ and $A(\Omega)^*$ satisfy the Dunford–Pettis property. First, we dispose of the known case of the unit ball in $\mathbb{C}^n$.

EXAMPLE 4.1 [**4**]. *If $\Omega = B_n$, then both $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.*

Proof. On the boundary of B_n, we have a quasi-metric d defined by

$$d(z, w) = |1 - \langle z, w\rangle|,$$

and the Bergman kernel $K(z, w)$ for B_n is:

$$K(z, w) = c_n^{-1} \frac{1}{(1 - \langle z, w\rangle)^{n+1}} \quad \text{for } z, w \in B_n,$$

where c_n is the volume of the unit ball B_n.

One can easily check that

(a) K is a homogeneous kernel;
(b) $|z - w| \leqslant (2d(z, w))^{1/2}$;
(c) $K(z, w) \in C(\overline{B_n} \times (B_n)_\varepsilon)$ for all $\varepsilon > 0$.

Therefore, by Corollary 3.2, $A(B_n)$ and $A(B_n)^*$ have the Dunford–Pettis property.

EXAMPLE 4.2. *If Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, then $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.*

Proof. We define the quasi-metric d on $\partial\Omega$ as follows. For $x \in \partial\Omega$, let π_x denote the complex tangent plane in $\mathbb{C}^n$ at x. For $t > 0$ let $A_{x,t}$ denote the set of points in $\mathbb{C}^n$ having distance at most t from the ball in the plane π_x with centre at x and radius $\sqrt{t}$. Let $B_{x,t} = A_{x,t} \cap \partial\Omega$. This is equivalent to the definitions given in [**24, 19**]. The quasi-metric on $\partial\Omega$ is defined by

$$d(x, y) = \inf\{t > 0; y \in B_{x,t}, x \in B_{y,t}\}.$$

One can easily check that

(a) $|z - w| \leqslant Cd(z, w)^{1/2}$ for all $z, w \in \partial\Omega$;
(b) the Bergman kernel $K(z, w)$ is a homogeneous kernel (apply Fefferman's asymptotic expansion of the Bergman kernel given in [**14**]);
(c) $K(z, w) \in C^\infty(\bar{\Omega} \times \Omega_\varepsilon)$ for all $\varepsilon > 0$ (apply the result of Kerzman in [**16**]).

By Corollary 3.2, $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.

EXAMPLE 4.3. *If Ω is a bounded pseudoconvex domain of finite type in C^2 with smooth boundary, then $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.*

Proof. We let d be the quasi-metric defined in [**20, 21**]. For convenience, we recall the definition. Let $p \in \partial\Omega$, and let $\Lambda(p, \delta)$ be the quantity defined in [**19**, p. 116]. Fix $\delta > 0$. Since $\Lambda(p, \tau)$ is strictly increasing in τ and $\Lambda(p, 0) = 0$, there is a unique $\tau = \tau(\delta, p)$ such that $\Lambda(p, \tau) = \delta$. Let X_1, X_2 be real vector fields such that X_1, X_2 and T span the real tangent space to $\partial\Omega$ at each point p. Here X_1, X_2 span the complex tangent space over R at each point and T points in the 'complex normal' direction. Then we define the ball $B(p, \delta)$ on $\partial\Omega$ by

$$B(p,\delta) = \{q \in \partial\Omega : q = \exp_p(\alpha_1 X_1 + \alpha_2 X_2 + \zeta T), |\alpha_j| \leqslant \tau(p,\delta) \text{ for } j = 1,2, |\zeta| \leqslant \delta\}.$$

Notice that $|B(p, \delta)| \approx \tau^2\delta$.

The quasi-metric d on $\partial\Omega$ is defined as follows:

$$d(z, w) = \inf\{t : z, w \in B(z, t) \text{ and } z \in B(w, t)\}.$$

Suppose that Ω is a domain of type m. Then, using [**19**, Theorems 3.1 and 3.2], one can easily check that

(a) $|z - w| \leqslant Cd(z, w)^{1/m}$ for all $z, w \in \partial\Omega$;
(b) the Bergman kernel $K(z, w)$ is a homogeneous kernel;
(c) $K(z, w) \in C(\bar{\Omega} \times \Omega_\varepsilon)$ for all $\varepsilon > 0$.

Therefore, by Corollary 3.2, $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.

5. *Application of $\bar{\partial}$-theory*

In many cases, the integral representing kernels for solutions of the $\bar{\partial}$-equations are simpler than the Bergman kernel, for example, for a bounded domain in the complex plane. In this section, we shall apply $\bar{\partial}$-theory (for which see [**17**]) and the ideas from Section 2 to prove that $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property for bounded Ω with smooth boundary. Let us start with the following more general theorem.

THEOREM 5.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary. If the $\bar{\partial}$ operator $T: C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous (here $\bar{\partial}T(f) = f$), then $A(\Omega)$ has the Dunford–Pettis property.*

Proof. According to Theorem 1.2, it suffices to prove that for each weakly null sequence $(f_n)_{n=1}^{\infty} \subset A(\Omega)$, we have

$$\lim_{n\to\infty} \text{dist}\,(\phi f_n, A(\Omega)) = 0 \tag{5.1}$$

for all $\phi \in C(\bar{\Omega})$. Since $C^1(\bar{\Omega})$ is dense in $C(\bar{\Omega})$, it suffices to prove that (5.1) holds for all $\phi \in C^1(\bar{\Omega})$.

Let $\phi \in C^1(\bar{\Omega})$. For each n, we let $u_n = T(f_n \bar{\partial}\phi)$. Since $T: C(\bar{\Omega}) \to C(\bar{\Omega})$, it follows that $u_n \in C(\bar{\Omega})$ for all n and $g_n := f_n\phi - u_n \in A(\Omega)$. Since $f_n \bar{\partial}\phi \to 0$ weakly and $T: C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous from $A(\Omega)$ to $C(\bar{\Omega})$, we have

$$\lim_{n\to\infty} \|u_n\|_\infty = 0.$$

However,

$$\text{dist}\,(\phi f_n, A(\Omega)) \leqslant \|\phi f_n - g_n\|_\infty = \|u_n\|_\infty,$$

completing the proof.

If we know the integral kernel of the solutions of the $\bar{\partial}$-equation on Ω, then we can check for complete continuity to obtain the Dunford–Pettis property. For example, in the case of the complex plane, we can prove the following theorem.

THEOREM 5.2. *Let Ω be a bounded domain in the complex plane with C^1 boundary. Then $A(\Omega)$ and $A(\Omega)^*$ have the Dunford–Pettis property.*

Proof. The $\bar{\partial}$-solution operator $T: C(\bar{\Omega}) \to C(\bar{\Omega})$ is given by

$$T(f)(z) = \frac{1}{\pi}\int_{\Omega} \frac{f(w)}{z-w}\, dA(w),$$

where dA is Lebesgue area measure. It is easy to see that $\{1/(w-z): z \in \Omega\}$ is a relatively compact set in $L^1(\Omega)$. Thus $T: C(\bar{\Omega}) \to C(\bar{\Omega})$ is completely continuous. By Theorem 5.1, $A(\Omega)$ has the Dunford–Pettis property. Alternatively, one can see the complete continuity by recalling that $T: C(\bar{\Omega}) \to \mathrm{Lip}_\alpha(\bar{\Omega})$ for any $\alpha < 1$ and that the embedding $\mathrm{Lip}_\alpha(\bar{\Omega}) \to C(\bar{\Omega})$ is compact.

In order to prove that $A(\Omega)^*$ has the Dunford–Pettis property, we need to repeat some parts of the proof of Theorem 2.3. In fact, for $\varepsilon > 0$ arbitrary, if we replace $K_\phi(z,w)$ in (2.4) by $\bar{\partial}\phi(w)/(z-w)$, then we have

$$\sup_{z\in\Omega} \left| \left\langle \frac{\bar{\partial}\phi(w)}{z-w}, x_n^{**} \right\rangle \right| < \varepsilon$$

if $n > N_\varepsilon$, where N_ε is some integer depending only on ε. By the related arguments in the proof of Theorem 2.3, if we let $x_\alpha \in A(\Omega)$ with $\|x_\alpha\|_\infty \leqslant 1$ be such that $\lim_\alpha \langle x_\alpha, x^* \rangle = \langle x_n^{**}, x^* \rangle$ for all $x^* \in A(\Omega)^*$, and if we define

$$u_\alpha(z) = \phi(z)\, x_\alpha(z) - T(x_\alpha \bar{\partial}\phi)(z),$$

then we have $u_\alpha \in A(\Omega)$ and

$$\begin{aligned} \operatorname{dist}(\phi x_\alpha, A(\Omega)) &\leqslant |\phi x_\alpha(z) - u_\alpha(z)| = |T(x_\alpha \bar{\partial}\phi)(z)| = \left| \int_\Omega \frac{x_\alpha(w)\,\bar{\partial}\phi(w)}{w-z}\, dA(w) \right| \\ &\leqslant \sup_{z\in\Omega} \left| \left\langle \frac{\bar{\partial}\phi(w)}{z-w}, x_n^{**} \right\rangle \right| + \varepsilon < 2\varepsilon. \end{aligned}$$

Therefore, we have

$$\operatorname{dist}(\phi x_n^{**}, A(\Omega)^{**}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Applying Theorem 1.2, the proof of Theorem 5.2 is complete.

REMARK. The main results of this paper can be proved alternatively by using the technique of $\bar{\partial}$-theory as represented in the proof of Theorem 5.2. The complete continuity of the relevant $\bar{\partial}$ solution operators can be found in **[23, 5, 17]**.

It was pointed out by the referee that Theorem 5.2 can be obtained in another way, using a result by Cima and Timoney in **[9]**.

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