Structure of JB^*-Triples

Bernard Russo

Abstract. This paper is a summary of the known structure of JB^*-triples. Its central focus consists of two structure theorems, due jointly to the author and Yaakov Friedman. Sections 2 and 3 discuss these results in detail. Sections 1 and 4 play somewhat different roles. The former discusses some general aspects of the subject, and gives some background. The latter discusses some topics which are especially interesting to the author.

The style of this survey is informal. Proofs of some major theorems are given in outline, together with the background material. In several places, a preview of forthcoming work is described and some problems are proposed.

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Introduction and overview

Why study JB^*-triples?

Here are two reasons from the functional analyst's point of view. In the first place, by the fundamental result of Kaup ([84]), JB^*-triples are in one to one correspondence with bounded symmetric domains in complex Banach spaces. In the finite dimensional case this is due to Koecher [86], cf. Loos [88]. Operators on Hilbert spaces of analytic functions is a mainstream topic in operator theory and has blossomed in the last 25 years. Although most attention has been focused on functions defined on the unit disk, recently much attention has been devoted to functions defined on the unit ball or on the unit polydisk in $\mathbb{C}^n$. More generally, Toeplitz, Hankel, and composition operators have been considered on spaces of holomorphic functions defined on Cartan domains (the finite dimensional bounded symmetric domains). A fundamental result here is the complete structure theory of the Toeplitz C^*-algebra of a Cartan domain, due to Upmeier (cf. Lecture 8 of [116]). More recently, the notion of quantization is playing a key role here (Upmeier [117] and Coburn).

A second justification for the study of JB^*-triples arises from the fact that the category includes C^*-algebras, JB^*-algebras, Hilbert spaces and spin factors. There is reason to believe that the well known Jordan algebra approach to quantum mechanics can be broadened in such a way that Jordan structures other than the binary ones will be significant. This is currently being explored by Friedman and Naimark [46].

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In a lecture at the second U.S.-Japan conference on operator algebras in 1977 ([39]), E. Effros introduced these three categories to study injectivity and nuclearity:

$\mathcal{N}$: real normed spaces, contractions;
$\mathcal{F}$: function systems, positive unital maps;
$\mathcal{O}$: operator systems, completely positive unital maps.

It was known for a long time that the appropriate algebraic models for the categories $\mathcal{F}$ and $\mathcal{O}$ are Jordan C^*-algebras and C^*-algebras, respectively. Investigation into the extent that the algebraic structure within these categories is influenced by geometric properties alone was carried out by Friedman–Russo ([49]). This investigation suggested that in the categories $\mathcal{F}$ and $\mathcal{O}$, geometric properties do not depend on the order structure, and at the same time introduced the appropriate algebraic model for the category $\mathcal{N}$, namely the JB^*-triple.

In order to see why a triple product rather than a binary one determines the geometry, we focus our attention on mappings of the algebraic models. If A is a unital C^*-algebra and w is any fixed unitary element in A, then T defined by $Tx = wx, x \in A$ is a linear isometry. Moreover, $\{Tx, Ty, Tz\} = \{xyz\}$, where

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x),$$

i.e., T preserves the Jordan triple structure. However, T does not preserve the order structure on A. In general, it is known that a completely positive unital isometry of a C^*-algebra preserves the (associative) C*-structure (Choi, Størmer, unpublished), a positive unital isometry of a C^*-algebra (more generally of a "JB-algebra") preserves the Jordan structure (Kadison [80], Wright-Youngson [121]), and an arbitrary isometry of a JC^*-triple (resp. JB^*-triple) preserves the Jordan triple structure (Harris [68], resp. Kaup [84]). Thus in each algebraic model the (surjective) norm preserving linear maps respect the algebraic structure.

Consider next the norm decreasing idempotent mappings in each model. Before 1980, it was known that the image of a completely positive unital projection on a C^*-algebra is a C^*-algebra ([20]), and that the image of a positive unital projection on a "JC-algebra" is a "JC-algebra" ([41]). Friedman–Russo have shown that the image of an arbitrary contractive projection on a JC^*-triple (a "concrete" JB^*-triple) is a JC^*-triple ([52]). Since in particular, the image of a contractive projection on a C^*-algebra has, in general, only a Jordan triple structure, this is further evidence that a JB^*-triple is the appropriate algebraic model in the category $\mathcal{N}$.

These three categories are the framework, and the following three tables, in which [ ] refers to literature references, constitute a guide to this survey of "classical associative and non-associative operator systems".

Table 1. Representation theorems

C^*-algebra	JB^*-algebra	JB^*-triple
	Gelfand Naimark theorems	
[60] Gelfand & Naimark 1943	[5] Alfsen, Shultz & Stormer 1978	[54] Friedman & Russo 1986
	State space characterization	
[6] Alfsen–Shultz & H.–Olsen 1980	[4] Alfsen & Shultz 1978	[59] Friedman & Russo 1992
	Equivalent geometric category	
[25] Connes	[76] Iochum 1984	[84][118][115] [82] Kaup 1983 Upmeier, Vigué 1976

Table 2. Structure theory

C^*-algebra	JB^*-algebra	JB^*-triple
	Factor classification	
[35][24][26][65] von Neumann 1929 Connes 1976 Haagerup 1983	[110][106][107] Topping 1965 Stormer 1968	[71][73][74][29] Horn 1984 Dang & Friedman 1987
	Duality	
[40] Effros 1963 [101] Sakai 1960	[103] Shultz 1979 [51] Friedman–Russo 1985	[17] Barton–Timoney 1986 [72] Horn 1987

Table 3. Linear mappings

C^*-algebra	JB^*-algebra	JB^*-triple
	Isometries	
[80] Kadison 1951	[80] Kadison 1951 [121] Wright & Youngson 1978	[84] Kaup 1983 [68] Harris 1973
	Contractive projections	
[20] Choi & Effros 1977	[41] Effros & Stormer 1979	[52][85] Kaup 1983 Friedman–Russo 1985
	Derivations	
[99][100][81] Kadison, Sakai 1966	[114] Upmeier 1980	[14][70] Ho 1992 Barton–Friedman 1990
	Amenability	
[27][63][19] Connes 1978 Haagerup 1983 Bunce–Paschke 1980		[70] Ho 1992
	Bilinear forms	
[64][93] Pisier 1978 Haagerup 1985	[23] Chu–Iochum & Loupias 1989	Barton–Friedman 1987 [13][15] B.F.–Russo 1992

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1. Contractive projections, quantum mechanics, bounded symmetric domains

We now know that the notion of contractive projection plays an (unexpected) ubiquitous role in the representation theory of Jordan triple systems. In the first two subsections of this section, the main results on contractive projections are described, setting the scene for the representation theory of section 2.

The third and fourth subsections introduce the quantum mechanical setting which is relevant to the representation theory of section 3.

In the final subsection, a beautiful and deep application of JB^*-triples to operator theory, due to Upmeier, is described.

1.1. Contractive projections on C^*-algebras. The commutative case

By a *contractive projection* on a normed space X we mean a continuous linear idempotent map $P : X \to X$ with $\|P\| = 1$. We are interested in the case that X is a Banach algebra A.

Let $P(A)$ denote the range of P, which is simply a closed linear subspace of A. If A is a C^*-algebra and $P(A)$ happens to be a C^*-subalgebra, then it is a well-known result of Tomiyama [108], that P is positive and satisfies the conditional expectation formula $P(axb) = aP(x)b$ for $a, b \in P(A)$ and $x \in A$.

In general, we can ask for the structure of P and of $P(A)$. That is, what algebraic properties do P and $P(A)$ have?

In 1980, the literature on this problem consisted of three papers:

- Choi–Effros 1974 [20]: If A is any unital C^*-algebra and P is completely positive and unital, then $P(A)$ is a C^*-algebra under the product $(a, b) \mapsto P(ab)$. The crux of the matter here was to prove the associative law, which amounts to a conditional expectation type formula:

$$P(P(ab)c) = P(aP(bc)), \quad a, b, c \in P(A).$$

- Effros–Stormer 1979 [41]: If A is any unital C^*-algebra and P is positive and unital, then $P(A_h)$ is a Jordan algebra under the product $(a, b) \mapsto P(a \circ b)$, where $a \circ b = (ab + ba)/2$. The main point here was to prove the Jordan identity, again, a conditional expectation formula:

$$P(a \circ P(b \circ a^2)) = P(P(a \circ b) \circ a^2), \quad a, b \in P(A).$$

- Arazy–Friedman 1978 [9]: If A is the C^*-algebra of compact operators on a separable Hilbert space, and P is any contractive projection on A, then $P = \oplus_{k=1}^{\infty} P_k$, $P_k = Q_k + T_k$ where Q_k has six possible forms: $Qx = x$, $Qx = (x + x^T)/2$, $Qx = (x - x^T)/2$, and three others which we will not describe here.

In each of these examples, the image of P, or of Q (=the "essential part" of P) was either a Jordan subalgebra or a Lie subalgebra. In 1980, it was natural to try to prove that if $P1 = 0$, then $P(A)$ should be a Lie algebra under the product $(a, b) \mapsto P([a, b])$. Failing this, it was "decided" that $P(A)$ must have an algebraic structure that includes both Jordan algebras and Lie algebras. It was also decided to work out completely the cases in which A is commutative or A is finite dimensional. This was done in [47] and [47a].

The following two theorems appear in Friedman–Russo 1982 [47]. The first one was the first detailed analysis of non-Markovian projections in this setting. It was extended to commutative real C^*-algebras (and corrected) in [21].

Theorem 1.1.1. *Let P be a contractive projection on a commutative C^*-algebra $A = C_0(K)$. Then there exist a family of norm one complex Borel measures $\{\mu_i\} \subset A^*$, with polar decompositions $\mu_i = \varphi_i.|\mu_i|$, and a contractive linear map $T : A^{**} \to A^{**}$ such that*

- *for $f \in A$,*

$$Qf := \sum_i \langle f, \mu_i \rangle \overline{\varphi_i} \text{ is continuous on } S = \text{the union of the supports of } \mu_i;$$

- *$\chi_S(T\xi) = 0$ for $\xi \in A^{**}$;*
- *$P = Q + T$ on A.*
The converse also holds.

The proof of this theorem revealed the following interesting facts:

- Pf is an isometric extension of Qf.
- $\overline{S}$ is a boundary for $P(A)$ in the sense of Shilov.

A *ternary algebra* is a linear space X with a trilinear map

$$[\cdot, \cdot, \cdot] : X \times X \times X \to X$$

satisfying

- $[a, a, a] = 0 \Leftrightarrow a = 0$;
- $[[a, b, c], d, e] = [a, [b, c, d], e] = [a, b, [c, d, e]]$.

X is *commutative* if $[a, b, c] = [c, b, a]$. The main literature references on this are [69], [87], and [122].

If a ternary algebra X is also a Banach space, then we have a *C^*-ternary algebra* if in addition

- $\|[a, b, c]\| \leq \|a\|\|b\|\|c\|$;
- $\|[a, a, a]\| = \|a\|^3$.

Theorem 1.1.2. *Let P be a contractive projection on a commutative C^*-algebra $A = C_0(K)$. Then $P(A)$ is a C^*-ternary algebra under the triple product $(f, g, h) \mapsto P(f\overline{g}h)$.*

The proof of this theorem revealed the following:

- $P(A)_{|\overline{S}}$ is a ternary <u>sub</u>-algebra of $C_0(\overline{S})$
- $P(A)$ is a subalgebra of $A \Leftrightarrow P$ is averaging in the sense of Birkhoff.

1.2. Algebraic and holomorphic methods in operator systems

The Riemann Mapping theorem states that any simply connected domain in the complex plane $\mathbb{C}$ is holomorphically equivalent either to $\mathbb{C}$ or to the open unit disk. For domains in $\mathbb{C}^n$, a classification is not possible without further assumption, such as the existence of an isolated holomorphic symmetry at each point.

For bounded domains, a consequence of Cartan's Uniqueness Theorem (1931) is that the set $\mathcal{G} = \mathrm{aut}\,(D)$ of complete holomorphic vector fields on D is a real Lie algebra making the group $G = \mathrm{Aut}\,(D)$ of all holomorphic automorphisms of D a real Lie group. From this the irreducible bounded symmetric domains in $\mathbb{C}^n$ were classified (1935).

Is there a Riemann mapping theorem or classification of bounded symmetric domains in an arbitrary complex Banach space? Although such a project is not feasable, the following progress was made.

• Upmeier 1976 [115]: the set $\mathcal{G} = \mathrm{aut}\,(D)$ of complete holomorphic vector fields on a bounded symmetric domain D in a complex Banach space is a real Banach Lie algebra and the group $G = \mathrm{Aut}\,(D)$ of all holomorphic automorphisms of D a real Banach Lie group.

• Vigué 1976 [118]: every bounded symmetric domain in a complex Banach space has a Harish-Chandra realization, that is, is holomorphically equivalent to a circled domain containing the origin.

• Kaup 1982 [83]: classified the bounded symmetric domains in a Hilbert space.

• Kaup 1983 [84]: (Riemann mapping theorem) every bounded symmetric domain in a complex Banach space is holomorphically equivalent to the open unit ball of some Banach space (a JB^*-triple).

Kaup's Riemann mapping theorem set up an equivalence of categories

$$\{\text{Bounded Symmetric Domains}\} \leftrightarrow \{JB^*\text{-triples}\}.$$

For a summary of the construction of the triple product on the JB^*-triple corresponding to a bounded symmetric domain, see Loos 1977 [88], or Friedman–Russo 1986 [54].

A forerunner of Kaup's theorem in one direction, in the special case of a C^*-algebra, was proved by Harris 1973 [68], thereby providing a link between infinite dimensional holomorphy and functional analysis: the open unit ball of a C^*-algebra is a bounded symmetric domain, a transitive family of automorphisms being provided by the Möbius transformations

$$a \mapsto (1 - bb^*)^{-1/2}(a + b)(1 + b^*a)^{-1}(1 - b^*b)^{1/2}.$$

Since this formula involves only the symmetrized triple product $ab^*c + cb^*a$, the result holds for *J^*-algebras* (now called JC^*-triples), that is, norm closed subspaces of $\mathcal{L}(H, K)$ stable for the map $a \mapsto aa^*a$.

Harris's result suggested that it is fruitful to find connections between functional analysis and other fields, and to exploit the connection. For example, index theory for elliptic operators connects algebraic topology and operator theory, and Connes' theory of cyclic cohomology connects differential geometry and operator algebras.

Kaup's Riemann mapping theorem suggests developing the theory of JB^*-triples and exploiting their relation with bounded symmetric domains.

As a step in this direction, Friedman–Russo solved the contractive projection problem in 1983 [52].

Theorem 1.2.1. *Let P be a contractive projection on a JC^*-triple M. Then $P(M)$ is a JB^*-triple under the triple product $(a, b, c) \mapsto P((ab^*c + cb^*a)/2)$ and $P(M)$ is isomorphic to a JC^*-triple.*

Friedman–Russo also pointed out how choosing the correct category (in this case, JC^*-triples instead of C^*-algebras) leads to insight and in some cases simplification.

After receiving a preprint of this work, Kaup [85] discovered an elegant short proof of the first statement of Theorem 1.2.1, valid in the category of JB^*-triples.

Theorem 1.2.2. *Let P be a contractive projection on a JB^*-triple U. Then $P(U)$ is a JB^*-triple under the triple product $(a, b, c) \mapsto P(\{abc\})$ where $\{abc\}$ is the original triple product in U.*

The proof of Theorem 1.2.1 is functional analytic, strongly dependent on the underlying Hilbert spaces, and is long. Kaup's proof of Theorem 1.2.2 uses holomorphic methods almost exclusively and is quite short.

We conclude this subsection with the following moral. Theorem 1.2.1 shows that results can be improved, clarified, and unified by a consideration of the appropriate category, in this case the one consisting of systems with a triple product rather than a binary one. Theorem 1.2.2 shows that further improvement and elegance can be obtained by working in an equivalent category, in this case a geometric one rather than an algebraic one.

1.3. State spaces of quantum mechanics

In the period 1978-1982 several papers appeared which characterized the state spaces of various topological algebraic structures in terms of affine geometric physically motivated axioms on a compact convex set.

The cornerstone result in this direction was the work of Alfsen–Shultz in 1978 [4]. We do not define all the terms here.

Theorem 1.3.1. *A compact convex set K is the state space of some JB-algebra if and only if it satisfies the following four properties:*

- *Hilbert ball property;*
- $K = K_1 \oplus K_2$, *K_1 atomic, K_2 non-atomic;*
- *all norm exposed faces are projective;*
- $A^{\mathbf{R}}(K) = A^+(K) - A^-(K)$.

At the same time, Alfsen and Shultz introduced the three pure state properties, which can replace the first two of the above properties:

- extreme points are norm exposed;
- P-projections preserve extreme rays;
- symmetry of transition probabilities.

Applications of this result to the cases of C^*-algebras and W^*-algebras were done by Alfsen–Hanche-Olsen–Shultz 1980 [6] and Iochum–Shultz 1982 [77]. A finite dimensional application for JB-algebras, but with more physicallly appealing axioms was done by Araki 1980 [8].

In the process of solving the contractive projection problem (Theorem 1.2.1), Friedman–Russo proved analogs of six out of the above seven properties in the case $K = \text{ball}\, Q(M^*)$, where Q is a contractive projection on the dual M^* of a JC^*-triple M.

In 1983 the following were three outstanding problems in the theory of JB^*-triples. All three are motivated by the classical theory of operator algebras, and have now been solved.

- Characterize the unit ball of the dual of a JB^*-triple. Give physical meaning to the axioms. (Note that this involves the case Q=identity, M= a JB^*-triple, of the contractive projection problem). The solution to this problem, due to Friedman and Russo [59], is described in section 3. See also the discussion in subsection 1.4.
- Find a universal enveloping object in the category of JB^*-triples, that is, show that the second dual of a JB^*-triple is a JB^*-triple. This was solved by Dineen [32] and is a key step in the solution of the next problem.
- Find a Gelfand-Naimark type representation theorem for JB^*-triples. The solution to this problem, due to Friedman and Russo [54], is described in section 2.

1.4. Geometric spectral theory

The spectral theorem says that for an operator x in a Hilbert space,

$$x = x^* \Rightarrow x = \int_{-\infty}^{\infty} \lambda \, de_\lambda$$

where $\{e_\lambda\}$ is a resolution of the identity supported on the spectrum of x. In particular,

$$x \geq 0 \Rightarrow x = \int_0^{\infty} \lambda \, de_\lambda.$$

Thus any self-adjoint $x \in \mathcal{L}(\mathcal{H})$, can be approximated in norm by operators of the form $\sum \lambda_i e_i$ with $\lambda_i \in \mathbb{R}$ and $\{e_i\}$ a finite family of pairwise orthogonal projections.

Using the polar decomposition $z = u|z|$ for an arbitrary bounded operator leads to the approximation in norm of z by finite sums $\sum \lambda_i v_i$ where $\lambda_i \geq 0$ and $\{v_i\}$ is a family of pairwise orthogonal partial isometries.

In an arbitrary normed space X, when does a (geometric) spectral theorem hold? That is, under what conditions can we write, for any element $x \in X$, uniquely,

$$x = \sum \lambda_i u_i$$

where $\lambda_i \geq 0$ and the u_i form an "orthogonal" family of "distinguished elements" (building blocks). The appropriate meanings of "orthogonal" and "distinguished element", as well as that of the sum, will depend on the underlying metric structure (norm, affine geometry) of X.

The natural setting in which to discuss spectral theory and polar decompostion of operators on Hilbert space is that of a von Neumann algebra, and more generally a C^*-algebra, or their non-associative analogs (for instance, Jordan operator algebras). In these settings, another important tool is a polar decompostion for a (not necessarily positive) normal functional.

In the algebraic approach to quantum mechanics, observables correspond to self-adjoint operators, and states correspond to positive normalized functionals. Making use of the order structure inherent in these models, Alfsen–Shultz, and later Alfsen–Hanche-Olsen–Shultz, Araki, and Iochum–Shultz showed (cf. subsection 1.3) that in an ordered Banach space, a compact convex set which satisfied a set of axioms, some with physical significance, was affinely isomorphic to the state space of an operator algebra of one of the above types. An initial tool in the proofs of these results was a spectral theorem for arbitrary elements in ordered Banach spaces satisfying some of the axioms referred to above.

In view of the above discussion, it is reasonable to ask whether a Banach space in which the unit ball satisfies some physically meaningful affine geometric axioms is isomorphic to the unit ball of the dual space of some C^*-algebra. Presumably, some preliminary steps would be the investigation of that validity of a polar decomposition of an arbitrary element (thought of as a "normal functional" on the dual space), and a spectral decomposition of an arbitrary element of the dual space. Since no order structure is assumed, it is more realistic to expect to obtain characterizations of the unit ball of the dual (respectively predual) of the non-ordered analogs of operator algebras, namely the JB^*-triples (respectively JBW^*-triples). Thus a possible interpretation of this subsection is a formulation of a geometric and order-free approach to quantum mechanics, in which observables correspond to operators (not necessarily self-adjoint), and states correspond to normalized functionals (not necessarily positive).

In fact, all of the properties considered by Alfsen–Shultz, with one exception (which has no meaning in an order-free context), have analogs which are satisfied by the dual spaces of JB^*-triples.

In the rest of this subsection, we shall introduce, based on the quantum mechanical measuring process, a class of Banach spaces which admits a polar decomposition of an arbitrary element and a spectral decomposition of an arbitrary element of its dual space. The unit ball of a member of this class is then the natural candidate for the "geometric state space" of a quantum mechanical system, in the sense that, under the appropriate set of "Alfsen–Shultz" type axioms, this class should coincide with the class of all preduals of JBW^*-triples.

Let us begin by recalling the following affine geometric properties of an operator algebra. Recall first that for an element x of norm 1 in a von Neumann algebra M, F_x denotes the norm-exposed face in the unit ball of the predual M_* of M determined by x. Also, for elements $f, g \in M_*$ f is orthogonal to g ($f \perp g$) if

$$\|f \pm g\| = \|f\| + \|g\|. \tag{1.1}$$

We have the following [57], which we use as the basis for our model:

(1) An element $x \in M$ is a partial isometry if and only if $\|x\| = 1$ and $\langle x, F_x^{\perp} \rangle = 0$.
(2) Partial isometries u and v are orthogonal if and only if $F_u \perp F_v$.
(3) There is a bijection of the set of all partial isometries and the set of norm exposed faces of the unit ball of M_*, given by the map $u \mapsto F_u$.

For an arbitrary normed space X, define orthogonality of elements f, g by ((1.1)), and denote it by $f \Diamond g$. Then define a *projective unit* to be an element x satisfying $\|x\| = 1$ and $\langle x, F_x^{\Diamond} \rangle = 0$, and define *orthogonality* of two projective units by $F_u \Diamond F_v$.

Then we have the following two questions:

(A) Is there a bijection $u \mapsto F_u$ of the set of projective units and the set of norm exposed faces?
(B) Is every element of X approximable in norm by a linear combination of pairwise orthogonal projective units?

The terms and the result in the next two paragraphs are from [53]. They will be discussed more fully in section 3.

Let Z be a *weakly facially symmetric space* (WFS space). Then the map $u \mapsto F_u$ is not a bijection in general, and the spectral theorem fails. However, the map $u \mapsto F_u$ restricts to a bijection of the set of all *geometric tripotents* and the set of all *symmetric faces.*

On the other hand, suppose Z is a *strongly facially symmetric space* (SFS space). Then (assuming that Z is reflexive), every $x \in Z^*$ can be written uniquely in the form

$$x = \sum_{i=1}^{n} \lambda_i v_i$$

where $\lambda_i > 0$ and $\{v_i\}$ is a pairwise orthogonal family of geometric tripotents.

This spectral theorem is used only peripherally in the main result of section 3. By contrast, the following geometric polar decomposition, from [56], is of paramount importance.

Let Z be a *neutral* SFS space. Then for any non-zero $f \in Z$, there exists a unique geometric tripotent v with $\|f\|^{-1}f \in F_v$, and $\langle v, \{f\}^{\diamond}\rangle = 0$. Moreover, F_v is minimal ($\|f\|^{-1}f \in F_x \Rightarrow F_v \subset F_x$) and f is "faithful" and "positive" on F_v in the sense that $F_u \subset F_v \Rightarrow f(u) > 0$.

1.5. Toeplitz C^*-algebras. Application of finite dimensional JB^*-triples

Bounded symmetric domains are higher dimensional generalizations of the open unit disk. H. Upmeier, in the mid 1980's, obtained a complete structure theory for the C^*-algebra $\mathcal{T}$ generated by all Toeplitz operators $T_f h = P(fh)$ with continuous symbol $f \in C(S)$ on the Shilov boundary S of a bounded symmetric domain D of arbitrary rank r. Here, $h \in H^2(S)$, P is the orthogonal projection of $L^2(S, \mu)$ onto the Hardy space $H^2(S)$, and μ is the unique probability measure invariant under the compact group K of all holomorphic automorphisms of D fixing 0. Previous work of Berger, Coburn, Koranyi in the 1970's dealt with the rank one case and the tube domain case of rank 2.

The boundary structure of a bounded symmetric domain is more complicated than that of the strictly pseudoconvex case, for which Toeplitz operators have been studied extensively.

Upmeier's theory is based on the fact that the domain D can be realized as the open unit ball of a finite dimensional JB^*-triple Z. The boundary structure of D can be described by using the tripotents of Z. A new feature is that $\mathcal{T}/\mathcal{K}$, $\mathcal{K} =$ the compacts, is not generally abelian, so the concept of solvable C^*-algebra, due to Dynin [37] is used.

Theorem 1.5.1. *The Toeplitz C^*-algebra $\mathcal{T}$ associated with a bounded symmetric domain $D \subset Z$ of rank r is solvable of length r. That is, there exist closed two-sided ideals*

$$\{0\} = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_r \subset \mathcal{I}_{r+1} = \mathcal{T}$$

with

$$\mathcal{I}_{k+1}/\mathcal{I}_k \equiv C(S_k) \otimes \mathcal{K}(H_k),$$

where S_k is the compact manifold of all tripotents of rank k.

Moreover, the set of equivalence classes of irreducible representations of $\mathcal{T}$ can be identified with the set of tripotents of Z, $\mathcal{I}_r$ is the closed commutator ideal of $\mathcal{T}$, and $\mathcal{T}/\mathcal{I}_r \equiv C(S)$, where $S = S_r$ is the Shilov boundary. Further, $\mathcal{I}_1 = \mathcal{K}(H^2(S))$.

The proof of this result is contained in a series of three papers of H. Upmeier, the contents of which are summarized as follows.

In the first paper [111], the following is achieved:

- a description of the root spaces of the semi-simple Lie algebra $\mathbf{k}^{\mathbb{C}}$ in terms of Peirce decompositions (here $\mathbf{k}$ is the Lie algebra of K);

- a separation of variables formula for polynomials on JB^*-algebras;
- explicit Peter–Weyl decomposition of the unitary K-module $H^2(S)$.

In the second paper [112], the following is achieved:

- a characterization of the Poisson and Szego kernel functions in Jordan theoretic terms;
- a relation between Toeplitz operators and differential operators;
- a description of the irreducible representation σ_e corresponding to a tripotent e in terms of Toeplitz operators on a lower dimensional bounded symmetric domain.

In the third paper [113], Theorem 1.5.1 is obtained by setting $\sigma_k := \oplus_{e\in S_k}\sigma_e$ $(0 \leq k \leq r)$, and $\mathcal{I}_k := \ker \sigma_k$.

I consider this result of Upmeier's to be one of the deepest and most beautiful applications of Jordan theory to analysis. For an elementary approach to the Peter–Weyl decompositon, I strongly recommend the book by Faraut–Koranyi [44].

2. Gelfand Naimark theorem

This section is devoted to an exposition of a representation theorem of Gelfand-Naimark type for JB^*-triples.

The first subsection recalls the statements and ideas of the proofs of the earlier Gelfand-Naimark type theorems for the categories of C^*-algebras and JB^*-algebras. As in the proof of the Jordan algebra version and a later proof of the C^*-algebra version, a reduction is made to the case of the JB^*-triples which are dual spaces by passing to the second dual. Thus the structure theory of this special case is reviewed in the second subsection. The next two subsections outline proofs of the main theorem, due to Friedman–Russo and Dang–Friedman respectively. The section concludes with a candidate for the definition of real JB^*-triple, and a Gelfand representation theory in the commutative case. The infinite dimensional non-commutative real triples need to be further explored from the point of view of functional analysis.

2.1. Gelfand Naimark theorem for C^*-algebras and Jordan algebras

Let's begin by recalling the original Gelfand-Naimark theorem of 1943. Let A be a C^*-algebra, that is, a Banach *-algebra satisfying $\|x\|^2 = \|x^*x\|$. Each state $\varphi \in S(A)$ of A gives rise via the GNS (Gelfand–Naimark–Segal) construction to a representation (π_φ, H_φ). For each set S of states one forms a representation $\pi_S = \oplus_{\varphi \in S}\pi_\varphi$ which is faithful if $S \subset S(A)$ separates the points of A.

The following is the celebrated Gelfand–Naimark theorem for C^*-algebras. A complete proof can be found in most books on functional analysis, for example [97]

Theorem 2.1.1 (Gelfand–Naimark theorem for C^*-algebras). *Each C^*-algebra is isometrically isomorphic to a C^*-algebra of operators on a complex Hilbert space. Each commutative C^*-algebra is isometrically isomorphic to the C^*-algebra of all continuous complex valued functions vanishing at infinity on some locally compact Hausdorff space.*

As a by-product of the above proof we obtain the following, which was observed by Sherman and Takeda in 1954 [34]. Let $\pi_u = \oplus_{\varphi \in S(A)}\pi_\varphi$ be the *universal representation* of A. Then A^{**} is isometrically isomorphic to the von Neumann algebra which is the weak closure of $\pi_u(A)$.

Anticipating the rest of this subsection, we ask rhetorically at this point: Where do the axioms of a C^*-algebra come from?

In the middle of the 1960's, Topping [110] and Stormer [106], [107] began the study of real Jordan subalgebras of $\mathcal{L}(H)_{sa}$ (H a complex Hilbert space). These were called JC-algebras if norm closed and JW-algebras if weakly closed.

The abstract version of these Jordan operator algebras are the JB-algebras and were defined as early as 1948 by Segal [102]. Thus, the axioms of a JB-algebra came from physics; more recently, in view of Iochum's thesis 1982 [76], they can be said to come from geometry too.

The complexification of a JB-algebra is called a JB^*-algebra and previously went under the name of Jordan C^*-algebra (Kaplansky, Wright [120]).

In the 1930's two important steps on the algebraic side of Jordan algebra theory were the classification of finite dimensional formally real Jordan algebras over the reals (Jordan, von Neumann, and Wigner 1934 [79]) and the fact that exceptional Jordan algebras exist (Albert).

Since the finite dimensional JB-algebras coincide with the formally real ones, a Gelfand–Naimark theorem for Jordan Banach algebras must exclude the exceptional algebras. The following is due to Alfsen–Shultz–Størmer in 1978 [5], [67].

Theorem 2.1.2 (Gelfand–Naimark theorem for Jordan algebras). *If A is a JB-algebra, then there is unique closed ideal J such that A/J is isometrically isomorphic to a JC-algebra, and J is purely exceptional, that is, every representation of J into some $\mathcal{L}(H)_{sa}$ is zero.*

The original proof of this theorem follows a well known path, but is long, requiring new techniques to deal with the non-associativity. These techniques include the following:

- ordered Banach spaces
- topologies on the enveloping monotone completion $\tilde{A} \subset A^{**}$
- spectral theory (singly generated subalgebras are continuous function spaces)
- comparison and equivalence in the lattice of projections
- analysis of spin factors
- coordinatization.

2.2. Some structure theory for JBW^*-triples

We begin with the definition and some properties of JB^*-triples. The reader who finds this subsection uncomfortable might prefer the more leisurely discussion of the same topics in subsection 3.1. This subsection is essentially a summary of [51].

Definition 2.2.1. A Banach space U over $\mathbb{C}$ is said to be a *JB^*-triple* if it is equipped with a continuous triple product $(a, b, c) \mapsto \{abc\}$ mapping $U \times U \times U$ to U such that

(i) $\{abc\}$ is linear in a and c and conjugate linear in b;

(ii) $\{abc\}$ is symmetric in the outer variables, *i.e.*, $\{abc\} = \{cba\}$;

(iii) for any $x \in U$, the operator $\delta(x)$ from U to U defined by $\delta(x)y = \{xxy\}$, is hermitian (*i.e.*, $\exp it\delta$ is an isometry for all real t) with non-negative spectrum;

(iv) the triple product satisfies the following identity, called the "main identity":

$$\delta(x)\{abc\} = \{\delta(x)a, b, c\} - \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\}; \tag{2.1}$$

(v) the following norm condition holds:

$$\|\{xxx\}\| = \|x\|^3. \tag{2.2}$$

Define the quadratic operator $Q(x)$ by $Q(x)y = \{xyx\}$ and then set $Q(x,y)z = \{xzy\}$. A tripotent is an element e satisfying $e = Q(e)e = \{eee\}$. With each tripotent there are associated the Peirce projections:

- $P_2(e) = Q(e)^2$
- $P_1(e) = 2(\delta(e) - Q(e)^2)$
- $P_0(e) = I - 2\delta(e) + Q(e)^2$

so that we have

$$I = P_2(e) + P_1(e) + P_0(e) \text{ and } \delta(e) = P_2(e) + \frac{1}{2}P_1(e),$$

and the Peirce decomposition: $U = U_2(e) \oplus U_1(e) \oplus U_0(e)$.

The following are fundamental *algebraic* properties of a tripotent e in a Jordan triple system U.

- $\{U_i(e), U_j(e), U_k(e)\} \subset U_{i-j+k}(e)$
- $\{U_0(e), U_2(e), U\} = 0 = \{U_2(e), U_0(e), U\}$
- $U_2(e)$ is a complex unital Jordan algebra with involution:

$$x \circ y := \{xey\} \qquad x^\sharp := \{exe\}.$$

The following are fundamental *topological* properties of a tripotent e in a JB^*-triple U.

- $U_2(e)$ is a JB^*-algebra, and $U_2(e)_{sa}$ is a JB-algebra
- The Peirce projections are contractive, as is $P_2(e) + P_0(e)$
- The family $S_\lambda(e) := \lambda^2 P_2(e) + \lambda P_1(e) + P_0(e)$ for $\lambda \in T$ is a one parameter group of *isometries*.

The following basic proposition has the interpretation that the Peirce spaces $U_2(e)$ and $U_0(e)$ have the *unique Hahn-Banach extension property.* It goes under the name "neutrality".

Proposition 2.2.2 (Proposition 1 of [51]). *Let U be a JB^*-triple, $e \in U$ a tripotent, and $f \in U^*$. Then $f \circ P_2(e) = f$ if and only if $\|f \circ P_2(e)\| = \|f\|$.*

A partial order for tripotents is defined by: $e \leq e'$ if $e' - e$ is a tripotent orthogonal to e, where a orthogonal to b means $\{ab\cdot\} = 0$.

Corollary 2.2.3. *$e \leq e'$ if and only if $P_2(e)e' = e$.*

A *JBW^*-triple* is a JB^*-triple which is the dual of some Banach space. It is known that a JBW^*-triple has a unique predual and that the triple product in a JBW^*-triple is separately weak*-continuous. This will be discussed in subsection 2.3. The predual of U is denoted by U_*.

By the Gelfand theory of commutative JBW^*-triples, one has a polar and spectral decomposition of an arbitrary element of a JBW^*-triple. For functionals on the other hand, we have the following ([51, Proposition 2]):

Proposition 2.2.4 (Polar decomposition of a normal functional). *Let U be a JBW^*-triple and let $f \in U_*$. Then there is a unique tripotent e, called the support tripotent of f, such that $f \circ P_2(e) = f$ and $f|_{U_2(e)}$ is a faithful positive normal functional.*

Another key property of the predual of a JBW^*-triple is the fact ([51, Proposition 8]) that every norm exposed face of ball U_* is "projective" (cf. Theorem 1.3.1). In this context this simply means that for every norm exposed face F_x, that is, with $x \in U$ of norm 1,

$$F_x = \{\rho \in (U_*)_1 : \|\rho\| = 1 = \rho(x)\},$$

there is a tripotent w such that $F_x = F_w$.

The Peirce projections are fundamental operators on JB^*-triples. When do they commute? The most general condition has been given in [90]. For our purposes here, the following sufficient condition is adequate.

Proposition 2.2.5 (Lemma 1.10 of [51]). *If e and v are tripotents in a JB^*-triple U and if one of them belongs to one of the Peirce spaces of the other, then $[P_\alpha(e), P_\beta(v)] = 0$ for all $\alpha, \beta \in \{0, 1, 2\}$.*

Atomic decompositions

This subsection is devoted to the decompositions of a JBW^*-triple U and its predual into atomic and purely non-atomic parts. We therefore next explain some of the definitions and tools needed in the proofs.

A tripotent e in a JB^*-triple U is *minimal* if the Peirce 2-space $U_2(e)$ is one-dimensional. In any JBW^*-triple, there is a bijection between minimal tripotents and extreme points of the unit ball of the predual, and each such extreme point is norm exposed ([51, Proposition 4]).

The map

$$\sum_i \alpha_i f_i \mapsto \sum_i \overline{\alpha}_i e_i,$$

where e_i is the support tripotent of the extreme point f_i, is a conjugate-linear bijection of the finite span of the extreme points onto the finite span of the minimal tripotents ([51, Lemma 2.11]).

The following four properties, analogs of results for JB-algebras [4], are instrumental for the main decomposition theorems.

Symmetry of Transition Probabilities [51, Lemma 2.2]
If f_1, f_2 are extreme points with support tripotents e_1, e_2 respectively, then

$$f_1(e_2) = \overline{f_2(e_1)}.$$

Hilbert Ball Property [51, Proposition 5]
If u and v are minimal tripotents in any Jordan triple system, then the smallest Jordan triple system containing these two elements is of dimension at most 4, and is thus isomorphic to one of the following spaces:

$$\mathbb{C},\ M_{1,2}(\mathbb{C}),\ \mathbb{C} \oplus \mathbb{C},\ M_2(\mathbb{C}),\ S_2(\mathbb{C}).$$

Extreme Ray Property [51, Proposition 7]
If e is any tripotent and f is an extreme point, then $P_2(e)^* f$ is a scalar multiple of an extreme point.

Minimal Ray Property [51, Proposition 6]
If e is any tripotent and u is a minimal tripotent, then $P_2(e)u$ is a scalar multiple of a minimal tripotent.

Theorem 2.2.6 (Atomic decomposition of U_* [51, Theorem 1]). *If U is a JBW^*-triple, then*

$$U_* = \mathcal{A} \oplus^{\ell^1} \mathcal{N},$$

where $\mathcal{A}$ is the norm-closure of the span of the extreme points of the unit ball of U_*, and $\mathcal{N}$ is a closed subspace of U_* whose unit ball has no extreme points.

Proof. If $\varphi \in U_*$ has support tripotent e, then the polar decomposition says that φ restricts to a normal faithful state on the JBW^*-algebra $U_2(e)$. Thus φ decomposes locally. The minimal ray property is then used to show that this is a global decomposition. □

Note that the corresponding result for JBW^*-algebras is elementary since the projections form a complete lattice.

The original proof of the next theorem uses the map

$$\sum_i \alpha_i f_i \mapsto \sum_i \overline{\alpha}_i e_i,$$

and the extreme ray property. Later, proofs were given in [33], [91], and [29].

Theorem 2.2.7 (Atomic decomposition of U [51, Theorem 2]). *If U is a JBW^*-triple, then*

$$U = A \oplus^{\ell^\infty} N,$$

where A is an ideal which is the weak-closure of the span of the minimal tripotents of U, and N is a weak*-closed ideal with no minimal tripotents.*

Some further structure theory for JBW^*-triples will be discussed in subsections 2.3, 2.4 (type I) and 4.3 (continuous).

Facial structure in a JBW^*-triple and its predual

The facial structure of the closed unit balls in JBW-algebras and their preduals were described by Edwards–Rüttimann by means of elements of the complete lattice of idempotents. One of their main methods, which is also available in the complex case, is the use of the mappings $E \mapsto E'$ and $F \mapsto F_{\prime}$ between subsets of the unit balls $\mathrm{ball}\, V$ and $\mathrm{ball}\, V^*$ in a Banach space V and its dual V^* defined by

$$E' = \{a \in \mathrm{ball}\, V^* : a(x) = 1, \forall x \in E\}$$

and

$$F_{\prime} = \{x \in \mathrm{ball}\, V : a(x) = 1, \forall x \in F\}.$$

The following two theorems appear in [38]. Let $\mathcal{U}(A)$ denote the set of tripotents of the JBW^*-triple A with a largest member adjoined.

Theorem 2.2.8. *Let A be a JBW^*-triple with predual A_*. Then the mapping $u \mapsto u_{\prime}$ is an order isomorphism from the complete lattice $\mathcal{U}(A)$ onto the complete lattice of norm-closed faces of the unit ball of A_*. In particular, every norm closed face of $\mathrm{ball}\, A_*$ is norm-exposed.*

Theorem 2.2.9. *Let A be a JBW^*-triple. Then the mapping $u \mapsto u_{\prime}{}'$ is an anti-order isomorphism from the complete lattice $\mathcal{U}(A)$ onto the complete lattice of weak*-closed faces of the unit ball of A. Moreover, $u_{\prime}{}'$ coincides with $u + \mathrm{ball}\, A_0(u)$.*

2.3. Gelfand Naimark theorem for JB^*-triples

JB^*-triples are generalizations of JB^*-algebras and hence of C^*-algebras. The axioms can be said to come from geometry in view of Kaup's Riemann mapping theorem. JB^*-triples first arose in M. Koecher's proof ([86]) of the classification of bounded symmetric domains in $\mathbb{C}^n$. The original proof of this fact, done in the 1930's by Cartan, used Lie algebras and Lie groups, techniques which do not extend to infinite dimensions. On the other hand, to a large extent, the Jordan algebra techniques do so extend, as shown by Kaup and Upmeier.

The following is due to Friedman–Russo ([54]). The Cartan factors are defined in subsection 3.1.

Theorem 2.3.1 (Gelfand-Naimark for JB^*-triples). *Every JB^*-triple is isometrically isomorphic to a subtriple of an ℓ^∞-direct sum of Cartan factors.*

This theorem is not unexpected. However, the proof required new techniques because of the lack of an order structure on a JB^*-triple. Here is a chronology of the proof of the theorem. Some of the steps have been mentioned already.

step 1: February 1983 Friedman–Russo ([52])
Let $P : A \to A$ be a linear projection of norm 1 on a JC^*-triple A. Then $P(A)$ is a JB^*-triple under $\{xyz\}_{P(A)} := P(\{xyz\}_A)$ for $x, y, z \in P(A)$. (This was new for $A =$ a C^*-algebra.)

step 2: April 1983 Friedman–Russo ([50])
Same hypotheses. Then P is a conditional expectation, that is, for $a, b, c \in A$,

$$P\{PaPbPc\} = P\{PabPc\} \text{ and } P\{PaPbPc\} = P\{aPbPc\}.$$

step 3: May 1983 Kaup ([85])
Let $P : U \to U$ be a linear projection of norm 1 on a JB^*-triple U. Then $P(U)$ is a JB^*-triple under $\{xyz\}_{P(U)} := P(\{xyz\}_U)$ for $x, y, z \in P(U)$. Also $P\{PaPbPc\} = P\{PabPc\}$ for $a, b, c \in U$, which extends one of the formulas in the previous step.

step 4: February 1984 Friedman-Russo ([51])
Every JBW^*-triple splits into atomic and purely non-atomic ideals.

step 5: August 1984 Dineen ([32)]
The bidual of a JB^*-triple is a JB^*-triple.

step 6: October 1984 Barton–Timoney ([17])
The bidual of a JB^*-triple is a JBW^*-triple, that is, the triple product is separately weak*-continuous.

step 7: December 1984 Horn ([71],[72],73],74])
Every JBW^*-triple factor of type I is isomorphic to a Cartan factor. More generally, every JBW^*-triple of type I is isomorphic to an ℓ^∞-direct sum of L^∞ spaces with values in a Cartan factor.

step 8: March 1985 Friedman–Russo ([54])
Putting it all together:

$$\pi : U \to U^{**} = A \oplus N = (\oplus_\alpha C_\alpha) \oplus N = \sigma(U^{**}) \oplus N$$

implies that $\sigma \circ \pi : U \to A = \oplus_\alpha C_\alpha$ is an isometric isomorphism.

Here are some consequences of the Gelfand-Naimark theorem for JB^*-triples, found in [54].

- Every JB^*-triple is isomorphic to a subtriple of a JB^*-algebra.
- In every JB^*-triple, $\|\{xyz\}\| \leq \|x\|\|y\|\|z\|$.
- Every JB^*-triple U contains a unique norm-closed ideal J such that U/J is isomorphic to a JC^*-triple and J is purely exceptional, that is, every homomorphism of J into a C^*-algebra is zero.

The following two properties of JBW^*-triples, suggested by the Gelfand–Naimark theorem, were established by Barton–Dang–Horn [12].

- Every JBW^*-triple splits into a direct sum $U = J \oplus [U/J]$ where J is purely exceptional and U/J is isomorphic to a weakly closed JC^*-triple. (For JBW-algebras this is due to Shultz 1979 [103].)
- Every JBW^*-triple which is isomorphic to a JC^*-triple is isomorphic to a weakly closed JC^*-triple. (For W^*-algebras this is due to Sakai 1957 [101].)

2.4. Classification of atomic factors

There is a second proof of the Gelfand-Naimark theorem for JB^*-triples which is due to Dang–Friedman [29]. It relies on their new and transparent proof of the classification of Cartan factors of type I. This latter proof is based on the following three works:

- (idea) 1934 Jordan–von Neumann–Wigner [9]: classification of formally real Jordan algebras;
- (technique) 1978 Arazy–Friedman [9]: classification of the ranges of contractive projections on C_1 and C_∞;
- (relations between tripotents) 1985 Neher [91]: Jordan triple systems with enough tripotents.

The building blocks of the algebraic structure of a Jordan triple system are the tripotents and their corresponding Peirce projections, and there are important relations between pairs, triples, and quadruples of tripotents (orthogonal, colinear, governing, trangle, quadrangle,...). These terms will not be defined here. The relations are fundamental tools in the Dang–Friedman proof. Entirely similar ideas are instrumental in the proof of the main result of [59], which is the topic of section 3.

The Dang–Friedman classification of the Cartan factors of type I begins with an irreducible JBW^*-triple U with a minimal tripotent v.

Proposition 2.4.1 ([29]). *If u is any tripotent in the Peirce 1-space $U_1(v)$ of v, then one of the following holds:*

- *u is minimal in U* (this holds if and only if u and v are colinear)*;*
- *u is minimal in $U_1(v)$ but not minimal in U* (this holds only if u governs v)*;*
- *u is not minimal in $U_1(v)$* (this implies u is the sum of two minimal tripotents of U).

Corollary 2.4.2. *The rank of $U_1(v)$ is at most* 2.

Proposition 2.4.3 ([29]). *If $v, \tilde{v}, u, \tilde{u}$ are the minimal tripotents forming a quadrangle, and if $U_1(v + \tilde{v}) \neq \{0\}$, then* $\dim U_2(v + \tilde{v}) \in \{4, 6, 8, 10\}$.

The Dang–Friedman classification scheme ([29]) is now the following: let $J(v)$ denote the weak*-closed ideal generated by v. Then

case 0: Rank $U_1(v) = 0$; then $J(v) \simeq \mathbb{C}$;
case 1: Rank $U_1(v) = 1, u$ a tripotent of $U_1(v)$ minimal in U; then $J(v)$ is a Hilbert space (Cartan factor of type 1);
case 2: Rank $U_1(v) = 1, u$ a tripotent of $U_1(v)$ minimal in $U_1(v)$; then $J(v)$ is a Cartan factor of type 3 (symmetric operators).

In cases 3–7, Rank $U_1(v) = 2$, u is a non-minimal tripotent in $U_1(v)$, and $\tilde{v} := \{uvu\}$.
case 3: $U_1(v + \tilde{v}) = \{0\}$; then $J(v)$ is a Cartan factor of type 4 (spin factor).

In cases 4–7, $U_1(v + \tilde{v}) \neq \{0\}$ and so by Proposition 2.4.3, these are all the cases possible.

case 4: $\dim U_2(v + \tilde{v}) = 4$; then $J(v)$ is a Cartan factor of type 1 (all operators).
case 5: $\dim U_2(v + \tilde{v}) = 6$; then $J(v)$ is a Cartan factor of type 2 (anti-symmetric operators).
case 6: $\dim U_2(v + \tilde{v}) = 8$; then $J(v)$ is a Cartan factor of type 5 (1 by 2 matrices over the Octonions).
case 7: $\dim U_2(v + \tilde{v}) = 10$; then $J(v)$ is a Cartan factor of type 6 (3 by 3 Hermitians over the Octonions).

In each case, $J(v)$ is a summand in U.

The ideas just discussed have application to the study of isometries of real and complex triples and algebras.

- All the JB^*-triples for which every real linear surjective isometry preserves the triple product can be determined, and as a corollary it follows that all real linear isometries of any (complex) C^*-algebra preserve the triple product [28]. This will be discussed below in subsection 4.1.
- Surjective isometries of *real* C^*-algebras preserve the triple product [21]. This will be discussed below in subsection 4.1.
- Do the isometries of a real JB^*-triple preserve the triple product? Since we do not yet have a workable definition of real JB^*-triple, I won't say much here. However, see subsection 2.5.

This subsection also suggests the following problem, which is of great interest for C^*-algebras. If A is any C^*-algebra, then A and its bidual $M := A^{**}$ can be considered as JB^*-triples, M being a JBW^*-triple with predual A^*. As in any JBW^*-triple, there is a bijection between minimal tripotents of M and extreme points of the unit ball of A^*. Also M has an atomic part spanned in the weak*-topology by the minimal tripotents and equal to an ℓ^∞-sum of Cartan factors of types 1–6. By the work of Horn, Neher, and Dang–Friedman, each Cartan factor is spanned by a "grid", and thus elements of M may be considered as functions on the set S of extreme points of the unit ball of A^*.

Problem 1. *Put a "topological-like" structure on S so that $A \subset M$ is identified as the set of all "continuous-like" functions on S.*

A partial solution to this problem will be discussed in subsection 3.6. The problem is significant because even for C^*-algebras, it is necessary to take advantage of the triple product structure in order to guarantee that S will consist of extreme points. In other words, *colinearity* doesn't exist for *projections*, which are the building blocks for binary structures.

2.5. Real JB^*-triples

In contrast to the situation for JB^*-algebras (and to some extent for C^*-algebras), Jordan triple systems over the reals have played no role in the analytic theory of JB^*-triples. This is due to the history of the area: JB^*-triples were born of an investigation into certain aspects of several complex variables ([86]). However, a theory of real Jordan triples and real bounded symmetric domains in finite dimensions was developed by Loos ([88]). This, together with the observation that many of the more recent techniques in Jordan theory ([51], [84], [17]) rely on functional analysis and algebra rather than holomorphy, suggests that it may be possible to develop a real theory and to explore its relationship with the complex theory.

In this subsection we employ a Banach algebraic approach to real Banach Jordan triples. Because of our recent observation on commutative JB^*-triples (see a subsection below), we can now propose a new definition of real JB^*-triple, which we call *J^*B-triple*. Our J^*B-triples include real C^*-algebras and complex JB^*-triples. The main result of [31], which will be described in this subsection, is a structure theorem of Gelfand-Naimark type for commutative J^*B-triples.

Real Banach Jordan triples

Definition 2.5.1. A *Banach Jordan triple* is a real or complex Banach space U equipped with a continuous bilinear (sesquilinear in the complex case) map

$$U \times U \ni (x, y) \mapsto x \square y \in \mathcal{L}(U)$$

such that, with $\{xyz\} := x \square y(z)$ we have

$$\{xyz\} = \{zyx\}; \tag{2.3}$$

$$\{x, y, \{uvz\}\} + \{u, \{yxv\}, z\} = \{\{xyu\}, v, z\} + \{u, v, \{xyz\}\}. \tag{2.4}$$

Recall that a Banach Jordan triple U over $\mathbb{C}$ is said to be a *JB^*-triple* if

(a) for any $x \in U$, the operator $x \square x$ from U to U (that is, $x \square x(y) = \{xxy\}$, $y \in U$) is hermitian (i.e., $\exp it(x \square x)$ is an isometry for all real t) with non-negative spectrum;

(b) the following norm condition holds:

$$\|x \square x\| = \|x\|^2.$$

The proof of the following theorem was suggested by Jonathan Arazy. Since it is so short and elegant, we include it here.

Theorem 2.5.2. *Let U be a complex Banach Jordan triple. Suppose that*

1. $\|\{xxx\}\| = \|x\|^3$;
2. $\|\{xyz\}\| \le \|x\|\,\|y\|\,\|z\|$;
3. *U is positive, i.e.,* $\sigma_{\mathcal{L}(U)}(x \square x) \subset [0,\infty)$ *for each* $x \in U$.

Then U is a JB^*-triple.

Proof. We only need to show that $x \square x$ is hermitian, for each $x \in U$.

Since $\delta := ix \square x$ is a continuous derivation, $\alpha := e^{t\delta}$ is a continuous automorphism for each real t. Thus, for each $x \in U$,

$$\|\alpha(x)\|^3 = \|\{\alpha(x), \alpha(x), \alpha(x)\}\| = \|\alpha(\{xxx\})\| \le \|\alpha\|\|x\|^3$$

and therefore, by iteration,

$$\|\alpha(x)\| \le \|\alpha\|^{1/3^n}\|x\|,$$

that is, $\|\alpha\| \le 1$. □

The terminology in the next definition was motivated by [7], and the spectral conditions were inspired by [115].

Definition 2.5.3. A *J^*B-triple* is a real Banach space A equipped with a structure of real Jordan triple system which satisfies

1. $\|\{xxx\}\| = \|x\|^3$;
2. $\|\{xyz\}\| \le \|x\|\,\|y\|\,\|z\|$;
3. $\sigma^c_{\mathcal{L}(A)}(x \square x) \subset [0,\infty)$ for $x \in A$;
4. $\sigma^c_{\mathcal{L}(A)}(x \square y - y \square x) \subset i\mathbb{R}$ for x, y $\in A$.

Over the complex field, JB^*-triples are the same as J^*B-triples.

A closed subtriple B of a J^*B-triple A is a J^*B-triple. In particular, a closed *real* subtriple of a JB^*-triple is a J^*B-triple.

A real C^*-algebra is a closed subalgebra of its complexification, which is a complex C^*-algebra in some norm. Thus, a real C^*-algebra, with the triple product

$$\{xyz\} = \frac{1}{2}(xy^*z + zy^*x).$$

is a closed real subtriple of a JB^*-triple, and hence a real C^*-algebra is a J^*B-triple.

Two important and natural problems left open in the paper [31] are the following.

Problem 2. Is the complexification of a J^*B-triple is a JB^*-triple in some norm extending the original norm. (This is solved for commutative J^*B-triples in Theorem 2.5.8.)

Problem 3. Is the bidual of a J^*B-triple a J^*B-triple with a separately weak*-continuous triple product.

Commutative complex triples

We are going to use Theorem 2.5.2 to modify the treatment in [84, Section 1] by not requiring that $x \square x$ be hermitian. Theorem 2.5.5 below is needed to prove the main result of this subsection, namely Theorem 2.5.8, which leads to a Gelfand-Naimark Theorem for commutative real J^*B-triples.

Definition 2.5.4. A Banach Jordan triple is *commutative* if

$$\{\{xyz\}uv\} = \{xy\{zuv\}\} = \{x\{yzu\}v\}.$$

For example, any commutative C^*-algebra $C_0(\Omega)$ is a commutative Banach Jordan triple with $f \square g(h) = f\overline{g}h$.

Throughout this subsection U will denote a commutative complex Banach Jordan triple.

Let $B = B(U) :=$ the closed span of $U \square U$ in $\mathcal{L}(U)$. Then B is a commutative Banach subalgebra of $\mathcal{L}(U)$. Denote the Gelfand Transform of B by

$$\Gamma_B : B(U) \to C_0(X),$$

where $X = X_B$ is the maximal ideal space of B. Let $\Lambda = \Lambda(U) :=$ the set of all non-zero triple homomorphisms $\lambda : U \to \mathbb{C}$. Precisely,

$$\Lambda = \{\lambda : U \to \mathbb{C} : 0 \neq \lambda \text{ linear}, \lambda(\{abc\}) = \lambda(a)\overline{\lambda(b)}\lambda(c)\}.$$

According to [84, Lemma 1.6], Λ is a bounded subset of $\mathcal{L}(U, \mathbb{C})$. Thus, Λ is a weak*-locally compact space and a "principle $\mathbb{T}$-bundle" ($\mathbb{T}$=unit circle) under the action

$$T \times \Lambda \ni (t, \lambda) \mapsto t.\lambda \in \Lambda,$$

where $(t.\lambda)(x) = t\lambda(x)$.

Define a norm closed subtriple of $C_0(\Lambda)$:

$$C_{\text{hom}}(\Lambda) := \{f \in C_0(\Lambda) : f(t.\lambda) = tf(\lambda), \forall (t, \lambda) \in T \times \Lambda\}$$

and a Gelfand Transform $U \ni x \mapsto \hat{x} = \Gamma_U(x) \in C_{\text{hom}}(\Lambda)$ by $\Gamma_U(x)(\lambda) = \lambda(x)$. Thus

$$\Gamma_U : U \to C_{\text{hom}}(\Lambda)$$

is a continuous triple homomorphism. The proof of the following theorem is immediate from [84, §1], since the assumptions imply that U is a JB^*-triple.

Theorem 2.5.5. *Let U be a commutative complex Banach Jordan triple. Suppose that*
1. $\|\{xxx\}\| = \|x\|^3$;
2. $\|\{xyz\}\| \leq \|x\|\,\|y\|\,\|z\|$;
3. *U is positive, i.e., $\sigma_{\mathcal{L}(U)}(x \square x) \subset [0,\infty)$ for each $x \in U$.*

Then the Gelfand representation $U \to C_{hom}(\Lambda)$ is an isometric surjective triple isomorphism.

For a generalization of this theorem, see [48].

Commutative Real triples

Now let A be a commutative real Banach Jordan triple, that is, a real Banach space A, together with a tri-linear map

$$A \times A \times A \ni (x,y,z) \mapsto \{xyz\} \in A$$

which satisfies

$$\{xyz\} = \{zyx\};$$

$$\{\{xyz\}uv\} = \{xy\{zuv\}\} = \{x\{yzu\}v\}.$$

We shall define a natural Gelfand transform and state a representation theorem of Gelfand-Naimark type.

By analogy with the complex case, let $B(A)$ be the Banach subalgebra of $\mathcal{L}(A)$ generated by $A \square A$. Then $B(A)$ is a commutative real Banach algebra (not necessarily unital, cf. [62, p. 63]). Let $X^c_{B(A)}$ denote the space of complexified characters (cf. [62, p. 82]), that is

$$X^c_{B(A)} = \{\tau : B(A) \to \mathbb{C}, 0 \neq \tau \text{ real-linear}, \tau(ST) = \tau(S)\tau(T)\}.$$

By analogy we define Λ^c_A to be the collection of all non-zero real-linear triple homomorphisms of A into $\mathbb{C}$; precisely,

$$\Lambda^c_A = \{\lambda : A \to \mathbb{C} : \lambda \text{ real-linear}, \lambda \neq 0, \lambda(\{abc\}) = \lambda(a)\overline{\lambda(b)}\lambda(c)\}.$$

By the proof of [84, Lemma 1.6], each such λ is automatically continuous and Λ^c_A is contained in a bounded subset of $\mathcal{L}_{\mathbb{R}}(A,\mathbb{C})$. Note that $e^{i\theta}\Lambda^c_A = \Lambda^c_A$, that Λ^c_A is closed under complex conjugation, and that Λ^c_A is locally compact in the topology of pointwise convergence on A.

In order to obtain the analogue of Theorem 2.5.5 we need to consider the complexification of A.

Let $U := A^{\mathbb{C}} = \phi(A) + i\phi(A)$ be the complexification of A and let $\phi : A \to U$ be the natural embedding. The space U becomes a complex commutative Jordan triple system in the natural way and ϕ is a real-linear triple isomorphism into.

The given norm on A can be used to define a norm on U as described in [62]. With this norm, U is a commutative complex Banach Jordan triple.

As above, let $B(U)$ be the closed complex subalgebra of $\mathcal{L}(U)$ generated by $U \square U$ and define $B(\phi(A))$ to be the closed real subalgebra of $\mathcal{L}(U)$ generated by $\phi(A) \square \phi(A)$. Then $B(A^{\mathbb{C}}) = (B(A))^{\mathbb{C}}$.

Proposition 2.5.6. *Suppose that A is a commutative J^*B-triple. Then $B(A)$, with the norm of $\mathcal{L}(A)$, is a commutative real C^*-algebra with involution determined by $(x \square y)^* = y \square x$. Consequently, $B(U)$ is a C^*-algebra in some norm extending the norm on $B(A)$* (by [62, 12.4]).

Let $\Lambda(U)$ be defined as above.

Lemma 2.5.7. *With the above notation,*
(i) *for each $\lambda \in \Lambda(U)$ there is $\lambda' \in \Lambda_A^c$ such that $\lambda(\phi(x) + i\phi(y)) = \lambda'(x) + i\lambda'(y)$ for $x, y \in A$. This correspondence establishes a bijection $\Lambda(U) \leftrightarrow \Lambda_A^c$;*
(ii) *for each $\tau \in X_{B(U)}$ there is $\tau' \in X_{B(A)}^c$ such that $\tau(T + iS) = \tau'(T) + i\tau'(S)$ for $T, S \in B(A)$. This correspondence establishes a bijection $X_{B(U)} \leftrightarrow X_{B(A)}^c$.*

We can now state the main result of [31].

Theorem 2.5.8. *Let A be a commutative J^*B-triple. There is a norm on the complexification U of A extending the norm on A and for which U is a JB^*-triple.*

We conclude by describing the Gelfand transform and stating and proving a Gelfand-Naimark type theorem for commutative J^*B-triples.

As noted earlier, the space Λ_A^c is a locally compact Hausdorff space in the topology of pointwise convergence on A. The bijection in Lemma 2.5.7(i) is a homeomorphism. Now let

$$C^*_{\text{hom}}(\Lambda_A^c) = \{f \in C_0(\Lambda_A^c) : f(e^{i\theta}\lambda') = e^{i\theta} f(\lambda') \text{ and } f(\overline{\lambda'}) = \overline{f(\lambda')}\}$$

and define a Gelfand transform $\Gamma_A^{\mathbb{R}} : A \to C^*_{\text{hom}}(\Lambda_A^c)$ by $\Gamma_A^{\mathbb{R}}(x)(\lambda') = \lambda'(x)$. Let $\rho : \Lambda_U \to \Lambda_A^c$ be the restriction map implicit in Lemma 2.5.7 and let $\rho^* : C^*_{\text{hom}}(\Lambda_A^c) \to C_{\text{hom}}(\Lambda_U)$ be its transpose.

Note that $\rho^{-1}(\overline{\lambda})(\phi(x) + i\phi(y)) = \overline{\lambda(x)} + i\overline{\lambda(y)}$ and therefore $\Gamma_A^{\mathbb{R}}$ maps A into $C^*_{\text{hom}}(\Lambda_A^c)$. Since

$$\Gamma_U^{\mathbb{C}} \circ \phi = \rho^* \circ \Gamma_A^{\mathbb{R}},$$

$\Gamma_A^{\mathbb{R}}$ is an isometry.

Finally, if $f \in C^*_{\text{hom}}(\Lambda_A^c)$ and $x, y \in A$ are such that $\rho^* f = \Gamma_U(\phi(x) + i\phi(y))$, the fact that $f(\overline{\lambda'}) = \overline{f(\lambda')}$ implies that $y = 0$, hence $\Gamma_A^{\mathbb{R}}(A) = C^*_{\text{hom}}(\Lambda_A^c)$. This proves

Theorem 2.5.9. *Let A be a commutative J^*B-triple. Then the Gelfand transform is an isometric triple isomorphism of A onto $C^*_{\text{hom}}(\Lambda^c_A)$.*

3. State spaces of JBW^*-triples

The main purpose of this section is to outline a proof of Theorem 3.0.2 below.

In the first subsection, we take a more leisurely look at the definition and examples of JB^*-triples than we did in subsection 2.2. In the second subsection we examine those properties of a JBW^*-triple that serve as the basis for the axioms that will characterize them. These axioms and their appropriate framework (the facially symmetric spaces) are introduced and discussed in the third subsection. The proof of the main result is outlined in the fourth subsection and in the following one, the spin factor, which is fundamental to the whole construction, is discussed in detail. The section closes with a discussion of the Stone-Weierstrass problem for JB^*-triples, which is the key to obtaining a duality in terms of extreme points ("pure states").

To start, let's state a special case of the main result. In the following, Z denotes a complex Banach space. The terms will be defined later in this section.

Theorem 3.0.1. *Let Z be an atomic neutral strongly facially symmetric space which satisfies the four "pure state properties" and has spin degree 4. Then there exist Hilbert spaces H and K, and a closed subspace M of Z^* such that*

$$Z^* \simeq \mathcal{L}(H,K) \oplus^{\ell^\infty} M.$$

In particular, if Z is irreducible, then $Z^* \simeq \mathcal{L}(H,K)$, that is, Z^* is isometrically (linearly) isomorphic to a Cartan factor of type 1.

More generally, we have the following non-ordered analog of the fundamental result of Alfsen–Shultz 1978 for Jordan algebras discussed in subsection 1.3.

Theorem 3.0.2 ([59]). *Let Z be an atomic neutral strongly facially symmetric space which satisfies the four "pure state properties". Then Z^* is isometrically isomorphic to an atomic JBW^*-triple.*

By work of Neher, Horn, and Dang–Friedman, (see step 7 in the proof of Theorem 2.3.1, and subsection 2.4),

$$Z^* \simeq \oplus_\alpha^{\ell^\infty} C_\alpha \tag{3.1}$$

where each C_α is a Cartan factor. Actually, (3.1) is proved directly, which implies Theorem 3.0.2.

3.1. What is a JB^*-triple?

Consider a complex Banach space U equipped with a triple product

$$\{xyz\} : U \times U \times U \to U$$

which satisfies two algebraic properties:

- $\{xyz\} = \{zyx\}$ is linear in x and z and conjugate linear in y.
- $\delta(x) := iD(x) : z \mapsto i\{xxz\}$ is a derivation, that is,

$$\delta\{xyz\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\};$$

and two topological or norm properties:

- $\|D(x)\|_{B(U)} = \|x\|_U^2$.
- $D(x) : z \mapsto \{xxz\}$ is hermitian positive, that is,

$$\sigma_{B(U)}(D(x)) \subset [0, \infty) \text{ and } \|e^{itD(x)}\|_{B(U)} \leq 1 \quad \forall t \in \mathbb{R}.$$

That is what a JB^*-triple is. Here are some examples:

- Hilbert space: $\{abc\} = [\langle a|b\rangle c + \langle c|b\rangle a]/2$.
- C^*-algebra: $\{abc\} = [ab^*c + cb^*a]/2$.
- JB^*-algebra: $\{abc\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$.

Now you ask, what is a JBW^*-triple? This is a JB^*-triple which is the dual of some Banach space. From the above examples of JB^*-triples, we see that Hilbert spaces, von Neumann algebras, and JBW^*-algebras are examples of JBW^*-triples. Perhaps the most important class of examples are the Cartan factors, which are defined as follows:

type 1 $\mathcal{L}(H, K)$
type 2 $\{x \in \mathcal{L}(H) : x^t = -x\}$
type 3 $\{x \in \mathcal{L}(H) : x^t = x\}$
type 4 spin factor
type 5 $M_{1,2}(\mathcal{O})$
type 6 $(M_3(\mathcal{O}))_{s.a}$

The Cartan factors of types 1 to 4 are (realizable as) JC^*-triples, that is, norm closed subspaces M of $\mathcal{L}(H, K)$ stable for the "cubes": $x \in M \Rightarrow xx^*x \in M$. For types 2 and 3, $x^t = Jx^*J$ for some conjugation J on the Hilbert space H. For a description of the Cartan factors of types 5 and 6, see subsection 4.3.

Now we give a definition of the spin factor, the Cartan factor of type 4. This is a JB^*-triple U equipped with a complete inner product $\langle \cdot | \cdot \rangle$ and a conjugation J on the resulting Hilbert space such that

$$\{xyz\} = \frac{\langle x|y\rangle z + \langle z|y\rangle x - \langle x|Jz\rangle Jy}{2} \tag{3.2}$$

and such that the given norm $\| \cdot \|_U$ and the Hilbert space norm are equivalent.

Note that a spin factor is necessarily a reflexive Banach space. Some simple examples of spin factors are $S_2(\mathbb{C})$ (two by two symmetric complex matrices), $M_2(\mathbb{C})$ (two by two complex matrices), $A_4(\mathbb{C})$ (four by four anti-symmetric complex matrices). Of course these are also Cartan factors of types 3,1, and 2 respectively.

Let's analize the last example. Let

$$u_1 = u_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{u}_1 = u_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix};$$

$$u_2 = u_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{u}_2 = -u_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};$$

and

$$u_3 = u_{14} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{u}_3 = u_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we consider $\{u_j, \tilde{u}_j\}_{j=1}^3$ as an orthonormal basis and define $Ju_j = \tilde{u}_j$, it is a simple matter to check (3.2).

It is important to note the following properties of the basis in the above example:

- u_i and $\tilde{u}_i$ are orthogonal;
- For $i \neq j$, u_i and u_j are colinear (their non-zero entries are in common rows and columns);
- u_i and $\tilde{u}_j$ are colinear;
- For $i \neq j$, $\{u_i u_j \tilde{u}_i\} = -\tilde{u}_j/2$ (the quadruple $(u_i, \tilde{u}_i; u_j, \tilde{u}_j)$ forms an "odd quadrangle").

We shall see in subsection 3.5 below that all spin factors can be described by "spin grids", that is, orthonormal bases of the form

$$\{u_i, \tilde{u}_i\}_{i \in I} \text{ or } \{u_i, \tilde{u}_i\}_{i \in I} \cup \{u_0\}.$$

The spin grids for the other examples mentioned above are

$$M_2(\mathbb{C}): \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix};$$

and

$$S_2(\mathbb{C}): \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3.2. Algebraic and affine aspects of JBW^*-triples

Recall that a tripotent in a JB^*-triple U is an element u satisfying $u = \{uuu\}$, and that it gives rise to a Peirce decomposition of U (see subsection 2.2):

$$U = U_2(u) \oplus U_1(u) \oplus U_0(u).$$

Let's illustrate this with a JC^*-triple $M \subset \mathcal{L}(H, K)$, where a tripotent u is just a partial isometry. If we set $\ell = uu^*$ and $r = u^*u$, then ℓ and r are projections and the block matrix decomposition

$$x = \begin{bmatrix} \ell x r & \ell x(1-r) \\ (1-\ell)xr & (1-\ell)x(1-r) \end{bmatrix}$$

is what the Peirce decomposition is all about. Here of course,

- $P_2(u) = \ell x r$
- $P_0(u)x = (1-\ell)x(1-r)$
- $P_1(u)x = \ell x(1-r) + (1-\ell)xr.$

Let A be a C^*-algebra and let $S(A)$ denote the state space of A. The pure states $P(A)$, that is, the extreme points of $S(A)$, are important examples of faces of the convex set $S(A)$, and the latter is a norm-exposed face, namely it equals F_1 in the unit ball of the predual of the von Neumann algebra $M = A^{**}$, where for $x \in M$ of norm 1, recall that

$$F_x := \{f \in M_* : \|f\| = 1 = f(x)\}.$$

As will be noted later, M_* is a neutral, strongly facially symmetric space satisfying FE and STP. Since these properties do not depend on order, it will be natural to replace M by a JBW^*-triple.

Let's recall that in an arbitrary JBW^*-triple U, there is a one-to-one correspondence

$$\{\text{tripotents of } U\} \leftrightarrow \{\text{norm exposed faces of } (U_*)_1\} \tag{3.3}$$

given by $u \mapsto F_u$, where

$$F_x = \{\rho \in (U_*)_1 : \|\rho\| = 1 = \rho(x)\}.$$

Under this correspondence, minimal tripotents correspond to extreme points. Such a minimal tripotent is called the support tripotent of the extreme point. (This is a special case of the polar decompositon of a normal functional, see Proposition 2.2.4.) Note that if U is in particular a von Neumann algebra, and $u = 1$, then F_u is the normal state space of U.

We next introduce an important mapping called the *Peirce reflection.* Namely, for any tripotent u, set

$$S_u = P_2(u) - P_1(u) + P_0(u).$$

Note that

$$S_u^2 = I,\ P_2(u) + P_0(u) = (I + S_u)/2, \quad \text{and } P_1(u) = (I - S_u)/2. \tag{3.4}$$

Because of the "Jordan" decomposition (for hermitian functionals) in the "Jordan" algebra $U_2(u) = P_2(u)U$, we have

$$P_2(u)^*(U_*) = \mathrm{sp}_{\mathbb{C}}\, F_u.$$

(Note the different uses of the name "Jordan"). We also have $P_0(u)^*(U_*) = F_u^\perp$. Therefore we have the key motivating property (implicit in [51]) for the geometric characterization of JBW^*-triples, that is:

Proposition 3.2.1. *Let u be a tripotent in a JB^*-triple. Then the fixed point set of the symmetry S_u^* is*

$$\mathrm{sp}_{\mathbb{C}}\, F_u \oplus F_u^\perp.$$

The following is a converse to Theorem 3.0.2, which of course preceded that theorem. PE, STP, and ERP are proved in [51], FE is proved in [38].

Proposition 3.2.1. *Every JBW^*-triple satisfies the four pure state properties.*

What are the pure state properties? Although they were discussed in subsection 2.2, they are important enough to bear repeating.

- **(PE) Point Exposure (resp. (FE) Face Exposure):** Every extreme point (resp. norm closed face) in the unit ball of the predual is norm exposed.
- **(STP) Symmetry of Transition Probabilities:** For every pair of extreme points g, f of the unit ball of the predual with support tripotents u, v respectively,

$$f(u) = \overline{g(v)}.$$

  Example: $M = \mathcal{L}(H,K)$, $f = \omega_{\xi,\eta}$, $g = \omega_{\alpha,\beta}$. Then $v = \eta \otimes \overline{\xi}$, $u = \beta \otimes \overline{\alpha}$ and

$$f(u) = \langle (\beta \otimes \overline{\alpha})\xi | \eta \rangle = \langle \xi | \alpha \rangle \langle \beta | \eta \rangle = \overline{g(v)}.$$

- **(ERP) Extreme Rays Property:** The image of an extreme point of the unit ball of the predual under any Peirce 2 projection is a multiple of an extreme point, that is

$$P_2(u)^*[\mathrm{ext}\,(U_*)_1] \subset \mathbb{C}\mathrm{ext}\,(U_*)_1.$$

  Example: With $M = \mathcal{L}(H,K)$, $\ell = uu^*$, $r = u^*u$ and $\xi' = r\xi/\|r\xi\|, \eta' = \ell\eta/\|\ell\eta\|$, we have

$$P_2(u)\omega_{\xi,\eta}(x) = \langle \ell x r \xi | \eta \rangle = \|r\xi\| \|\ell\eta\| \omega_{\xi',\eta'}(x).$$

- **(JP) Joint Peirce decomposition:** The intersection of the Peirce 1-subspaces corresponding to two orthogonal minimal tripotents is contained in the Peirce

2-space of their sum, that is,

$$U_1(u) \cap U_1(v) \subset U_2(u+v).$$

For any two orthogonal tripotents e_1, e_2 (not necessarily minimal), the well known joint Peirce decomposition includes the formula

$$U_2(e_1+e_2) = U_2(e_1) + U_2(e_2) + U_1(e_1) \cap U_1(e_2).$$

The weaker version (JP) quoted above suffices for our purposes.

3.3. Affine geometric model for JBW^*-triples

What is an affine geometric model, based on physically significant axioms, for a JBW^*-triple? The answer is: a neutral strongly facially symmetric Banach space. These were discussed briefly in subsection 1.4. We now discuss in more detail this fundamental concept in an abstract normed space.

A basic notion in affine geometry is that of face. We consider faces F of the unit ball $(Z)_1$ of Z, so recall that this means that F is a non-empty convex subset of $(Z)_1$ with the following property: if $f \in (Z)_1$ and $f = \lambda g + (1-\lambda)h$ for some $g, h \in F$ and $\lambda \in (0,1)$, then $f \in F$.

Let Z be a normed space, with elements denoted by $f, g, \ldots$. We say that f and g are *orthogonal*, notation $f \Diamond g$, if

$$\|f \pm g\| = \|f\| + \|g\|.$$

A norm one element $u \in Z^*$ is called a *projective unit* if $\langle u, F_u^\Diamond \rangle = \{0\}$. Obviously, by $Y^\Diamond$ is meant the set of all elements orthogonal to each element of $Y \subset Z$. A face $F \subset (Z)_1$ is said to be a *symmetric face* if there is a surjective linear isometry S_F on Z such that $S_F^2 = I$ and $\overline{\mathrm{sp}}\, F \oplus F^\Diamond$ is the fixed point set of S_F (cf. Proposition 3.2.1). Then call Z *weakly facially symmetric* (WFS for short) if every norm exposed face in $(Z)_1$ is symmetric.

Every symmetric face F gives rise to the *"geometric" Peirce projections* $P_k(F)$ on Z defined (with an eye toward (3.4)) by $P_2(F) + P_0(F) = (I + S_F)/2$, and $P_1(F) = (I - S_F)/2$. Of course the meaning here is that $P_2(F)$ is the projection with range the closed span of F and $P_0(F)$ is the projection with range $F^\Diamond$.

Our definitions are not sharp enough to get a purely geometric analog of the correspondence (3.3). So let's introduce the right thing. A projective unit u is said to be a *geometric tripotent* if F_u is a symmetric face *and* $S_{F_u}^* u = u$. At this point it would be prudent to use the notation S_u for S_{F_u}.

Here is the promised one-to-one correspondence [56].

Proposition 3.3.1. *In a WFS space, there is a one to one correspondence between geometric tripotents and symmetric faces.*

What about the minimal geometric tripotents of Z^* (which are defined by the condition that the geometric Peirce 2-space be one-dimensional). Do they correspond to extreme points of the unit ball of Z. Only in strongly facially symmetric spaces satisfying the (geometric) pure state property (PE) (see the discussion and table below). By definition, a weakly facially symmetric space is *strongly facially symmetric* (notation, SFS) if whenever a symmetric face F is contained in a norm exposed face, say $F \subset F_y$ (with $y \in Z^*$ of norm 1), then $S_F^* y = y$. In such spaces, projective units are automatically geometric tripotents. Also, in such spaces, the one-to-one correspondence given in Proposition 3.3.1 restricts to a one-to-one correspondence between minimal geometric tripotents and *exposed* points. Thus, *a priori*, there may be extreme points whose "support geometric tripotent" is not a minimal geometric tripotent. What is meant by the support geometric tripotent of an element of Z? This is defined by the polar decomposition of an arbitrary element of Z, discussed in the next paragraph.

Note that three of the four pure state properties stated in Proposition 3.2.2, namely FE, ERP, and JP, make immediate sense in the category of WFS spaces. But so does the fourth, STP, in neutral SFS spaces because of the uniqueness of the geometric polar decomposition for any element of the space [56]: if $f \in Z$, then there is a unique geometric tripotent u such that

$$f/\|f\| \in F_u \text{ and } \langle u, \{f\}^{\diamond} \rangle = 0.$$

Since we mentioned the concept of *neutral*, and because it is fundamental in what follows, we better give its definition. The notion is based on a fundamental result of Effros (1963) for von Neumann algebras which was extended to JBW^*-triples by Friedman–Russo (1985), see Proposition 2.2.2. The result of Effros is the following: if ρ is a normal functional on a von Neumann algebra, if e is a projection in the von Neumann algebra, and if the functional $x \mapsto \rho(ex)$ has the same norm as ρ, then it equals ρ. This property is referred to as neutrality.

This suggests defining a space to be *neutral* if for every symmetric face F, $P_2(F)$ is a neutral projection, that is, $\|P_2(F)f\| = \|f\|$ implies $P_2(F)f = f$.

In view of the previous discussion, the following propostion should not be surprising.

Proposition 3.3.2. *A JBW^*-triple* (more precisely, its predual) *is an example of a neutral SFS space satisfying the four* (geometric) *pure state properties. In particular, geometric tripotents correspond to tripotents.*

Table 4. Summary of above discussion

von Neumann algebra	JBW^*-triple	SFS space
partial isometry	tripotent	geometric tripotent
block matrix	Peirce	geometric Peirce
decomposition	decomposition	decomposition
state	norm one functional	norm one functional
pure state	extreme point	extreme point
pure state property	pure state property	pure state property
(binary)	(ternary)	(geometric)
neutrality	neutrality	neutrality

It may be more natural to use the term "extreme point property" in the ternary and geometric categories, but we chose to stick with "pure state property", for physical reasons.

3.4. Steps in the main theorem

In this subsection, let Z denote an atomic neutral SFS space which is *irreducible.* Irreducible means that Z does not have a non-trivial L-summand, that is, there do not exist subspaces $Y \neq Z, \{0\}$ and Y' such that $Z = Y \oplus^{\ell^1} Y'$. Theorems 3.4.1 and 3.4.2 below appear in [58].

Theorem 3.4.1. *If Z satisfies PE, STP, and is of rank* 1, *then Z^* is isometric to a Hilbert space* (Cartan factor of type 1 and rank 1)

A normed space is said to be of *rank 1* if no two non-zero elements are orthogonal.

The key point in the proof of Theorem 3.4.1 (see [58, Section 2]) is the construction of a continuous symmetric sesquilinear form $\langle f|g\rangle_\pi = \langle f, \pi(g)\rangle$ induced by the densely defined conjugate linear map

$$\pi : Z \times Z \ni \sum_1^n z_j f_j \mapsto \sum_1^n \overline{z}_j v(f_j) \in Z^*,$$

where $v(f)$ denotes the support geometric tripotent of the extreme point f. Under the assumptions of the theorem, every element of Z is a scalar multiple of an extreme point, and it follows that the above sesquilinear form is positive definite and induces the given norm of Z.

Recall that every WFS space can be decomposed with respect to any geometric tripotent. In particular, if v and $\tilde{v}$ are a pair of minimal orthogonal geometric tripotents, then

$$Z = Z_2(v + \tilde{v}) + Z_1(v + \tilde{v}) + Z_0(v + \tilde{v}).$$

If $Z_1(v + \tilde{v}) = Z_0(v + \tilde{v}) = \{0\}$, we say that Z is of Type I_2. In that case, $Z = Z_2(v + \tilde{v})$ is the span of the rank 2 face $F_{v+\tilde{v}}$. A detailed study of rank 2 faces in SFS spaces is carried out in [58, Section 3], which leads to the proof of the following theorem (given in [58, Section 4]). Because of its fundamental importance, the spin factor, and the paper [58] will be discussed in some detail in the next subsection.

Theorem 3.4.2. *If Z satisfies FE, STP, and is of type I_2, then Z^* is isometric to a spin factor* (Cartan factor of type 4).

Theorems 3.4.3 and 3.4.4 below appear in [59]. These theorems are proved with the aid of an invariant called the *spin degree.* Roughly speaking, this is the dimension of a spin factor appearing in Z as a geometric Peirce 2-space. More precisely, Z has spin degree n ($3 \leq n \leq \infty$) if there exist orthogonal minimal geometric tripotents $v, \tilde{v}$ such that $Z_2(v+\tilde{v})$ has dimension n and $Z_1(v+\tilde{v}) \neq \{0\}$. Without the latter condition, Z would be a spin factor (by Theorem 3.4.2), since by irreducibility, it would follow that $Z_0(v + \tilde{v}) = \{0\}$.

Theorem 3.4.3. *If Z satisfies FE, STP, ERP, and is of spin degree 3, then Z^* is isometric to symmetric operators* (Cartan factor of type 3).

Theorem 3.4.4. *Suppose that Z satisfies FE, STP, ERP and JP. Then*

- *if Z has spin degree 4, then Z^* is isometric to $\mathcal{L}(H, K)$* (Cartan factor of type 1)*;*
- *if Z has spin degree 6, then Z^* is isometric to anti-symmetric operators* (Cartan factor of type 2)*;*
- *if Z has spin degree 8, then Z^* is isometric to $M_{1,2}(\mathcal{O})$* (Cartan factor of type 5)*;*
- *if Z has spin degree 10, then Z^* is isometric to $[M_3(\mathcal{O})]_{sa}$* (Cartan factor of type 6).

The other spin degrees (∞, odd greater than 3, even greater than 10) *do not occur.*

Theorems 3.4.1, 3.4.2, 3.4.3 and 3.4.4, together with Zorn's lemma yield a proof of Theorem 3.0.2. Two important steps in the proof, each involving the geometric Peirce 1-space are the following:

- If a space has finite spin degree n, then there is a minimal geometric tripotent whose geometric Peirce 1-space has spin degree $n - 2$.
- The geometric Peirce 1-space of a minimal geometric tripotent is of rank at most 2.

Here is a sketch of the proof of Theorem 3.4.4.

Let $M := Z^*$. Suppose first that Z has spin degree 4. Then there exist two orthogonal minimal geometric tripotents with $M_2(v + \tilde{v}) \simeq M_2(\mathbb{C})$. Let u and $\tilde{u}$ correspond in $M_2(v + \tilde{v})$ to the matrix units E_{12} and E_{21}. Then $u, \tilde{u} \in M_1(v)$, and it follows that $M_1(v)$ is the direct sum of two orthogonal Hilbert spaces, with orthnormal bases of the form $\{u = u_{12}, u_{13}, \ldots\}$ and $\{\tilde{u} = u_{21}, u_{31}, \ldots\}$. The fact that the triple v, u_{1j}, u_{i1} sits strategically in a spin factor implies, by Theorem 3.4.2, the existence of a geometric tripotent u_{ij} such that the quadruple $v, u_{1j}, u_{i1}, u_{ij}$ behaves like the two by two matrix units. It is now natural to set $v = u_{11}$, obtaining a family $\{u_{ij}\}_{i \in I, j \in J}$ in one-to-one correspondence with the matrix units $\{E_{ij}\}$, which are known to be weak*-total in $\mathcal{L}(H, K)$. Since it is easy to show abstractly that the family $\{u_{ij}\}$ is weak*-total in M, this sets up a natural bijection κ from a dense set in M onto a dense set in $\mathcal{L}(H, K)$.

It remains to show that the mapping κ is norm preserving, and this is the difficult part of the proof. First of all, you can reduce to the case that one of the index sets I or J is finite. Then you use the following fact, valid in this case: if $x \in M$ is of norm 1 and has "coordinates" x_{ij} (coordinates can be defined for an arbitrary element of M, not just a finite linear combination of the generating set), then x is a minimal geometric tripotent if and only if for all i, j, k, p,

$$\det \begin{bmatrix} x_{ij} & x_{ik} \\ x_{pj} & x_{pk} \end{bmatrix} = 0.$$

This implies that κ and κ^{-1} are contractive, completing the proof of the first statement of Theorem 3.4.4.

Now suppose that Z has spin degree 6. Then there is a minimal geometric tripotent v such that $M_1(v)$ has spin degree 4 and is of rank at most two. By the part of the theorem proved already, $M_1(v) \simeq \mathcal{L}(H, K)$, and hence there is a system of minimal geometric tripotents u_{ij} with $i \in \{1, 2\}$ and $j \in J$ corresponding to the matrix units in the space of 2 by $|J|$ matrices $(2 \leq |J| \leq \infty)$. By a completion argument as above, using spin factor structure, one obtains a "geometric symplectic grid", that is, a family of minimal geometric tripotents which behaves like the standard generators in the Cartan factor of type 2, the anti-symmetric operators on a Hilbert space. Again, the difficult part is to prove that the natural map of M into the Cartan factor is an isometry.

Next suppose that Z has spin degree 8. As above, there is a v with $M_1(v)$ of spin degree 6 and rank at most 2. This implies that $M_1(v)$ is either the 4 by 4 or 5 by 5 anti-symmetric matrices. Since it cannot be a spin factor, it is the latter. This time the completion process using the spin factor structure leads to a generating set for Z consisting of 16 elements. The natural map must now be shown to be an isometry of M onto the Cartan factor of type 5.

Finally suppose that Z has spin degree 10. As above, there is a v with $M_1(v)$ of spin degree 8 (and rank at most 2), hence must be the Cartan factor of type 5. In this case the completion process using the spin factor structure leads to a generating set for Z consisting of 27 elements. The natural map must now be shown to be an isometry of M onto the Cartan factor of type 6.

In summary, setting up the correspondences between a dense subset of M and a dense subset of the appropriate Cartan factor is (relatively) easy. Proving that this map is an isometry, equivalently, that minimal geometric tripotents are mapped to minimal tripotents, is the hard part.

3.5. Geometry of the dual ball of the spin factor

We are going to discuss in more detail the complex spin factor, which is the Cartan factor of type 4. It is a JC^*-triple which in dimensions 3,4,6, and 8 is isomorphic to $S_2(\mathbb{C}), M_2(\mathbb{C}), A_4(\mathbb{C})$ and the Cayley numbers, respectively. It is an example of a JBW^*-algebra, it is related to Clifford algebras, and to the CAR C^*-algebra and it is the complexification of the real spin factor (which is itself a JBW-algebra).

In what follows, we shall construct the complex spin factor. We begin with an outline of the construction. In this subsection, following [58] and [29], we outline a proof of Theorem 3.4.2 and discuss a new property of the spin factor (facial decomposition). The proof consists of four steps:

step 1 Hilbert ball property of a rank 2 face
step 2 $\dim Z = 3$
step 3 $\dim Z = 4$
step 4 dual spin grids

Assume the notation of Theorem 3.4.2.

Step 1: Hilbert ball property

The starting point is the following lemma.

Lemma 3.5.1. *Let v and $\tilde{v}$ be orthogonal minimal geometric tripotents and let ρ and σ be any two orthogonal extreme points of the rank 2 face $F_{v+\tilde{v}}$. Then $v(\rho) + v(\sigma) = v + \tilde{v}$ and $\rho + \sigma = f + \tilde{f}$.* (Here, $v(g)$ denotes the support geometric tripotent of the functional g and $f, \tilde{f}$ are extreme points with $v = v(f)$ and $\tilde{v} = v(\tilde{f})$.)

This lemma allows us to unambiguously define the "center" of the rank 2 face F to be the average of any two orthogonal extreme points of the face.

The following is the basic construction of a real Hilbert space, which justifies the name Hilbert ball property.

Proposition 3.5.2. *Let v and $\tilde{v}$ be orthogonal minimal geometric tripotents and let ξ be the center of the rank 2 face $F_{v+\tilde{v}}$. Then $F_{v+\tilde{v}} - \xi$, with the norm of Z is the unit ball of a real Hilbert space $\mathcal{H}$ which is the completion of*

$$\mathcal{H}_0 := \{t(g - \xi) : t \geq 0, g \in co \quad ext F_{v+\tilde{v}}\}$$

with respect to the inner product

$$\langle t(g-\xi)|s(h-\xi)\rangle := ts(2\langle\tau|\rho\rangle - 1).$$

(Here, $\langle\tau|\rho\rangle$ denotes transition probability)

Step 2: Two-by-two symmetric matrices

Let Z be the closed span of the rank 2 face F_ξ with center ξ and let $\mathcal{H}_\xi$ denote the Hilbert space $F_\xi - \xi$. Let e_1, e_2 be any orthonormal set in $\mathcal{H}_\xi$.

Lemma 3.5.3. *There exist orthogonal extreme points $f_1, \tilde{f}_1$ such that*

$$\operatorname{ext} F_{e_2} \cap \operatorname{span}\{e_1, e_2, \xi\} = \{(e^{i\theta} f_1 + e^{-i\theta} f_2)/2 + e_2 : \theta \in \mathbb{R}\}.$$

Thus we have a rich supply of extreme points, which is enough to prove

Theorem 3.5.4. $\operatorname{span}\{f_1, \tilde{f}_1, e_2\} \simeq S_2(\mathbb{C})_*$.

Step 3: Two-by-two matrices

Same setting as the previous step. Let e_1, e_2, e_3 be any orthonormal set in $\mathcal{H}_\xi$.

Lemma 3.5.5. *There exist orthogonal extreme points $f_2, -\tilde{f}_2$ in F_{e_2} such that*

$$e_2 = \frac{f_2 - \tilde{f}_2}{2}, \quad e_3 = i\frac{f_2 + \tilde{f}_2}{2}.$$

By using this lemma and the known structure of the state space of the C^*-algebra $M_2(\mathbb{C})$, we can prove

Theorem 3.5.6. $\operatorname{span}\{f_1, \tilde{f}_1, f_2, \tilde{f}_2\} \simeq M_2(\mathbb{C})_*$.

In any JB^*-triple, the following corollary is well known and easily seen to be valid for arbitrary tripotents. Here, it is obtained in a more general setting (SFS space) for a particular kind of geometric tripotent (minimal).

Corollary 3.5.7. *If Z is a neutral strongly facially symmetric space of type I_2 satisfying FE and STP, and if u is a minimal geometric tripotent, then the one parameter group $S_\lambda(u) = \lambda P_2(u) + P_1(u) + \overline{\lambda} P_0(u), \lambda \in T$ consists of isometries of Z.*

Before going into step 4, we are first going to construct the triple product and norm in a concrete spin factor in an elementary way by using properties of a spin grid. There are two parts to the construction, namely, the Hilbert space structure and the norm ($=JB^*$-triple structure). Keep in mind that until we reach step 4, we are in a concrete setting.

Part 1 of the construction: Hilbert space structure

The assumptions that we make in the following definition are known to hold for a spin grid in a spin factor ([29]).

Definition 3.5.8. Let I be an index set of arbitrary cardinality. A *basis*, or *spin grid* is a collection $\mathcal{G}$ of linearly independent elements $\{u_i, \tilde{u}_i\}_{i\in I}$ or $\{u_0, u_i, \tilde{u}_i\}_{i\in I}$. Define a triple product $\{uvw\}$ for elements of the basis by:

1. $\{uuu\} = u$ for all $u \in \mathcal{G}$ (the basis will consist of tripotents).
2. For distinct non-zero i, j,

$$\{u_iu_iu_j\} = \{\tilde{u}_i\tilde{u}_iu_j\} = \frac{1}{2}u_j, \{u_ju_ju_i\} = \{\tilde{u}_j\tilde{u}_ju_i\} = \frac{1}{2}u_i,$$

$$\{\tilde{u}_i\tilde{u}_i\tilde{u}_j\} = \frac{1}{2}\tilde{u}_j, \text{ and } \{\tilde{u}_j\tilde{u}_j\tilde{u}_i\} = \frac{1}{2}\tilde{u}_i$$

(u_i will be *colinear* with u_j and with $\tilde{u}_j$, and $\tilde{u}_i$ will be *colinear* with $\tilde{u}_j$); and

$$\{u_iu_j\tilde{u}_i\} = -\frac{1}{2}\tilde{u}_j, \ \{u_j\tilde{u}_i\tilde{u}_j\} = -\frac{1}{2}u_i$$

(the quadruple $(u_i, u_j, \tilde{u}_i, \tilde{u}_j)$ will be an *odd quadrangle*).
3. In case u_0 exists, for each $i \neq 0$,

$$\{u_iu_iu_0\} = \{\tilde{u}_i\tilde{u}_iu_0\} = \frac{1}{2}u_0, \ \{u_0u_0u_i\} = u_i, \ \{u_0u_0\tilde{u}_i\} = \tilde{u}_i$$

(u_0 *governs* u_i and $\tilde{u}_i$), and

$$\{u_0u_iu_0\} = -\tilde{u}_i, \ \{u_0\tilde{u}_iu_0\} = -u_i.$$

4. $\{uvw\} = \{wvu\}$ for all $u, v, w \in \mathcal{G}$.
5. All other products $\{uvw\}$ where u, v, w are from the basis, are zero. In particular, for each $i \neq 0$,

$$\{u_i\tilde{u}_iu\} = 0 = \{\tilde{u}_iu_iu\} \text{ for all } u \in \mathcal{G}$$

($u_i, \tilde{u}_i$ will be *orthogonal*).

It follows from these properties that the set of all scalar multiples of basis elements is closed under the triple product $\{\cdot,\cdot,\cdot\}$. Hence, the triple product $\{\cdot,\cdot,\cdot\}$ can be extended to the real or complex span of the basis elements to be linear in the outer variables and (in the complex case) conjugate linear in the middle variable.

Define an inner product on $\operatorname{sp}\mathcal{G}$ by

$$\langle a|b\rangle = \sum a_i\bar{b}_i + \sum \tilde{a}_i\bar{\tilde{b}}_i + 2a_0\bar{b}_0. \tag{3.5}$$

where $a = \sum a_iu_i + \sum \tilde{a}_i\tilde{u}_i + a_0u_0$ and $b = \sum b_iu_i + \sum \tilde{b}_i\tilde{u}_i + b_0u_0$ are two elements of $\operatorname{sp}\mathcal{G}$.

Definition 3.5.9. The completion of $\operatorname{sp}\mathcal{G}$ with respect to the norm $\|\cdot\|_2$ determined by the inner product (3.5) is called a *(concrete) spin factor*, and will be denoted by $\mathcal{C}$.

If I is finite with n elements, the dimension of $\mathcal{C}$ is $2n$ or $2n+1$. Otherwise, $\mathcal{C}$ is infinite dimensional, and (3.5) is then a convergent sum.

Part 2 of the Construction: JB^*-triple structure (norm)

The norm on the spin factor $\mathcal{C}$ which will make it into a JB^*-triple is not the Hilbert space norm used in the definition, since that norm does not satisfy (2.2). In order to define the correct norm, which will be equivalent to the Hilbert norm, we introduce the following concepts. Define a conjugation $\sharp$ on basis elements by $u_i^\sharp = \tilde{u}_i$, $\tilde{u}_i^\sharp = u_i$, and $u_0^\sharp = u_0$, and extend this to the linear span in a conjugate linear way.

The connection between the triple product, inner product, and conjugation is given by

$$2\{abc\} = \langle a|b\rangle c + \langle c|b\rangle a - \langle a|c^\sharp\rangle b^\sharp. \tag{3.6}$$

For each element a of $\mathcal{C}$, the notion of *determinant* is defined by

$$\det a := \sum a_i\tilde{a}_i + a_0^2 = \frac{1}{2}\langle a|a^\sharp\rangle. \tag{3.7}$$

Proposition 3.5.10 (Proposition 3.3, Lemma 3.4 of [29]). *Let $\mathcal{C}$ be a spin factor.*

1. *If $a \in \mathcal{C}$, then a is a scalar multiple of a minimal tripotent if and only if* $\det a = 0$ *and in this case, from* (2.2) *and* (3.6), *the norm must be defined by* $\|a\| = \langle a|a\rangle^{\frac{1}{2}} = \|a\|_2$ *for such a;*
2. *Elements a and b in $\mathcal{C}$ with* $\det a = \det b = 0$ *are scalar multiples of orthogonal tripotents if and only if there is $\lambda \in \mathbb{C}$ such that $b = \lambda a^\sharp$.*

Proposition 3.5.11 (Proposition 3.6 of [29]). *For any element a in a spin factor $\mathcal{C}$, with* $\det a \neq 0$, *there is a unique set of non-negative numbers $\{s_1, s_2\}$ determined by*

$$s_1^2 + s_2^2 = \langle a|a\rangle, \quad s_1 s_2 = |\det a|. \tag{3.8}$$

Also, if $s_1 \neq s_2$, two orthogonal minimal tripotents e, f are determined uniquely by a such that

$$a = s_1 e + s_2 f. \tag{3.9}$$

Corollary 3.5.12 (Corollary 3.7 of [29]). *If a has decomposition* (3.9), *then from* (2.2) *and* [51, Lemma 1.3(a)], *$\|a\|$ must be defined as* $\max\{s_1, s_2\}$. *From* (4.4) *it follows that this norm is equivalent to the Hilbert norm $\|\cdot\|_2$, and therefore $\mathcal{C}$ is complete and reflexive in this norm.*

Hence every element $a \in C$ has unique coordinates $\{a_i, \tilde{a}_i, a_0\}$ with

$$a = \sum a_i u_i + \sum \tilde{a}_i \tilde{u}_i + a_0 u_0, \tag{3.10}$$

where convergence is in $\|\cdot\|$ (u_0 may not exist).

Step 4: Dual spin grids

Definition 3.5.13. A *dual spin grid* in a facially symmetric space $Z = Z_2(F_\xi)$ of type I_2 is a family $\{f_j, \tilde{f}_j\}_{j \in I \cup \{1\}}$, or $\{f_j, \tilde{f}_j\}_{j \in I \cup \{1\}} \cup \{f_0\}$, where I is an index set not containing 0 or 1, and for each $j \in I$, $f_j, \tilde{f}_j$ are a pair of orthogonal extreme points of ball Z such that $\xi = (f_1 + \tilde{f}_1)/2$, and with $e_1 = (f_1 - \tilde{f}_1)/2$, and

$$e_j = (f_j - \tilde{f}_j)/2 \text{ and } e'_j = i(f_j + \tilde{f}_j)/2 \ (j \in I), \tag{3.11}$$

the collection $\{e_1, e_j, e'_j\}_{j \in I}$ or $\{e_1, e_j, e'_j\}_{j \in I} \cup \{f_0\}$, is an orthonormal basis in the Hilbert space $\mathcal{H}_\xi$.

We can now prove Theorem 3.4.2. Label any orthonormal basis of $\mathcal{H}_\xi$ as $\{e_1, e_j, e'_j\}_{j \in I}$. For each j construct a pair $f_j, \tilde{f}_j$ satisfying (3.11). Next set $f_1 = e_1 + \xi$, and $\tilde{f}_1 = e_1 - \xi$. Then $\{f_j, \tilde{f}_j\}_{j \in I \cup \{1\}}$ is a dual spin grid with the same linear span as $\{e_1, e_j, e'_j, \xi\}_{j \in I}$. There is now an obvious map from this linear span to the linear span of a spin grid in a complex spin factor. It must be shown that this map is isometric, and therefore extends to the desired isomorphism. As usual, this is the difficult part.

This completes the proof of Theorem 3.4.2.

Facial decomposition

We next discuss the facial decomposition in a concrete spin factor. We expect this to be a useful tool in the relation between spin factors and physics. It already helped in the proof of Theorem 3.5.6.

Definition 3.5.14. Let Z be the predual of a concrete spin factor. (Z itself can therefore be identified with a spin factor.) An element $\xi \in Z$ is said to be *unitary* if $\|\xi\|_Z = 1$ and $\xi^\sharp = \lambda\xi$ for some $\lambda \in T$. For $0 \neq a \in Z$ define the *phase* $\zeta(a)$ of a to be

$$\zeta(a) = \det a / |\det a| \text{ if } \det a \neq 0; \qquad \zeta(a) = 1 \text{ if } \det a = 0.$$

Proposition 3.5.15. *Let a be any norm 1 element of Z and let $\lambda = \zeta(a)$. Then the center ξ of a rank 2 face F_v containing a is given by*

$$\xi = \frac{a + \lambda a^\sharp}{2}. \tag{3.12}$$

Moreover, ξ is unitary and $\zeta(\xi) = \lambda$.

The following proposition is the facial decomposition of an arbitrary element of the predual of a spin factor.

Proposition 3.5.16 (Facial decomposition). *Let Z be the predual of a spin factor. For each non-zero $a \in Z$, there are unique elements $\xi, h \in Z$ with*

(i) *ξ, h are scalar multiples of unitaries and $a = \xi + h$;*
(ii) $\zeta(\xi) = -\zeta(h) = \zeta(a)$.

Moreover,

(iii) $\langle \xi | h \rangle = 0$, *and hence* $\|a\|_2^2 = \|\xi\|_2^2 + \|h\|_2^2$;
(iv) $\|a\|_Z = \|\xi\|_Z \geq \|h\|_Z$;
(v) $|\det a| = (\|\xi\|_Z^2 - \|h\|_Z^2)/4$, *and hence* $\|\xi\|_Z = \|h\|_Z \Leftrightarrow \det a = 0$.

In Proposition 3.5.16, if $\|a\|_Z = 1$, then a and ξ belong to the face F_ξ containing a whereas $h = a - \xi$ is "parallel" to F. Moreover, by this proposition, the face F_ξ is a real Hilbert ball defined by

$$F_\xi = \{\xi + h : h \in Z, \|h\|_2 \leq 1, \langle \xi | h \rangle = 0, h^\sharp = -\zeta(\xi) h\}.$$

Note that since $\langle h | h' \rangle = \langle h'^\sharp | h^\sharp \rangle = \langle h' | h \rangle$, the inner product $\langle \xi + h | \xi + h' \rangle = \langle \xi | \xi \rangle + \langle h | h' \rangle$ is real. This should be compared with Proposition 3.5.2.

The following consequence of the facial decomposition in the concrete spin factor is needed in the proof of Theorem 3.5.6. A direct proof is very computational.

Lemma 3.5.17. *Let A be an extreme point of the unit ball of $M_2(\mathbb{C})_*$ (that is, $M_2(\mathbb{C})$ with the trace norm). Then there exists $\mu \in T$ and orthogonal projections $P, Q \in M_2(\mathbb{C})$ with*

$$A \in F_{\mu P + \bar{\mu} Q}.$$

3.6. Stone Weierstrass theory

JB^*-triples as complex Banach spaces

Sakai [101] initiated the study of C^*-algebras and W^*-algebras as Banach spaces.

Since the algebraic structure of a JB^*-triple is determined by holomorphy and geometry, and does not depend on an order structure, it seems appropriate to analyze the Banach space structure of a JB^*-triple. This can be illustrated in the relationship between RNP (Radon–Nikodym property) KMP (Krein–Milman property), and atomicity (Chu–Iochum [22], Barton–Godefrey [16]).

The following definition is motivated by ideas of Shultz ([3],[104]). In it we use the notion of atomic decomposition of a Banach space, as formulated in [33]. We denote the atomic part of a Banach space X by $(X)_a$.

Definition 3.6.1. A Banach space is said to be *weakly perfect* if for every $\xi \in (X^{**})_a$, if ξ is weak*-uniformly continuous on $\text{ext}\,(X^*)_1 \cup \{0\}$, then $\xi \in X$.

Theorem 3.6.2 (Shultz [104], Brown [18]). *Every C^*-algebra is weakly perfect.*

Proof. Shultz showed that if $\xi, \xi^*\xi$ and $\xi\xi^* \in A^{**}$ are all uniformly continuous on $P(A) \cup \{0\}$, then $\xi \in A$. Brown removed the assumptions on $\xi^*\xi$ and $\xi\xi^*$. But $P(A) \subset \text{ext}\,(A^*)_1$. □

Problem 4. Which JB^*-triples are weakly perfect?

Besides C^*-algebras, examples of weakly perfect JB^*-triples are Hilbert spaces, spin factors, elementary JB^*-triples, and commutative JB^*-triples. The answer is not known for JB^*-algebras. To understand why weakly perfect JB^*-triples are important, see Corollary 3.6.5 below.

Extreme points as a dual object for JB^*-triples

The following theorem was proved using a classical theorem of Wigner, together with Glimm's Stone-Weierstrass theorem (for C^*-algebras) and some representation theory of C^*-algebras. It can be summarized as follows (in the category of C^*-algebras): in the presence of Wigner's theorem,

$$\text{Stone–Weierstrass} \Rightarrow \text{Weakly Perfect} \Rightarrow \text{Dual Object.}$$

Theorem 3.6.3 (Shultz [104]). *If A and B are C^*-algebras, and $\psi : P(A) \cup \{0\} \to P(B) \cup \{0\}$ is a weak*-uniformly bicontinuous bijection preserving orientation and transition probabilities, then A and B are isomorphic* (as C^*-algebras).

The following theorem is an unpublished analog of Wigner's theorem for JBW^*-triples. In it, $E(A)$ denotes the set of extreme points of the unit ball of the predual A_* of the JBW^*-triple A.

Theorem 3.6.4 (Dang–Russo). *If A and B are atomic JBW^*-triples, and $\psi : E(A) \cup \{0\} \to E(B) \cup \{0\}$ is a bijection preserving transition probabilities, then A and B are isomorphic* (as JB^*-triples).

Corollary 3.6.5. *If A and B are JB^*-triples, one of which is weakly perfect, and $\psi : E(A) \cup \{0\} \to E(B) \cup \{0\}$ is a uniform homeomorphism in the weak*-topology preserving transition probabilities, and if $\psi(0) = 0$, then A and B are isomorphic* (as JB^*-triples).

In view of the above discussion, a Stone-Weierstrass theorem becomes important for JB^*-triples.

Problem 5 (Stone–Weierstrass). *If A is a JB^*-subtriple of B, and A separates $E(B) \cup \{0\}$, then $A = B$.*

The Stone-Weierstrass conjecture is known to be true for

- some C^*-algebras (Glimm, Sakai, Popa, Bunce, ..., Fujimoto);
- all commutative JB^*-triples (Friedman–Russo [48]);
- Hilbert spaces and spin factors.

4. Linear mappings of JB^*-triples

The content of this section is obvious from the title and the table of contents. We discuss, in turn, isometries, contractive projections, derivations, and bilinear forms.

4.1. Isometries

Isometries of JB^*-triples

In 1951, Kadison [80] proved the following non-commutative extension of the Banach-Stone theorem, thereby showing that the geometry of a C^*-algebra determines somes aspects of its algebraic structure.

Theorem 4.1.1. *Let T be a surjective linear isometry of a unital C^*-algebra A onto a unital C^*-algebra B. Then there is a unitary element $u \in B$ and a Jordan *-isomorphism ρ of A onto B such that*

$$Tx = u\rho(x), \quad x \in A.$$

The proof of the original Banach-Stone theorem, that is, the case in which A and B are abelian, say $A = C(X)$, $B = C(Y)$, uses duality and the intimate relation between the topological space X and the algebra $C(X)$.

Instead, Kadison gives an intrinsic proof, depending mainly on spectral theory and the underlying Hilbert spaces on which A and B act. He also points out that the Jordan *-isomorphism ρ preserves the "quantum mechanical structure", that is, the linear structure and the power structure of self-adjoint elements. It follows that it preserves the symmetrized triple product $\{abc\} = (ab^*c + cb^*a)/2$, that is, if $\tilde{\rho}(a + ib) := \rho(a) + i\rho(b)$, then $\tilde{\rho}\{abc\} = \{\tilde{\rho}(a), \tilde{\rho}(b), \tilde{\rho}(c)\}$, for all $a, b, c \in A$, not necessarily self-adjoint.

As an early example of the use of triple products, Kadison also shows that a quantum mechanical isomorphism preserves commutativity. We sketch the elegant argument.

- ρ preserves the Lie triple product.
  Reason: $[[a, b], c] = [ab - ba, c] = 2\{abc\} - 2\{bac\}$.
- ρ preserves the square of commutators.
  Reason: $[a, b]^2 = (ab - ba)^2 = 2\{a1\{bab\}\} - \{a\{b1b\}a\} - \{b\{a1a\}b\}$.
- If $ab = ba$, then $[a, b] = 0$ so $[[a, b], c] = 0 \ \ \forall c$ which implies that $[\rho(a), \rho(b)]$ belongs to the center of B. But $[\rho(a), \rho(b)]^2 = 0$ so $[\rho(a), \rho(b)] = 0$.

Using some basic results in operator algebras from 1951–1963, a proof of Theorem 4.1.1 can be given [30] which

- is similar to the commutative proof in that it uses affine geometric properties of the convex set of states; more precisely, it consider faces (=extremal subsets) instead of pure states (=extreme points), where by "state" is meant any norm one functional;

- is independent of order, hence applies to JB^*-triples to give a different proof of Kaup's 1983 generalization [84] of Theorem 4.1.1.

Theorem 4.1.2 (Kaup). *Let T be a surjective linear isometry of a JB^*-triple A onto a JB^*-triple B. Then T is an isomorphism.*

The basic results referred to above are

- bidual;
- polar decomposition of normal functional;
- Jordan decomposition of normal hermitian functional;
- neutrality.

Prior to Theorem 4.1.2, versions of it were obtained for JC^*-triples (Harris 1973 [68]) and for JB^*-algebras (Wright-Youngson 1978 [121]).

In the rest of this subsection, we outline the proof of Theorem 4.1.2 obtained in [30] in the special case of a C^*-algebra, that is, of Theorem 4.1.1.

If T is a surjective linear isometry of a unital C^*-algebra A onto a unital C^*-algebra B, then its adjoint T^* maps faces to faces and preserves orthogonality. The proof is then achieved by

- connecting the algebraic structure to faces;
- characterizing partial isometries in terms of faces and orthogonality;
- using polarization and approximation (spectral theorem); namely with $z^{(3)} := zz^*z$ and $x = \sum \lambda_j u_j$ (finite sum, $\lambda_j \geq 0$ u_j orthogonal partial isometries) we have $x^{(3)} = \sum \lambda_j^3 u_j$, $Tx = \sum \lambda_j T u_j$, $T(x^{(3)}) = \sum \lambda_j^3 T u_j = (Tx)^{(3)}$. This, together with the polarization formula
$$\{abc\} = \frac{1}{8} \sum_{\alpha^4=1,\beta^2=1} \alpha\beta(a + \alpha b + \beta c)^{(3)} \tag{4.1}$$
completes the proof.

Besides the basic results referred to above, the following five lemmas, some of independent interest, are also needed in the proof.

Algebraic structure of the Peirce 2-space

For any partial isometry v in a von Neumann algebra M, $M_2(v)$ is an abstract W^*-algebra ($a \cdot b := av^*b$, $a^\sharp := va^*v$, unit v) with normal state space F_v.

Norm condition for orthogonality

For $f, g \in M_*$ with polar decompositions $f = u|f|$ and $g = v|g|$, $u \perp v$ if and only if $\|f \pm g\| = \|f\| + \|g\|$.

Norm exposed faces and partial isometries

Every norm-exposed face F_x of ball M_* is given by a unique partial isometry w: $F_x = F_w$.

Geometric condition for orthogonality
$u \perp v \Leftrightarrow F_u \perp F_v$.

Geometric characterization of partial isometry
An element $x \in M$ is a partial isometry if and only if $\|x\| = 1$, $F_x \neq \phi$, and $\langle x, F_x^{\perp} \rangle = 0$.

The genesis of the above proof is contained in subsection 1.4.

Real isometries of JB^*-triples

As suggested by section 3, a natural candidate for the "state space" of a JB^*-triple is the entire unit sphere of its dual. Motivated by the symmetry transformation in quantum mechanics, one would like to consider invertible affine maps on the unit sphere of the dual of a JB^*-triple. Unlike the situation in C^*-algebras and JB^*-algebras, these maps turn out to be the adjoints of real-linear (not complex-linear) surjective isometries. Thus it is meaningful to study such maps. The result described below is due to Dang [28] and can be viewed as an extension of a famous theorem of Wigner or of the previously mentioned theorem of Kaup. It also provides a partial converse to a theorem of Friedman–Hakeda [45].

The first step is to exploit the one-to-one correspondence between tripotents in a JBW^*-triple and the norm-exposed faces in the unit ball of its predual, as in the previous subsection, to show that a real-linear isometry preserves cubes and orthogonality of elements. Since the map is assumed to be only real-linear however, the polarization formula (4.1) is no longer applicable. Thus a new technique is needed. In view of the Gelfand-Naimark theorem of section 2, the next step is to analyze real-linear isometries between Cartan factors.

Theorem 4.1.3. *Let U and V be Cartan factors with the rank of U at least 2 and let $\phi : U \longrightarrow V$ be a real-linear surjective isometry. Then ϕ is either linear or conjugate linear and preserves the triple product.*

The following is the main result of [28].

Theorem 4.1.4. *Let M and N be JB^*-triples and let $\phi : M \longrightarrow N$ be a real-linear surjective isometry. If M^{**} does not have a nontrivial Cartan factor of rank 1 as a direct summand, then ϕ is the sum of a linear isometry and a conjugate linear isometry.*

The result has immediate application to all C^*-algebras and all JB^*-algebras since these categories do not contain Hilbert spaces of dimension larger than 1. For a further extension of Kadison's isometry theorem, see the next subsection.

Isometries of real C^*-algebras

The paper [21] arose from a desire to define and study infinite dimensional real JB^*-triples via functional analysis (cf. subsection 2.5). The first attempt to formulate a definition came from a consideration of the range of a contractive projection on a real C^*-algebra. Although this can be analyzed easily in the commutative case, see [21 section 7], the general case poses serious obstacles, and it remains open as to whether this range is isomorphic to a norm closed subspace of another real C^*-algebra stable for the triple product in that C^*-algebra (see [52] for the case of a complex C^*-algebra).

Upmeier ([115, §20]) has proposed a definition of a real JB^*-triple. His spaces include real C^*-algebras, JB^*-triples considered as vector spaces over the reals, and the bounded operators between real or quaternionic Hilbert spaces. They also have the property that their open unit balls are real bounded symmetric domains. Since a real C^*-algebra is a real JB^*-triple, hence essentially a geometric object, a natural test for its structure theory is whether the surjective linear isometries preserve the triple product. This is the main problem solved in [21] and discussed in this subsection.

The main result is the analog, for real C^*-algebras, of Kadison's celebrated theorem (4.1.1), and is based, in outline, on the recent affine geometric proof of that theorem ([30]). Accordingly, the tools needed for that proof, which are standard results in the theory of (complex) C^*-algebras, need to be found for real C^*-algebras. Although some of these results were expected or could be predicted, some of the proofs contain new ideas.

The ingredients of the proof are the following:

- the bidual of a real C^*-algebra is a real C^*-algebra;
- definition of a real W^*-algebra;
- the complexification of a real W^*-algebra is a W^*-algebra;
- standard spectral theoretic type results are formulated for a real W^*-algebra;
- relation between partial isometries and norm exposed faces which connects the algebraic structure of a real W^*-algebra with the geometric structure of the unit ball of its predual;
- an isometry preserves orthogonality and "cubes", and sends partial isometries to partial isometries;
- the special case of the main result in which the two real C^*-algebras are W^*-factors of type I, *i.e.*, of the form $\mathcal{L}(H)$ for some real, complex, or quaternionic Hilbert space H;
- reduction of the main result to the special case.

In the rest of this subsection, we elaborate on some of these steps.

A *real C^*-algebra* is a real Banach *-algebra A such that $\|a^*a\| = \|a\|^2$ and $1 + a^*a$ is invertible in A if A has a unit. If A is not unital we require that $1 + a^*a$ be invertible for all a in the unit extension $\tilde{A}$ of A. We note that, by [96, 4.1.13], if A is a non-unital real C^*-algebra, then the unit extension $\tilde{A} = A \oplus \mathbb{R}$ is a real C^*-algebra under the norm

$$\|(x, \lambda)\| = \sup\{\|xu + \lambda u\| : u \in A, \|u\| = 1\}.$$

A real Banach algebra A is *Arens regular* if the two Arens products on the second dual A'' coincide. If A is a real Banach *-algebra which is Arens regular, then the involution * on A extends naturally to A'' and A'' becomes a real Banach *-algebra. Moreover, the extended involution is $\sigma(A'', A')$-$\sigma(A'', A')$ continuous.

If A is a real C^*-algebra, then its complexification $\mathcal{A} = A + iA$ can be given a norm so that it becomes a complex C^*-algebra, and A embeds isometrically as a real C^*-subalgebra of $\mathcal{A}$ ([62, 15.4]).

In [21], it is shown that a real C^*-algebra is Arens regular. Hence, there exists a natural $\sigma(A'', A')$-$\sigma(A'', A')$-continuous involution * on A'' which extends the involution * on A: for $x \in A''$, $x^*(f) := \langle x, f^* \rangle$ where $f^* \in A'$ is defined by $f^*(a) := f(a^*)$ for $a \in A$.

Theorem 4.1.5. *Let A be a real C^*-algebra. Then its second dual A'', equipped with the Arens product and natural involution, is a real C^*-algebra.*

Let A be a real C^*-algebra. We call A a *real W^*-algebra* if A is linearly isometric to the dual space E' of a real Banach space E such that multiplication in A is separately $\sigma(A, E)$-continuous.

Theorem 4.1.6. *Let A be a real W^*-algebra. Then its complexification $\mathcal{A}$ is a W^*-algebra. Morever, A is $\sigma(\mathcal{A}, \mathcal{A}_*)$-closed in $\mathcal{A}$, and for $a, a_\alpha \in A$,*

$$\sigma(A, E)\text{-}\lim_\alpha a_\alpha = a \Leftrightarrow \sigma(\mathcal{A}, \mathcal{A}_*)\text{-}\lim_\alpha a_\alpha = a.$$

Theorem 4.1.7. *Let H and K be Hilbert spaces over the same set of scalars which is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $\phi : \mathcal{L}(H) \to \mathcal{L}(K)$ be a weak*-weak*-continuous surjective real-linear isometry. Then ϕ preserves the triple product: for $a, b, c \in \mathcal{L}(H)$,*

$$\phi(ab^*c + cb^*a) = \phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a).$$

Theorem 4.1.8. *A surjective linear isometry ϕ between two real C^*-algebras preserves the triple product: $\phi(ab^*c + cb^*a) = \phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a)$.*

4.2. Contractive projections

We have witnessed the important role played by contractive projections in the structure theory of JB^*-triples, see subsections 1.1, 1.2 and 2.3. This subsection contains information on contractive projections which is a result of that structure theory.

Conditional expectation and bicontractive projections

Let X be an arbitrary Banach space. We consider two questions regarding idempotent linear maps on X.

Problem 1. If $P : X \to X$ is a contractive linear projection, that is $\|P\| \leq 1$, then describe P and its range $P(X)$.

Problem 2. Define the class $\mathcal{S}$ to be those Banach spaces X for which every bicontractive projection P, that is $\|P\| \leq 1$ and $\|I - P\| \leq 1$, is of the form $P = (I + \theta)/2$ where θ is an involutive isometry. Describe the class $\mathcal{S}$.

Here is the history of Problem 1 prior to 1980. References can be found in [47].

1965 Douglas: $X = L^1(\mu) \Rightarrow P = QR$, Q Markov, and $P(X) \simeq L^1(\nu)$.

1966 Ando: $X = L^p(\mu) \Rightarrow P(X) \simeq L^p(\nu)$.

1969 Wulbert–Lindenstrauss: $X = C^{\mathbb{R}}(K) \Rightarrow P(X) \simeq$ a C_σ-space.

On $C(K)$, most results assumed $P1 = 1$, that is, P is Markov.

1978 Arazy–Friedman: $X = C_\infty$ (=compact operators on a separable Hilbert space)$\Rightarrow$ complete classification of $P(X)$ (Cartan factors of types 1,2,3,4).

Two other results involving *positive* and *completely positive* projections on C^*-algebras, and progress after 1980 have been discussed in subsections 1.1 and 1.2.

Here is the history of Problem 2. References can be found in [55].

1977 Bernau–Lacey: $C_0(K), L^p \in \mathcal{S}$.

1978 Friedman–Arazy: C_∞, C_1 (trace class) $\in \mathcal{S}$.

A natural question: Which C^*-algebras belong to $\mathcal{S}$? In particular, does $\mathcal{L}(H) \in \mathcal{S}$? The answer is yes, see Theorem 4.2.1.

1982 Størmer: Partial result: If P is a *positive* bicontractive projection on a C^*-algebra, then $2P - I$ is an isometry.

In their study of a contractive projection P on a commutative C^*-algebra A, as noted in subsection 1.1, Friedman–Russo also proved the following:

- $P(A)$ is an abstract C^*-ternary ring.
- $P(A)$ is a subalgebra if and only if P is averaging $(P(fPg) = PfPg)$.
- $P(A) \simeq$ a unital Banach algebra if and only if $\operatorname{ext} \operatorname{ball} P(A) \neq \phi$.
- $P(A) \simeq$ a C_σ space always.

With this information, it was easy to give another proof of the result of Bernau–Lacey, namely that $C_0(K) \in \mathcal{S}$.

The bicontractive projection problem was completely solved in the category of JB^*-triples by Friedman–Russo in 1985 ([55]). Earlier it was solved for JC^*-triples, hence C^*-algebras, in 1983 ([50]).

Theorem 4.2.1. *Every JB^*-triple belongs to the class $\mathcal{S}$.*

Here are the main steps in the proof. Let P be a bicontractive projection on a JB^*-triple U.

- Let $\theta = 2P - I$. To prove $\|\theta\| \leq 1$, it suffices to show that
$$\theta(\{xxx\}) = \{\theta x \theta x \theta x\}. \tag{4.2}$$
- Two conditional expectation formulas (for contractive projections):

  Kaup: $P\{PaPbPc\} = P\{PaPbc\}$;
  Friedman–Russo: $P\{PaPbPc\} = P\{PabPc\}$.
- $P(U)$ is a subtriple of U.

  A simple calculation now yields (4.2).

As a by-product of the above proof, it can be shown that the image $P(U)$ of a contractive projection on a JB^*-triple U is isomorphic to a subtriple of the JB^*-triple U^{**}. Thus, the abstract triple product on $P(U)$ can be realized in a concrete way, as was the case with JC^*-triples.

Application to Banach space theory: contractive projections on C_p

Let H be a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(H)$ denote the space of all bounded linear operators on H. Let $C_\infty = C_\infty(H)$ denote the subspace of $\mathcal{L}(H)$ consisting of all compact operators.

For $1 \leq p < \infty$ let $C_p = C_p(H)$ be the Banach space of all $x \in C_\infty$ for which
$$\|x\|_p := (tr|x|^p)^{1/p}$$
is finite, where $|x| = (x^*x)^{1/2}$ and "tr" denotes the usual trace.

The spaces C_p are called "von Neumann–Schatten p-classes"; C_2 is the Hilbert space of Hilbert–Schmidt operators and C_1 is the trace class. The standard references for the basic properties of C_p are [61], [105], [89], and [36].

An important application of JB^*-triple-theory is the following result of Arazy-Friedman 1992, [10], [11]. It extends and continues their work on contractive projections in C_1 and C_∞, [9]. While this result is similar to those of [9], the methods used are very different. We shall not define all of the terms here.

Theorem 4.2.2 (Arazy-Friedman). *Let X be a closed subspace of $C_p, 1 < p < \infty, p \neq 2$. Then the following four properties are equivalent:*

(1) *X is the range of a contractive projection from C_p;*

(2) *X is the ℓ_p-sum of subspaces, each of which is canonically isometric to the C_p ideal of a Cartan factor of one of the types 1 - 4;*
(3) $X^{p-1} := \{v(x)|x|^{p-1}; x \in X\}$ *is a closed linear subspace of* C_q, $(p^{-1} + q^{-1} = 1)$ *where* $x = v(x)|x|$ *is the polar decomposition of the operator* x;
(4) $V := \overline{span}^{w^*}\{v(x); x \in X\}$ *is closed under the triple product*

$$\{u, v, w\} = (uv^*w + wv^*u)/2,$$

and is an atomic JC^*-subtriple of $\mathcal{L}(H)$. Moreover, X is a module over V, namely $\{VVX\} \subseteq X$ and $\{VXV\} \subseteq X$.

As a corollary it follows that the category of C_p ideals of atomic JCW^*-triples is stable under contractive projections.

This work is obviously related to the theory of contractive projections on JB^*-triples (see subsections 1.1 and 1.2). The Jordan triple product is the algebraic structure which respects the geometry (in JB^*-triples as well as in C_p). Therefore the JB^*-formalism can be used effectively in this work. In this formalism the results have a relatively simple form, contrary to the work [9] in which the ranges of contractive projections in C_1 and C_∞ are described without this formalism in a complicated way.

The main tools in studying the contractive projections in C_1 and C_∞ in [9] are the fact that contractive projections on these spaces respect the rich facial structures of their unit balls, and the intimate connection of these structures to the Peirce projections. In the present context C_p is uniformly convex and uniformly smooth (see [89], [109]), and thus each non-trivial face of the unit ball is a single point. Instead, the basic technique is the study of the differentiation of the support-functional map $N_p : C_p \to C_q$ (for $2 < p$). This map is defined by $N_p(0) = 0$ and

$$N_p(x) = x^{p-1}/\|x\|_p^{p-2} \quad \text{if } x \neq 0,$$

where for x with the polar decomposition $x = v(x)|x|$ the powers x^α are defined by $x^\alpha := v(x)|x|^\alpha$.

Here are some of the topics needed for the proof of Theorem 4.2.2.

- Properties of contractive projections on C_p which depend on smoothness, strict convexity and reflexivity.
- Differentiation formulas and Schur multipliers.
- Connection between a contractive projection and Pierce projections associated with elements in its range.
- Existence of atoms and basic relations between atoms.
- Structure of N-convex subspaces of C_p.

4.3. Derivations

In a regional conference in 1985 held at UC Irvine [116], H. Upmeier posed three basic questions regarding derivations:

Q1: Are everywhere defined derivations bounded?
Q2: When are all bounded derivations inner?
Q3: Can bounded derivations be approximated by inner derivations?

The meaning of derivation depends on category. All definitions will be given, and usually the meaning will be clear from the context. The property in Q2 is abbreviated i.d.p. (inner derivation property).

As shown in Table 3 in the introduction, these three questions have all been answered in the binary cases. Q1 has been answered affirmatively by Sakai in [98] for C^*-algebras and affirmatively by Upmeier in [114] for JB-algebras. Q2 has been answered affirmatively by Sakai in [99] and Kadison in [81] for von Neumann algebra; Sakai in [100], Elliott in [42], and Akemann–Pedersen in [2] have all worked out results for certain C^*-algebras, while Upmeier has also answered it in [114] for "JW-algebras". Q3 has been answered affirmatively by Upmeier in [114] for JB-algebras, and it follows trivially from the Kadison–Sakai answer to Q2 in the case of C^*-algebras.

In the ternary case, both Q1 and Q3 have been answered affirmatively by Barton and Friedman in [14] for JB^*-triples. Here, we discuss the answer to Q2 for "JBW^*-triples", due to Ho [70]. There are three subsections. The first two concern derivations of a JBW^*-triple into itself. The third involves derivations of a JB^*-triple into its dual space, which is an example of a "module."

Derivations of JBW^*-triples of type I

There are two main results in Ho's dissertation. We state the first one now, and then define some terms and briefly describe what is involved.

Theorem 4.3.1. *Let A be a JBW^*-triple of type* I (by known structure theory,

$$A = \oplus^{\infty}_{\alpha\in\mathbf{I}}\mathbf{L}^{\infty}(\Omega_\alpha, C_\alpha),$$

where C_α's are Cartan factors of types 1-6). *Let*

$$K = \{\alpha \in \mathbf{I} : C_\alpha \textit{ is type } 1 \textit{ non-square or type } 4\}.$$

Then every derivation on A is inner if and only if $\sup_{\alpha\in K} \dim C_\alpha < \infty$.

A tripotent u is said to be *abelian* if $\{uAu\}$ is a unital associative Jordan algebra. By definition, a Type I JBW^*-triple A is one which is generated by its abelian tripotents. It was proved in Horn's dissertation [71] that a type I JBW^*-triple A can be decomposed as follows: there exist measure spaces Ω_α and Cartan factors C_α such that

$$A = \oplus^{\infty}_{\alpha\in\mathbf{I}}\mathbf{L}^{\infty}(\Omega_\alpha, C_\alpha).$$

Horn also showed that for an arbitrary JBW^*-triple $A, A = A_I \oplus A_C$, where A_C (for continuous) means that A_C contains no abelian tripotents, and A_I is of type I.

The Cartan factors of types 1–4 were described in subsection 3.1. To define the remaining two "exceptional" Cartan factors we first need a definition.

Definition 4.3.2. The *Octonions* $\mathcal{O}$ is an eight dimensional, complex (nonassociative) algebra with standard basis $\{e_0, \cdots, e_7\}$ satisfying

(i) $e_0 e_j = e_j = e_j e_0, \quad j = 0, \cdots, 7;$

(ii) $e_j^2 = -e_0, \; e_j e_k = -e_k e_j, \quad j, k = 1, \cdots, 7, \; j \neq k$; and

(iii) $e_1 e_2 = e_3, \; e_1 e_4 = e_5, \; e_6 e_7 = e_1, \; e_2 e_5 = e_7, \; e_6 e_4 = e_2, \; e_3 e_4 = e_7, \; e_3 e_5 = e_6$.

For $a = \sum_{i=0}^{7} t_i e_i \in \mathcal{O}$, define $a^j := t_0 e_0 - \sum_{i=1}^{7} t_i e_i, \bar{a} := \sum_{i=0}^{7} \bar{t}_i e_i$, and $a^* := \bar{a}^j$.

Cartan factor of type 5: $\mathbf{M}_{1,2}(\mathcal{O}) :=$ the space of all 1×2 matrices over $\mathcal{O}$, the Octonions. The triple product on $\mathbf{M}_{1,2}$ is

$$\{xyz\} := \frac{1}{2}[x(y^* z) + z(y^* x)].$$

For $x = (a, b) \in \mathbf{M}_{1,2}(\mathcal{O})$, x^* is defined as $\begin{pmatrix} a^* \\ b^* \end{pmatrix}$.

Cartan factor of type 6: $\mathbf{H}_3(\mathcal{O}) :=$ the space of all 3×3 hermitian matrices over $\mathcal{O}$. The triple product is defined via the Jordan product

$$\{xyz\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

The Jordan product is defined here as $x \circ y := \frac{1}{2}(xy + yx)$, and for $(a_{mn})_{1 \le m,n \le 3}$ in $\mathbf{H}_3(\mathcal{O})$, $(a_{mn})^*_{1 \le m,n \le 3}$ is defined as $(\bar{a}_{mn})_{1 \le m,n \le 3}$.

A *derivation* δ of a Jordan algebra or an associative algebra A is a linear map $\delta : \mathrm{Dom}(\delta) \longrightarrow A$ satisfying for all $a, b \in \mathrm{Dom}(\delta) \subseteq A$, $\delta(ab) = (\delta a)b + a(\delta b)$. An *inner derivation* of a Jordan algebra A is a finite sum of the form $[M_a, M_b]$, where $M_a x = ax$ and $[S, T] = ST - TS$. If $\delta = \sum_{i=1}^{n} [M_{a_i}, M_{b_j}]$, we shall say that δ is "a sum of n commutators". An *inner derivation* of an associative algebra A is one of the form $\delta = \mathrm{ad}h, h \in A$ where $\mathrm{ad}h(x) = hx - xh, x \in A$.

A derivation δ of a JB^*-triple A is defined for all $a, b, c \in \mathrm{Dom}(\delta)$, by the requirement that

$$\delta\{abc\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\}.$$

Examples of derivations on JB^*-triples are $ix \square x$ and inner derivations, where by an inner derivations we mean a finite sum of derivations of the form $a \square b - b \square a$. That fact that these maps are derivations follow from the main identity, that is, equation (2.1). The inner derivation property, abbreviated i.d.p. on a JB^*-triple A (or any algebraic system) refers to the property that every (continuous) derivation on A is inner.

In finite dimensions, it is known on purely algebraic grounds ([78] and [88]), that all derivations of the relevant Jordan structures are inner.

We shall see later that examples of outer derivations come from Hilbert spaces, spin factors, and infinite dimensional (non-square) $\mathcal{L}(H, K)$.

The following known theorems related to derivations for C^*-algebras and Jordan C^*-algebras motivated Theorem 4.3.1. The first is classical, and the second is a key step in the proof of Theorem 4.3.1.

Theorem 4.3.3 (Sakai [98], [99], Kadison [81]). *Any von Neumann algebra has i.d.p. Derivations of C^*-algebras are automatically continuous and "spatial" with generator in the weak closure* (or bidual).

Theorem 4.3.4 (Upmeier [114]). *A JB^*-algebra has i.d.p. if and only if all "spin representations" have bounded finite dimensions.*

We now start discussing the ingredients of the proof of Theorem 4.3.1. It is first necessary to show that the Cartan factors of type 2 with $\dim H$ even or infinite, and all Cartan factors of type 3 are reversible JC*-algebras (cf. Upmeier [114]).

Each JW-algebra $\mathbf{X}$ has a type decomposition of the form

$$I_{fin} \oplus I_\infty \oplus II_1 \oplus II_\infty \oplus III$$

([110, th. 13]). We do not define the meaning of these symbols here.

Remark 4.3.5. H. Upmeier has shown that each derivation on a properly non-modular JW-algebra, that is, its modular part $I_{fin} \oplus II_1$ vanishes, is the sum of 6 commutators ([114, th. 3.8]), and each derivation on a reversible JW-algebra of type I_{fin} is the sum of 5 commutators ([114], th. 3.9]).

We now state the following theorem, implicit from [114], for completeness. Its proof uses results of Ajupov, Fack–de la Harpe, and Størmer from [1], [43], [106], [107].

Theorem 4.3.6. *Let $\mathbf{X}$ be a reversible type* II_1 *JW-algebra. Then each derivation of $\mathbf{X}$ is inner. Moreover, each inner derivation is a sum of at most* 80 *commutators.*

By combining Remark 4.3.5 and Theorem 4.3.6, it follows that each derivation on a reversible JW-algebra is a sum of at most 91 commutators.

Now we go back to $\mathbf{U}$, a JBW^*-triple of type I. Recall that by Horn's classification, $\mathbf{U} \simeq \oplus_\alpha^\infty \mathbf{C}(\mathbf{X}_\alpha, C_\alpha)$. If C is a reversible JC^*-algebra, then $\mathbf{C}(\mathbf{X}, C)$, with pointwise algebraic operations and supremum norm, is also reversible. As stated earlier, each derivation on a reversible JW-algebra is a sum of at most 91 commutators. Since this is stated in the real binary case, i.e., as JB-algebra, an argument is necessary to handle the ternary complex version. Fortunately this has been provided by Barton and Friedman in [14]. We omit the details.

We can classify all Cartan factors as follows:

(a) non-square $\mathcal{L}(H,K)$,
(b) reversible unital JC^*-algebras,
(c) antisymmetric operators in $\mathcal{L}(H)$, H odd finite dimensional,
(d) triple spin factors,
(e) finite dimensional (irreducible) JB^*-triples.

Now the algebraists have shown how to handle case (e). Upmeier has solved both (b) and (c) in [114] for JW-algebras. In [70], it is shown how Upmeier's solution leads to a solution for JB^*-triples in both (b) and (c); both (a) and (d) are solved directly, and finally the global (non-factor) result for type I JBW^*-triples is obtained by patching. The details are in [70]. To conclude this subsection, we sketch the proof of (a), which is really the heart of the matter.

There are three possibilities for "non-square" infinite dimensional $\mathcal{L}(H,K)$. It is either $2\times\infty$-matrices, finite$\times\infty$-matrices, or countable $\times$uncountable-matrices. The first case is typical, and in fact the last case is simpler. So let $\{u_{1j},u_{2j}\}_{j\in J}$ be a rectangular grid of matrix units, where without loss of generality, J is countably infinite. Set

$$\delta^0 := \sum_{j=1}^{\infty}\sqrt{-1}\frac{u_{1j}\square u_{1j}}{2^j}.$$

Since δ^0 is a pointwise limit of derivations, it is a derivation. Let's use the notation $\delta(a,b)$ for the inner derivation $a\square b - b\square a$, so that for any isometry (=automorphism) T we have $T\delta(a,b)=\delta(Ta,Tb)T$. Suppose that δ^0 is inner, say

$$\delta^0 = \sum_{i=1}^{N}\delta(a_i,b_i).$$

One can construct an isometry T which is a product of symmetries in such a way that the elements Ta_i and Tb_i have only finitely many non-zero coordinates. Now consider the derivation $\tilde{\delta}=T\delta^0T^{-1}$. If H denotes the Hilbert space which is the closed span of $\{u_{1j}\}$ and H_L denotes the span of $u_{11},\ldots,u_{1L}$, then it can be shown that

$$H_{4N}\supset\tilde{\delta}H\supset T\delta^0H\supset TH=H,$$

a contradiction.

Derivations of continuous JBW^*-triples

We first describe the structure of the continuous summand of a JBW^*-triple, which is due to Horn–Neher [75]: if $\mathbf{U}$ is a continuous JBW^*-triple, then

$$\mathbf{U}\simeq\mathbf{R}\oplus^{\infty}\mathbf{H}(A,\alpha).$$

Here $\mathbf{R}=p\mathbf{B}$ where $\mathbf{B}$ is a continuous von Neumann algebra and p is a projection in $\mathbf{B}$, and $\mathbf{H}(A,\alpha)=\{a\in A : a^\alpha = a\}$ where α is a $\mathbb{C}$-linear involution of the continuous W^*-algebra A. Since $1 \in \mathbf{H}(A, \alpha), \mathbf{H}(A, \alpha)$ is actually a reversible unital JC^*-algebra. As for $\mathbf{R}$, it can be decomposed further as follows:

A JBW^*-triple is called *type* II_1^a (resp. $\mathrm{II}_{\infty,1}^a$, II_{∞}^a, III^a) if it is isomorphic to pA, p a projection in a von Neumann algebra A, where

type II_1^a: A is of type II_1 and p is (necessarily) finite,
type $\mathrm{II}_{\infty,1}^a$: A is of type II_∞ and p is finite,
type II_{∞}^a: A is of type II_∞ and p is properly infinite, and
type III^a: A is of type III and p is (necessarily) purely infinite.

Furthermore, Horn and Neher have shown that if $\mathbf{U}$ is a JBW^*-triple of type II_∞^a or III^a with a separable predual, then $\mathbf{U}$ is isomorphic to a von Neumann algebra. Since every von Neumann algebra is a unital JB^*-algebra, every derivation here is also inner.

Here are the positive results on derivations of continuous JBW^*-triples.

Proposition 4.3.7. *Let M be a continuous von Neumann algebra.*

(a) *If M if of type* III *and either countably decomposable or a factor, then the JBW^*-triple pM has i.d.p.*
(b) *If M is of type* II_∞ *and has a separable predual, then pM has i.d.p.*

Here is another approach which may be successful in the remaining (open) cases. Horn–Neher also show that if the JBW^*-triple pM is of type $\mathrm{II}_{\infty,1}$, II_∞, or III (that is, not finite), then

$$pM \cong \oplus_\alpha^\infty B_\alpha \otimes H_\alpha \subset \oplus_\alpha^\infty B_\alpha \otimes \mathcal{L}(H_\alpha)$$

where B_α is a von Neumann algebra of the same type, and H_α are Hilbert spaces. The property i.d.p. is then equivalent to the solvability of a system of operator equations. For details we refer to [70]. Although this approach omits the II_1 case, it is known to be true in some of the III and II_∞ cases.

Weakly amenable Banach Jordan triples

We now state and discuss, without detail, the second main result of Ho's dissertation.

Theorem 4.3.8. *Every derivation from a commutative JB^*-triple into its dual is inner.*

A commutative JB^*-triple is a JB^*-triple satisfying the property that

$$[D(a,b), D(c,d)] = 0.$$

where $D(a,b)$ is the linear operator $z \mapsto \{abz\}$. In order to define a derivation (and inner derivation) from a JB^*-triple into its dual denoted by A', we need to show that A' is a module over A. We refer the interested reader to Ho's dissertation for these definitions and proof of the above result.

We next state the known result for C^*-algebras which motivated Theorem 4.3.8.

Theorem 4.3.9 (Haagerup [63], 1983). *All derivations from any C^*-algebra into its dual are inner.*

It is also shown in [70] that this result is not true for Jordan C^*-algebras. The proof involves some results on commutators of operators on a separable Hilbert space, as in [95].

4.4. Bilinear forms

In this subsection, we give a preview of the paper [15].

For C^*-algebras, the important functionals are the states, giving rise in the commutative case to probability measures. These can be decomposed into discrete and continuous parts, and further into absolutely continuous and singular parts. It is this kind of phenomenon that we wish to consider in a setting in which positivity, commutativity, associativity, and even the binary product are absent.

For von Neumann algebras, the use of the trace in the semifinite cases, and of extreme points in the atomic cases has facilitated their study. In the purely infinite cases, where no trace is present, the Tomita-Takesaki theory showed how to effectively use non-tracial normal states. For JBW^*-triples, the Hilbertian seminorms introduced below will be shown to be useful for obtaining structural information on the triple and the functional.

The main theorem of [15] (Theorem 4.4.11 below) gives a fundamental relation between two basic kinds of sesquilinear forms (called OP for operator positivity, and RN for Radon–Nikodym). To appreciate the level of abstraction in this theorem, it is necessary to introduce the algebraic inner product arising from a normal functional, and recall the theorems of Grothendieck type in C^*-algebras and JB^*-triples.

Grothendieck's inequality

Let A be a JBW^*-triple and A_* its predual. We want to consider local properties of A, A_* starting from a functional $\varphi \in A_*$. Order structure plays a key role when it exists *a priori.* When it does not exist, one can produce a local order structure by means of the polar decomposition of normal functionals on a von Neumann algebra (Proposition 2.2.4).

Definition 4.4.1. Let A be any complex vector space. A map $b : A \times A \to \mathbb{C}$ is called a *sesquilinear form* if it is linear in the first argument and conjugate linear in the second one. A sesquilinear form is said to be *positive* if $b(x,x) \geq 0$ for all $x \in A$. Since a positive sesquilinear form is automatically hermitian, it defines a seminorm called a *Hilbertian seminorm.* If b is a sesquilinear form, then $b^*(y)$

denotes the linear functional on A defined by $b^*(y) = b(\cdot, y)$:

$$\langle x, b^*(y)\rangle = b(x, y).$$

The following proposition may be used to show that each normal functional φ on a JBW^*-triple A gives rise to a Hilbertian seminorm, which retains much of the information supplied by φ.

Theorem 4.4.2 ([13]). *Let A be a JBW^*-triple. For $\varphi \in A_*$, let $e = e(\varphi)$ be the support tripotent of φ. Define a sesquilinear form a_φ by*

$$a_\varphi(x, y) = \varphi(\{xye\}) \quad x, y \in A. \tag{4.3}$$

Then a_φ is positive on A, $a_\varphi^*(e) = \varphi$ and

$$\|x\|_\varphi := a_\varphi(x, x)^{\frac{1}{2}}$$

is a Hilbertian seminorm on A. Moreover, $\|x\|_\varphi \leq \|\varphi\| \cdot \|x\|$ and e in (4.3) can be replaced by any $a \in A$ satisfying $\|a\| = 1 = \varphi(a) = \|\varphi\|$.

The a in the notation a_φ for the form is used since it is based on the algebraic structure of A. One of the places where these seminorms appear is in the Grothendieck inequality for C^*-algebras.

Theorem 4.4.3 ([93], [64]). *There is a universal constant K such that for any two C^*-algebras A, B, and any bounded bilinear form $T : A \times B \to \mathbb{C}$ there exist states φ on A and ψ on B such that*

$$|T(x, y)| \leq K\|T\| \left[\varphi\left(\frac{x^*x + xx^*}{2}\right)\right]^{\frac{1}{2}} \left[\psi\left(\frac{y^*y + yy^*}{2}\right)\right]^{\frac{1}{2}} \quad \textit{for } x \in A, y \in B.$$

The Hilbertian seminorm $\left[\varphi\left(\frac{x^*x+xx^*}{2}\right)\right]^{\frac{1}{2}}$ is not one of the natural ones associated to C^*-algebras. The natural ones are the ones that occur in the G.N.S. construction, namely

$$\|x\|_\varphi^\sharp = \varphi(x^*x)^{\frac{1}{2}} \text{ and } \|x\|_\varphi^\flat = \varphi(xx^*)^{\frac{1}{2}}.$$

These "associative" seminorms were shown by Pisier to be insufficient for the inequality of Grothendieck. The seminorm that is needed in Grothendieck's inequality for C^*-algebras is the one introduced above, namely

$$\|x\|_\varphi = \left[\varphi\left(\frac{x^*x + xx^*}{2}\right)\right]^{\frac{1}{2}} = \varphi(\{xx1\})^{\frac{1}{2}},$$

indicating that the inequality does not rely on associativity. This also suggests that the inequality is independent of the order structure and is related to the geometry of the unit ball (which is a bounded symmetric domain). This was confirmed for JB^*-triples (corresponding to arbitrary bounded symmetric domains in complex Banach spaces) in the following result.

Theorem 4.4.4 ([14]). *There is a universal constant $\tilde{K}$ ($\geq K$) such that for any two JB^*-triples A, B, and any bounded bilinear form $T : A \times B \to \mathbb{C}$, there exist norm one functionals φ on A and ψ on B such that*

$$|T(x,y)| \leq \tilde{K}\|T\| \, \|x\|_\varphi \, \|y\|_\psi \quad x \in A, y \in B.$$

Hilbertian seminorms based on algebraic structure and local order

The following Proposition is a reformulation of the Radon–Nikodym theorem for JB^*-triples. The proof is a modification of the proof for von Neumann algebras ([92]).

Proposition 4.4.5. *Let A be a JBW^*-triple and let $\varphi \in A_*$ with support tripotent e. If $0 \leq \psi \leq \varphi$ (in $A_2(e)_*$), then there is a unique y such that $0 \leq y \leq e$ (in $A_2(e)$) and*

$$\psi(x) = \varphi(\{xye\}) \text{ for all } x \in A.$$

Definition 4.4.6. Let A be a JBW^*-triple and let e be a tripotent. We shall say that a sesquilinear form b satisfies the *Radon–Nikodym property with respect to e* (e-RN property for short) if for any $\psi \in A_2(e)_{*+}$ with $\psi \leq b^*(e)$, there exists a *unique* $h \in [0, e]$ such that $\psi = b^*(h)$. In particular, $[0, b^*(e)] \subset b^*[0, e]$.

Corollary 4.4.7. *Let A be a JBW^*-triple and let $\varphi \in A_*$ with support tripotent e. Then the form a_φ defined by Proposition* 4.4.2. *satisfies the e-RN property.*

The complementary property to e-RN property for sesquilinear forms is the Order Positivity property defined as follows:

Definition 4.4.8. Let A be a JBW^*-triple and let e be a tripotent. We shall say that a sesquilinear form b satisfies the *Order Positivity property with respect to e* (e-OP property for short) if $b^* = b^*P_2(e)$ and $b(x, y) \geq 0$ for all $x, y \in A_2(e)^+$.

Recall that a sesquilinear form b (for which $b^* = b^*P_2(e)$) is e-OP if

$$b^*[0, e] \subset [0, b^*(e)]$$

and a sesquilinear form is e-RN if

$$[0, b^*(e)] \subset b^*[0, e].$$

A form that combines both of these properties is the so called self-polar form, defined as follows:

Definition 4.4.9. A sesquilinear form s on a JB^*-triple A is called *self-polar* (resp. *weakly self-polar*) *relative to the tripotent* e if $s^* = s^*P_2(e)$ and

$$s^*[0, e] = [0, s^*(e)]$$

(resp. $s^*[0,e]$ is $\sigma(A_2(e)^*, A_2(e))$-dense in $[0, s^*(e)]$).

Self-polar forms were introduced by Connes ([25]) and Woronowicz ([120]). They were used by Connes to show that a von Neumann algebra can be represented as the set of derivations of a self-dual cone in a Hilbert space, a result which was generalized to Jordan algebras by Iochum ([76]).

In the case A is a JBW^*-algebra (with unit $1 = e$), or a von Neumann algebra, a self-polar form s_ψ with $s_\psi^*(e) = \psi$ exists and is unique for each positive faithful normal functional ψ on A. (See [120] for the von Neumann algebra case and [66] for the JBW-algebra case.)

Proposition 4.4.10. *Let A be a JBW^*-triple and let $\varphi \in A_*$ have support tripotent e. Then there is a unique self-polar form s_φ relative to e such that $s_\varphi^*(e) = \varphi$.*

Before stating the main theorem below, we must introduce some terminology. As in [94], we say a sesquilinear form a on a vector space A is *represented* by (π, T) if

$$a(x,y) = (T\pi(x)|\pi(y))_H, \quad x, y \in A.$$

Here π is a linear map of A onto a dense subset of a Hilbert space H and $T \in \mathcal{L}(H)$. Obviously, a is positive if and only if T is a positive operator.

Obviously, any positive sesquilinear form can be represented in this way (by the identity operator) and in fact, by [94, Theorem 1.1], any two such forms can be represented on the same Hilbert space by *commuting* positive operators. The *geometric mean* $\sqrt{\alpha\beta}$ of two sesquilinear forms α, β is defined by [94, Theorem 1.2] as follows: if α is represented by (π, S) and β is represented by (π, T), then $\sqrt{\alpha\beta}$ is represented (unambiguously) by $(\pi, (ST)^{1/2})$.

Theorem 4.4.11. *Let a and b be positive sesquilinear forms on a JBW^*-triple A and let $\varphi \in A_*$ have support tripotent e. Suppose that a and b satisfy the following:*

- $a^*(e) = b^*(e) = \varphi$;
- *a satisfies e-RN;*
- *b satisfies e-OP.*

Then there exists a positive sesquilinear form h with $h^*(e) = \varphi$ which satisfies e-OP, such that b is the geometric mean $\sqrt{ah}$ of a and h. Moreover

$$b(x,x) \le a(x,x) \tag{4.4}$$

for any $x \in A$.

Equation (4.4) states that the seminorm defined by any e-RN form is larger than the seminorm defined by any e-OP form. In particular, the seminorm defined by a self-polar form is larger than all seminorms defined by an e-OP form and is smaller than all seminorms defined by an e-RN form.

The Cauchy–Schwarz inequality holds for any positive sesquilinear form, and by Corollary 4.4.7, a_φ is e-RN. Therefore we can obtain another inequality of

Grothendieck type for JB^*-triples. Note that this version has the constant 1 and involves one functional and one JBW^*-triple.

Corollary 4.4.12. *Let b be a positive sesquilinear form on a JBW^*-triple A which is e-OP and satisfies $b^*(e) = \varphi$. Then*

$$|b(x,y)| \leq \|x\|_\varphi \|y\|_\varphi$$

for any $x, y \in A$, where the norm $\|\cdot\|_\varphi$ is defined as in Proposition 4.4.2.

Schur multipliers and Hilbertian seminorms of an atomic functional

Recall (Theorems 2.2.6 and 2.2.7) that every JBW^*-triple and its predual have atomic decompositions. Let A be an atomic JBW^*-triple and let φ be a normal functional on A. Then

$$\varphi = \sum_j s_j f_j \,,$$

where the f_j's form an orthogonal family of extreme points of the unit ball of A_* and the s_j's are nonnegative scalars with $\sum s_j = \|\varphi\|$.

Recall (see subsection 2.2) the contractive conjugate linear map $\pi : A_* \to A$ defined by

$$\pi(\sum \alpha_j f_j) = \sum \overline{\alpha_j} v_j$$

for each finite linear combination $\sum \alpha_j f_j$ of extreme points f_j of the unit ball of A_*, where $v_j = e(f_j)$ is the support tripotent of f_j. The map π is injective and gives rise in turn to a sesquilinear form $\langle f|g\rangle_\pi := \langle f, \pi(g)\rangle$ which is positive definite.

We call this inner product the *tracial* inner product. For $x, y \in \pi(A_*)$, which is known to be weak*-dense in A, define $\langle x|y\rangle_\pi$ to be $\langle \pi^{-1}x|\pi^{-1}y\rangle_\pi$. This is well defined since π is injective. A large number of Hilbertian seminorms associated to a given functional can be obtained in the following way:

Definition 4.4.13. Let $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables s, t and $P : \mathbb{R} \times \mathbb{R} \to \mathcal{L}(A)$ be a projection-valued function of the same variables. We shall call a linear map

$$\mu = \sum_{s,t} m(s,t) P(s,t)$$

on A (when defined) a *Schur multiplier* associated to the pair (m, P). We also define a sesquilinear form on the dense subset $\pi(A_*)$ of A by

$$\langle x|y\rangle_\mu = \sum_{s,t} m(s,t) \langle P(s,t)x|P(s,t)y\rangle_\pi = \langle \mu x|y\rangle_\pi \text{ for } x, y \in \pi(A_*).$$

Note that a Schur multiplier is positive if and only if $m(s,t) \geq 0$ whenever $P(s,t) \neq 0$.

We now consider several examples illustrating this construction. In these examples we shall have $\mu(\pi(A_*)) \subset \pi(A_*)$ and in fact $\mu(A) \subset \pi(A_*)$, which gives rise to a map $\tilde{\mu} := \pi^{-1} \circ \mu$ of A into A_*.

Example 4.4.14. Let A be an atomic von Neumann algebra and let φ be a positive normal functional on A given by a positive trace class operator $a = \sum s_j e_{s_j}$. Define $P(s,t)x = e_s x e_t$, where $e_s = e_{s_j}$ if $s = s_j$ and $e_s = 0$ otherwise.

a) If $m_1(s,t) := s$ and μ corresponds to (m_1, P), then
$$\langle x|y\rangle_\mu = \langle x|y\rangle^\flat = \varphi(xy^*).$$

b) If $m_2(s,t) := t$ and μ corresponds to (m_2, P), then
$$\langle x|y\rangle_\mu = \langle x|y\rangle^\sharp = \varphi(y^*x).$$

Example 4.4.15. Let A be an atomic JBW^*-triple and let $\varphi = \sum_j s_j f_j$ be a normal functional on A given by an orthogonal family f_j of extreme points of the unit ball of A_* and scalars $s_j \geq 0$ with $\sum s_j = \|\varphi\| = 1$. Let v_j denote the support tripotent of f_j and $e = \sum v_j$. Define the projection valued function $P(s,t)$ by the joint Peirce decomposition relative to the family $\{v_j\}$ as follows
$$P(s,t) = \begin{cases} P_2(v_j) & \text{if } s = t = s_j \\ P_1(v_i)P_1(v_j) & \text{if } s = s_i \neq s_j = t \\ P_1(e)P_1(v_j) & \text{if } s = 0,\ t = s_j \\ P_0(e) & \text{if } s = t = 0. \end{cases}$$
Then we have three examples corresponding to the choice of m.

a) If $m(s,t) = (s+t)/2$ is the *arithmetic mean* of s and t, then
$$\langle x|y\rangle_\mu = \varphi(\{xye\}) = a_\varphi(x,y)$$
(where a_φ is defined as in Proposition 4.4.2).

b) If $m(s,t) = \sqrt{st}$ is the *geometric mean* of s and t, then
$$\langle x|y\rangle_\mu = s_\varphi(x,y)$$
is the self-polar form of φ.

c) If $m(s,t) = 2(s^{-1} + t^{-1})^{-1}$ is the *harmonic mean* of s and t, then
$$\langle x|y\rangle_\mu = h_\varphi(x,y)$$
(where h_φ is the sesquilinear form h defined in Theorem 4.4.11).

In Example 4.4.15, three widely used means have appeared (geometric, arithmetic, harmonic). Two of them correspond to sesquilinear forms which can be described intrinsically (a_φ by the algebraic structure, s_φ by the order structure).

Problem 6. Can h_φ be described and constructed by some intrinsic properties?

A positive constructive answer would lead to a constructive method of obtaining the self-polar form.

We now indicate how to define a large family of Schur multipliers on an arbitrary JBW^*-triple A from a given functional $\varphi \in A_*$. We shall simply apply the functional calculus for sesquilinear forms ([94, Theorem 1.2]) to a_φ and s_φ. That is, for each f in the class J defined in [94], we obtain, by [94, Theorem 1.2], a sesquilinear form $b_f = f(a_\varphi, s_\varphi)$. We can then determine the conjugate linear map $\tilde{\mu} : A \to A_*$ by the rule $\langle x, \tilde{\mu} y \rangle = b_f(x, y)$. The class J consists of all Borel measurable functions on $[0, \infty) \times [0, \infty)$ which are homogeneous ($f(\lambda r, \lambda s) = \lambda f(r, s), \quad \lambda, r, s \in [0, \infty)$) and bounded on compact sets.

This definition is consistent with Definition 4.4.13.

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