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Applications of Factorization in the Hardy Spaces of the Polydisk

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1 INTRODUCTION AND PRELIMINARIES

1.1 Introduction

It is well known that factorization of a function in the Hardy space H^1 in the unit disc into a product of two H^2 functions fails in higher dimensions, but that for the unit ball in $\mathbf{C}^n$ a weak factorization exists and has many applications (Theorem of Coifman, Rochberg, and Weiss). The purpose of this paper is to give the corresponding applications of an analogous factorization theorem for the Hardy spaces on the polydisk (which is due to the first named author).

Classical Hankel operators

A Hankel matrix (finite or infinite) is a matrix of the form

$$(a_{k+n}) = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ a_4 & a_5 & a_6 & a_7 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$(k = 1, 2, ..., n = 0, 1, 2, ...)$. The matrix elements are constant on lines (diagonals) perpendicular to the main diagonal. A famous example is the Hilbert matrix

$$(\frac{1}{k+n}).$$

Hilbert, in his lectures on integral equations (1906), showed that this matrix is bounded in l^2; this result was first published by Weyl in his thesis (1908), and the exact norm was determined by Schur (1911) who showed it is π.

Thus the Hilbert matrix represents a bounded operator of norm π on l^2. This can be restated as the well known and famous Hilbert inequality, see Hardy-Littlewood-Polya (1934):

$$\sum_{m,n\geq 0} \frac{a_n a_m}{m+n+1} \leq \pi \sum a_n^2.$$

For purposes of generalization, it is convenient to realize Hankel matrices as linear operators acting on suitable function spaces. For example, let T denote the unit circle in $\mathbf{C}$, let $P = P_+$ be the Riesz projection, that is, the orthogonal projection of $L^2(T)$ onto the Hardy Space H^2:

$$P_+(\sum_{-\infty}^{\infty} \hat{\phi}(n) z^n) = \sum_{0}^{\infty} \hat{\phi}(n) z^n,$$

and let $P_- = I - P_+$. For a function f on T, define the Hankel operator with symbol f to be $H_f = P_- M_f$ where M_f is the operator of multiplication by f.

Hankel operators are intimately related to Toeplitz operators. A Toeplitz matrix is a matrix of the form

$$(a_{m-n}),$$

for $m, n = 0, 1, 2, \ldots$. We see that the matrix entries are constant along lines parallel to the main diagonal. An important example is

$$(\frac{1}{m-n}),$$

with the convention $\frac{1}{0} = 0$. For a function f on T, define the Toeplitz operator with symbol f to be $T_f = PM_f$, where M_f is multiplication by f.

Now the unit circle T is the Šilov boundary of the uniform algebra of all continuous functions on the closed unit disk $\overline{\triangle}$ which are analytic on the open unit disk $\triangle$; and it is the topological boundary of $\triangle$. This suggests that a fruitful direction of generalization lies in the realm of several complex variables. One can consider first the case where the "base space" $\triangle$ is replaced by the unit ball $B = B_n$ in $\mathbf{C}^n$. After that, one may ask what happens if one passes to general pseudoconvex domains, to the polydisk, to Siegel domains, etc.

In the case of the unit ball, the most natural candidate for a substitute for P seems to be the (Cauchy-) Szegö projection, that is, the orthogonal projection S of $L^2(\partial B)$ onto the Hardy space $H^2(\partial B)$ (∂B is the unit sphere in $\mathbf{C}^n$ and the Hardy spaces are defined below). Then Toeplitz and Hankel operators could be defined respectively by

$$T_f = SM_f \text{ and } H_f = (I - S)M_f.$$

This definition of Toeplitz operator is as good as it can be. For technical reasons, a Hankel operator with symbol f will be defined to be the conjugate linear map

$$K_f \phi = S(f\overline{\phi}).$$

We return to the case of the unit disc. It is natural to ask when a given Hankel operator H_f, intially defined on holomorphic polynomials, has an extension as a bounded operator from H^2 to $H^{2\perp}$,where $H^{2\perp} = \overline{zH^2}$ = the orthogonal complement of H^2. The answer is the classical theorem of Nehari, see Nehari (1957).

THEOREM 1.1 (Nehari 1957) H_f can be extended to a bounded operator from H^2 to $H^{2\perp}$ if and only if $f \in L^\infty$. Moreover,

$$\|H_f\| = \|f\|_{L^\infty/H^\infty} = \text{dist}\,(f, H^\infty).$$

The next question is: When is a Hankel operator compact? The answer is the classical theorem of Hartman, see Hartman (1958). Let C denote the space of continuous functions on T.

THEOREM 1.2 (Hartman 1958) H_f is compact operator from H^2 to $H^{2\perp}$ if and only if $f \in C + H^\infty$.

Note that, since the symbol of the Hilbert matrix is $f(\theta) = i(\theta - \pi)$, we obtain an elegant proof of Hilbert's inequality as a consequence of Nehari's theorem:

$$\|H_f\| = \|f + H^\infty\| \leq \|f\|_\infty = \pi.$$

The factor space L^∞/H^∞ can be identified with the space $\overline{zBMOA}$ ($f \in BMOA \leftrightarrow f \in BMO$ and $\hat{f}(n) = 0$ if $n < 0$). Thus the theorem of Nehari can also be expressed by saying that "H_f is a bounded operator from H^2 to $H^{2\perp}$ if and only if $P_- f \in \overline{zBMOA}$" (see below for the definition of BMO).

To prove this identification, recall first that $L^\infty \subset BMO$ and by a deep theorem of C. Fefferman, $P(L^\infty) = BMOA$. Hence $P_- f = f - Pf \in BMO$. On the other hand, for $f = \sum_{-\infty}^{\infty} a_n z^n$,

$$P_- f = f - Pf = \sum_{1}^{\infty} a_{-n}\bar{z}^n = \overline{g(z)},$$

where g is analytic and $g(0) = 0$. Thus $P_- f \in \overline{zBMOA}$.

Similarly, the factor space $(C + H^\infty)/H^\infty$ can be identified with the space $\overline{zVMOA}$ ($f \in VMOA \leftrightarrow f \in VMO$ and $\hat{f}(n) = 0$ if $n < 0$). Thus the theorem of Hartman admits the following formulation: "H_f is compact as an operator from H^2 to $H^{2\perp}$ if and only if $P_- f \in \overline{zVMOA}$".

As a consequence of a new factorization theorem in H^1, see Lin (1994a, 1994b), in this note, we shall extend Theorems 1.1 and 1.2 to the bidisc in $\mathbf{C}^2$.

For more information along these lines, see the interesting survey article of Peetre, Peetre (1983), from which the above was borrowed.

Harmonic analysis on the unit ball of $\mathbf{C}^n$

For comparison to our work on the unit bidisc, we present here the known results on the unit ball.

Let $B = B_n = \{z = (z_1, \ldots, z_n) : z_j \in \mathbf{C}, \sum |z_j|^2 < 1\}$ be the unit ball in complex n-space. Denote Lebesgue area measure on ∂B by σ. For $z \in B$, let $z' = z/|z|$. The Hardy space $H^p(\partial B)$, $1 \leq p < \infty$ is defined to be the space of functions F which are holomorphic on B and for which

$$\|F\|_p = \sup_{r<1} \left(\int_{\partial B} |F(rz')|^p \, d\sigma(z') \right)^{1/p} < \infty.$$

For $z, \zeta \in \partial B$, let $d(z,\zeta) = |1 - \overline{\zeta} \cdot z|^{1/2}$ and let $S_{\zeta,r}$ denote the sphere on ∂B of radius r in the d-metric. An *atom* is a function $a(\zeta)$, defined on ∂B, which is either identically 1 or is supported on a sphere S and satisfies

$$|a(\zeta)| \le \frac{1}{|S|}, \qquad \int_{\partial B} a\, d\sigma = 0.$$

A *holomorphic atom* is a function of the form $A = P(a)$, where a is an atom and P temporarily denotes the projection of $L^2(\partial B)$ onto $H^2(\partial B)$. More precisely,

$$A(z) = c \int_{\partial B} \frac{a(\zeta)}{(1 - \overline{\zeta} \cdot z)^n}\, d\sigma(\zeta) \qquad (|z| < 1).$$

The space BMO consists of all functions b on ∂B such that

$$\|b\|_{BMO} = \sup_{x \in S} \int_S |b(y) - m_S(b)|\, d\sigma < \infty,$$

where the supremum is taken over all spheres S, $|S|$ denotes the σ-measure of S, and

$$m_S(b) = \frac{1}{|S|} \int_S b.$$

Finally, we let $BMOA$ be the subspace of BMO consisting of those L^2-functions which have holomorphic extensions to B, that is $BMOA = H^2 \cap BMO$.

With all these definitions, we can now state a basic and standard theorem in the theory, see Coifman-Rochberg-Weiss (1976), Coifman (1974), and Fefferman-Stein (1972).

THEOREM 1.3 The following statements are true:

(a) Every $F \in H^1(\partial B)$ can be written as $F = \sum_1^\infty \lambda_i A_i$, where the A_i are holomorphic atoms and λ_i are complex scalars with $\sum |\lambda_i| \le c\|F\|_1$.

(b) The dual space of $H^1(\partial B)$ is $BMOA$.

(c) P is a continuous map of $L^\infty(\partial B)$ onto $BMOA$.

To show the importance of the atomic decomposition (part (a) of Theorem 1.3), we use it to prove that $BMOA \subset (H^1)^*$. Let $\phi \in BMOA$ and let $F = \sum \lambda_i a_i \in H^1$. Then

$$\begin{aligned}
\left| \int F\phi \right| &= \left| \sum \lambda_i \int_{S_i} a_i \phi \right| \\
&= \left| \sum \lambda_i \int_{S_i} a_i(\phi - m_{S_i}(\phi)) \right| \\
&\le \sum |\lambda_i| \frac{1}{|S_i|} \int_{S_i} |\phi - m_{S_i}(\phi)| \\
&\le \left(\sum |\lambda_i| \right) \|\phi\|_{BMO} \le c\|\phi\|_{BMO} \|F\|_{H^1}.
\end{aligned}$$

It is well known and easy to prove that on the unit disk, every H^1-function is the product of two H^2-functions. For the unit ball the following is the analogous factorization theorem, see Coifman-Rochberg-Weiss (1976).

THEOREM 1.4 Given $F \in H^1(\partial B)$, there are $G_i, H_i \in H^2(\partial B)$ such that

$$F = \sum_1^\infty G_i H_i$$

and

$$\sum_1^\infty \|G_i\|_2 \|H_i\|_2 \leq c\|F\|_1.$$

This theorem is proved by combining part (a) of Theorem 1.3 with the following proposition, as in Coifman-Rochberg-Weiss (1976).

PROPOSITION 1.5 Every holomorphic atom A can be written as

$$A = \sum_1^N B_i C_i,$$

with $B_i, C_i \in H^2(\partial B)$ and $\sum \|B_i\|_2 \|C_i\|_2 \leq C$, where C and N depend only on the dimension n.

Let $b \in H^2(\partial B)$ and let K_b be defined in $H^2(\partial B)$ formally by $K_b(f) = P(b\overline{f})$. As noted above, such operators will be called Hankel operators.

The next two theorems are generalizations of the theorems of Nehari and Hartman to the unit ball in $\mathbf{C}^n$, see Coifman-Rochberg-Weiss (1976).

THEOREM 1.6 For $b \in H^2(\partial B)$, the following are equivalent:

(a) K_b is a bounded map from $H^2(\partial B)$ into $H^2(\partial B)$.

(b) $b = P(F)$ for some $F \in L^\infty(\partial B)$

(c) $b \in BMOA$

For $b \in BMO$ define

$$M_r(b) = \sup_{|S| \leq r} \frac{1}{|S|} \int_S |b(z) - m_S(b)|\, d\sigma(z).$$

Here, the S are spheres on ∂B and $|S|$ is the measure of S. We say that b belongs to the space VMO (vanishing mean oscillation) if $\lim_{r \to 0} M_r(b) = 0$.

THEOREM 1.7 For $b \in H^2(\partial B)$, the following are equivalent:

(a) K_b is a compact map from $H^2(\partial B)$ into $H^2(\partial B)$.

(b) $b = P(F)$ for some $F \in C(\partial B)$

(c) b is an analytic function in VMO

Here is a related duality theorem, see Coifman-Weiss (1977).

THEOREM 1.8 Let $VMOA = H^2 \cap VMO$. Then the dual of $VMOA$ is $H^1(\partial B)$.

The material in this subsection is taken primarily from Coifman-Rochberg-Weiss (1976). As noted above, in this note, we shall obtain analogs of Theorems 1.6 and 1.7 in the context of the bidisc, as well as for Theorem 1.8.

1.2 Preliminaries on Hardy spaces

Hardy spaces of the bidisc

The following notation will be used: $\mathbf{R}$ denotes the real numbers; $\mathbf{C}$ denotes the complex numbers; $\triangle$ denotes the open unit disk, T denotes the unit circle.

We shall have occasion in this paper to use some standard concepts from the classical function theory of several complex variables, such as (Cauchy-) Szegö kernel, (Cauchy-) Szegö projection (in polydisc),.... A standard reference is the book by Krantz, see Krantz (1992).

The *Hardy spaces* H^p, for $0 < p < \infty$ are defined by

$$H^p(\triangle^n) = \{f \text{ is holomorphic in } \triangle^n : \|f\|_{H^p} < \infty\},$$

where

$$\|f\|_{H^p} \equiv \sup_{0<r<1} \left[\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} |f(r_1e^{i\theta_1},\cdots,r_ne^{i\theta_n})|^p d\theta\right]^{1/p}$$

and $1 > r = (r_1, r_2, ...r_n) > 0$ means $1 > r_j > 0$ for $j = 1, 2, ..., n$.

For $p = \infty$,

$$H^\infty(\triangle^n) = \left\{f \text{ is holomorphic in } \triangle^n : \sup_{\triangle^n} |f| \equiv \|f\|_{H^\infty} < \infty\right\}.$$

A fundamental result in the theory of Hardy spaces is that if $f \in H^p(\triangle^n)$, $0 < p \le \infty$, then the limit

$$\lim_{r\to 1^-} f(r_1e^{i\theta_1},\cdots,r_ne^{i\theta_n}) \equiv \tilde{f}(e^{i\theta_1},\cdots,e^{i\theta_n})$$

exists for a.e., $\theta_j \in [0, 2\pi)$, for $j = 1, 2, ..., n$, where $r = (r_1,\cdots,r_n) \to 1^-$ means $r_j \to 1^-, j = 1, 2, ..., n.$. For $1 \le p \le \infty$, the function f can be recovered from $\tilde{f}$ by way of the Cauchy or Poisson integral formulas.

Although we have stated the definition for the polydisk, we also need to consider Hardy spaces in the bi-upper half plane $\mathbf{R}^2_+ \times \mathbf{R}^2_+$. Hardy spaces can be defined in any reasonable domain in $\mathbf{C}^n$, see Chapter 8 of Krantz (1992).

Discussion of the real variable Hardy spaces

The theory of Hardy spaces that we are using will consist of holomorphic functions on $\triangle^2$ or, alternatively, their boundary values, which are measurable functions on T^2. There is a parallel theory of Hardy spaces, which is described entirely by real variable methods. Although we do not make use of this theory explicitly, we summarize it here because of its importance in the development of the subject.

For the *Hardy spaces* $H^p(\mathbf{R}^{n+1}_+)$, we consider first the case $n = 1$. For $0 < p < \infty$, an H^p-function is a complex analytic function $F(z)$ in the upper half-plane $\mathbf{R}^2_+$, such that the L^p-norms

$$\left(\int_{-\infty}^{\infty} |F(x+iy)|^p\, dx\right)^{1/p}$$

are bounded independent of $y > 0$.

There is an extension of the theory of Hardy spaces, due to E.M.Stein and Guido Weiss (Stein-Weiss (1960)). We denote by $\mathbf{R}^{n+1}_+$ the upper-space in $\mathbf{R}^{n+1}$, that is, $\{(x, y) \in \mathbf{R}^{n+1}$:

$x \in \mathbf{R}^n, y > 0\}$. Whereas Hardy spaces in $\mathbf{R}^2_+$ are just analytic functions, or pairs of conjugate harmonic functions, Stein and Weiss consider $H^p(\mathbf{R}^{n+1}_+)$ functions as a system of $n+1$ harmonic functions, $F(x,y) = \{u_i(x,y)\}, i = 0,1,2,...,n$, defined on $\mathbf{R}^{n+1}_+$, which are conjugate in the sense that they satisfy the generalized Cauchy-Riemann equations:

$$\frac{\partial u_i}{\partial x_j} \equiv \frac{\partial u_j}{\partial x_i} \text{ and } \sum_{i=0}^{n} \frac{\partial u_i}{\partial x_i} \equiv 0 \ (y = x_0)$$

and such that

$$\sup_{y>0} \left(\int_{\mathbf{R}^n} |F(x,y)|^p \right)^{1/p} < \infty$$

Here

$$|F(x,y)| = \left(\sum_{i=0}^{n} |u_i(x,y)|^2 \right)^{1/2}.$$

This theory is explained in the book Stein (1970), and in the fundamental paper Fefferman-Stein (1972), the latter containing the result that BMO is the dual of H^1.

1.3 Harmonic Analysis on the bi-upper half plane

The purpose of this subsection is to discuss the analog, for the bi-upper half plane, of Theorem 1.3, that is, the atomic decomposition for H^1 and the duality of that space with BMO, which is due to Chang and R. Fefferman.

Atomic decomposition for H^1

Recall that, in one variable (Coifman (1974)), if $f \in H^1(\mathbf{R}^1)$ then $f(x)$ can be written as

$$f(x) = \sum \lambda_k a_k(x)$$

where $\sum |\lambda_k| \le C\|f\|_{H^1}$ and $a_k(x)$ are particularly simple functions called "atoms".

An analogous decomposition holds for functions f defined on $\mathbf{R}^2$ which are boundary values of functions in $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, where $\mathbf{R}^2_+$ is the upper-half plane. Such an f is written as $\sum \lambda_k a_k(x,y)$ where again $\sum |\lambda_k| \le C\|f\|_{H^1}$ and a_k is an atom. But in the product case, an atom is supported not in a rectangle, as one might expect, but in an open set $\Omega \subseteq \mathbf{R}^2$ such that $\|a\|_{L^2} \le 1/|\Omega|^{1/2}$, and other conditions are satisfied, see below.

In what follows, we shall deal exclusively with the domain $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ and its Šilov boundary $\mathbf{R}^2$. A point of $\mathbf{R}^2_+ \times \mathbf{R}^2_+$ will be denoted by $z = (z_1, z_2)$ where $z_j = x_j + iy_j$ and $x_j \in \mathbf{R}$, $y_j > 0$, $j = 1,2$.

Now, we shall introduce some notation: let $\psi \in C^1(\mathbf{R})$ be supported on $[-1,1]$ with ψ even and $\int_{-1}^{1} \psi(x)dx = 0$; if $y > 0, \psi_y(x) = (1/y)\psi(x/y)$ and if $y = (y_1, y_2)$ and $x = (x_1, x_2) \in \mathbf{R}^2$ then $\psi_y(x) = \psi_{y_1}(x_1)\psi_{y_2}(x_2)$. If f is a function defined on $\mathbf{R}$ then we define $f(x,y) := f * \psi_y(x)$; if $x = (x_1, x_2) \in \mathbf{R}^2$, we denote $\Gamma(x) := \Gamma(x_1) \times \Gamma(x_2)$, where $\Gamma(x_j) := \{(t_j, y_j) \in \mathbf{R}^2 : |x_j - t_j| < y_j\}$, $j = 1,2$.

Given a function f on $\mathbf{R}^2$ we define its double Square function by

$$Q^2(f)(x) := \int\int_{\Gamma(x)} |f(t,y)|^2 \frac{dtdy}{y_1^2 y_2^2}.$$

There are alternative definitions of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$. Although we consider this at first as the set of boundary value functions on $\mathbf{R}^2$ of biholomorphic functions on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$, the work of Gundy-Stein shows that the various definitions via area integrals and nontangential maximal functions are equivalent. For example, we can define $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ as the class of functions on $\mathbf{R}^2$ for which $A(f) \in L^1(\mathbf{R}^2)$ where

$$A^2(f)(x) := \int\int_{\Gamma(x)} |\nabla_1 \nabla_2 u(t,y)|^2 dtdy,$$

and u is the multiple Poisson integral of f.

Also, if $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$, then $Q(f) \in L^1(\mathbf{R}^2)$ (see Gundy-Stein (1978) and Fefferman-Stein (1972)). On the polydisc Chang and Fefferman (see Chang-Fefferman (1980)) make the following definition.

DEFINITION 1.9 An *atom* is a function $a(x), x = (x_1, x_2) \in \mathbf{R}^2$ defined on $\mathbf{R}^2$ and supported in some open set Ω of finite measure, such that

1. $\|a\|_{L^2} \le |\Omega|^{-1/2}$
2. $\int_{I_j} a(x)dx_j = 0,\ j = 1, 2$, where I_j is any component interval of a set of the form $\{x_j \in \mathbf{R}^1 : x = (x_1, x_2) \in \Omega\}$ (Where x_k is fixed $k = 1, 2,\ k \neq j$), i.e., a has mean 0 over every component interval of every x_j–cross section of Ω.
3. a can be further decomposed into "elementary particles" a_R as follows:

   (a) $a = \sum_R a_R$, where a_R is supported in a rectangle $R \subseteq \Omega$ (say $R = I_1 \times I_2$) and the R in the sum have the property that no one R is contained in the triple of any other. (When convenient, we shall write $a = \sum_{R \in \mathcal{R}_a} a_R$. Note that the condition on the rectangles implies $\sum_{R \in \mathcal{R}_a} |R| \le 2|\Omega|$.)

   (b) $\int_{I_j} a_R(x)dx_j = 0,\ j = 1, 2$.

   (c) a_R is C^1 with

$$\|a_R\|_\infty \le c_R/|R|^{1/2}.$$
$$\|\partial a_R/\partial x_j\|_\infty \le c_R/|I_j||R|^{1/2}, j = 1, 2$$

   and $\sum_R c_R^2 \le A/|\Omega|$, where A is an absolute constant.

With this definition, Chang and Fefferman have shown following result:

THEOREM 1.10 Let $f \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$. Then (identifying f with its boundary values) f can be written as $f = \sum \lambda_k a_k$ where a_k are atoms and $\lambda_k \ge 0$ satisfy

$$\sum \lambda_k \le A\|f\|_{H^1}.$$

The Chang-Fefferman duality for the bi-upper half plane

Recall that, if ϕ is a locally integrable function on $\mathbf{R}$, it is said to be of *bounded mean oscillation* (abbreviated as $BMO(\mathbf{R})$) if

$$\|\phi\|_*^2 := \sup_I \frac{1}{|I|} \int_I |\phi - \phi_I|^2 dx < \infty,$$

where the supremum ranges over all finite intervals I in $\mathbf{R}$, and $\phi_I = \frac{1}{|I|}\int_I \phi(x)dx$. In this case, C.Fefferman and E.Stein proved in Fefferman-Stein (1972) that $BMO(\mathbf{R})$ is the dual space of $H^1(\mathbf{R})$. Chang and Fefferman gave two kinds of definitions for BMO functions defined on $\mathbf{R}^2$, and proved that both definitions characterize the dual space of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$.

We now state these two definitions and the Chang-Fefferman theorem.

DEFINITION 1.11 $BMO_{(a)}$ is the space of locally integrable functions ψ defined on $\mathbf{R}^2$, satisfying:

$$\|\psi\|_*^2 := \sup_{\Omega} \frac{1}{|\Omega|} \| \sum_{R \subset \Omega} \psi_R \|_2^2 < \infty,$$

where the supremum ranges over all open sets Ω of finite measure in $\mathbf{R}^2$, and for each dyadic rectangle R, ψ_R is defined with respect to ψ as

$$\psi_R(x) = \int\int_{(t,y) \in R_+} \psi(t,y) \psi_y(x-t) \frac{dt dy}{y_1 y_2},$$

where ψ_y is defined above, and if $R = I \times J$,

$$R_+ = \{(t,y) \in \mathbf{R}_+^2 \times \mathbf{R}_+^2 : t \in \mathbf{R}, |I|/2 < y_1 \leq |I|, |J|/2 < y_2 \leq |J|\}.$$

Recall that $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ is the disjoint union of all the R_+. The second definition of Chang and Fefferman is motivated by their work on atomic decomposition of H^1 described above.

DEFINITION 1.12 $BMO_{(b)}$ is the space of locally integrable functions ψ defined on $\mathbf{R}^2$ such that given any open set $\Omega \subset \mathbf{R}^2$, there exists a function $\tilde{\psi}_\Omega$ so that

$$\frac{1}{|\Omega|} \int_\Omega |\psi(t) - \tilde{\psi}_\Omega(t)|^2 dt \leq M,$$

for some M independent of Ω. The functions $\tilde{\psi}_\Omega$ satisfy the following three conditions.

1. $\tilde{\psi}_\Omega = \sum \tilde{\psi}_i$, where each $\tilde{\psi}_i$ is supported on the triple $\tilde{R}_i$ of distinct dyadic rectangles R_i with $|\tilde{R}_i \cap \Omega| < |\tilde{R}_i|/2$, and $\tilde{\psi}_i$ has mean value zero over each horizontal and vertical segment of $\tilde{R}_i$.

2. Furthermore, if $R_i = I_i \times J_i$,

$$\|\tilde{\psi}_i\|_\infty \leq \frac{C_{R_i}}{|R_i|^{1/2}}, \quad \|\frac{\partial \tilde{\psi}_i}{\partial x_1}\|_\infty \leq \frac{C_{R_i}}{|R_i|^{1/2}|I_i|},$$

$$\|\frac{\partial \tilde{\psi}_i}{\partial x_2}\|_\infty \leq \frac{C_{R_i}}{|R_i|^{1/2}|J_i|}, \text{and } \|\frac{\partial^2 \tilde{\psi}_i}{\partial x_1 \partial x_2}\|_\infty \leq \frac{C_{R_i}}{|R_i|^{3/2}}$$

   for some C_{R_i}.

3. $\sum_{|R_i' \cap \Omega| \sim (1/2^k)|R_i'|} C_{R_i}^2 \leq c 2^k k |\Omega|$ for each $k = 1, 2, \cdots$ and some absolute constant c.

In Chang-Fefferman (1980), the following theorem is proved.

THEOREM 1.13 Assume $\psi \in L^2(\mathbf{R}^2)$ satisfies

$$\int \psi(x_1, x_2) dx_1 = 0 = \int \psi(x_1, x_2) dx_2$$

for all $(x_1, x_2) \in \mathbf{R}^2$. Then the following conditions are equivalent:

1. $\psi \in BMO_{(a)}$;

2. $\psi \in BMO_{(b)}$.

3. $\sup 1/|\Omega| \sum_{R\subset\Omega} S_R^2(\psi) < \infty$, where the supremum ranges over all finite open sets Ω in $\mathbf{R}^2$, and for each dyadic rectangle R,

$$S_R^2(\psi) = \int\int_{R_+} |\psi(t,y)|^2 \frac{dtdy}{y_1 y_2};$$

4. ψ is in the dual of $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, that is, $|\int \psi f| \leq$ const $\|f\|_{H^1}$ for all $f \in H^1$.

Denote the space of functions defined by Theorem 1.13 by $BMO(\mathbf{R}^2)$ and denote a norm in this space by $\|g\|_*$. Each condition defines a norm, all equivalent.

REMARK 1.14 The integral in the fourth condition exists thanks to the argument on pages 192–193 of Chang-Fefferman (1980).

1.4 Factorization Theorem

We now state the factorization theorem of Lin (1994a,1994b), that we shall use in the next section. We state several versions of it. We shall use the version for $p = 1$ for the bidisc in our applications.

The bi-upper half plane, $p = 1$

THEOREM 1.15 Let a be an atom (see DEFINITION 1.9), $a = \sum_R a_R$, $A = S(a)$ and $A_R = S(a_R)$, where S denotes the Szegö projection. Then there exist $B_{j,R}, C_{j,R} \in H^2(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$ ($j = 1, 2, 3, 4$) such that

$$A = \sum_{R\in\mathcal{R}_a} \sum_1^4 B_{j,R} C_{j,R}$$

and $\sum_R \sum_1^4 \|B_{j,R}\|_{H^2} \|C_{j,R}\|_{H^2} \leq c$, where c is an absolute constant.

Theorem 1.10, 1.15 imply the following.

THEOREM 1.16 (Factorization Theorem for $\mathbf{R}_+^2 \times \mathbf{R}_+^2$) If $f \in H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, then there exist $g_j, h_j \in H^2(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, such that

$$f = \sum_{J=1}^{\infty} g_j h_j$$

in the sense of distributions, and $\sum \|g_j\|_{H^2} \|h_j\|_{H^2} \leq c\|f\|_{H^1}$.

The bidisc $(0 < p \leq 1)$

The following is proved first for the bi-upper half plane, and then transferred to the bidisc via the Cayley transform. See Lin (1994a,1994b) for details.

THEOREM 1.17 (Factorization Theorem for $\triangle^2$) If $f \in H^p(\triangle^2)$ for $0 < p \leq 1$ then there exist $g_j, h_j \in H^{2p}(\triangle^2)$, such that

$$f = \sum_{J=1}^{\infty} g_j h_j$$

and $\sum \|g_j\|_{H^{2p}} \|h_j\|_{H^{2p}} \leq c\|f\|_{H^p}$

2 DUALITY OF HARDY SPACES AND APPLICATIONS

In this section we begin by transferring the duality theorem of Chang and Fefferman from the bi-upper half plane to the bidisc. Then we use this duality, together with the factorization theorem to prove the analogue of Nehari's theorem for Hankel operators on the Hardy spaces of the bidisc. We also consider duality of VMO with H^1 and compactness of Hankel operators (Hartman's theorem).

2.1 Duality of BMO and H^1

In this subsection, as an application of the Chang-Fefferman result, we shall prove that the dual of $H^1(\triangle^2)$ is $BMO(T^2)$. Chang and Fefferman have shown that the dual of $H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ is $BMO(\mathbf{R}^2)$. We now use the Cayley transform to transfer this result to the bidisc.

DEFINITION 2.1 The *Cayley Transform* from $\triangle^n$ onto $(\mathbf{R}^2_+)^n$ is defined by

$$\tau : \triangle^n \to (\mathbf{R}^2_+)^n, \quad \tau \ = \ (\tau_1, \tau_2, ..., \tau_n),$$

where

$$\tau_j : \triangle \to \mathbf{R}^2_+, \quad \tau_j(w_j) \ = \ \frac{i(1-w_j)}{1+w_j}, \ j = 1, 2, ..., n.$$

If $n = 1$ we write for short that:

$$\tau : \triangle \to \mathbf{R}^2_+, \text{ where } \tau(w) \ = \ \frac{i(1-w)}{1+w},$$

and we note that $\tau^{-1}(z) = (i-z)/(i+z)$. The single variable version of the next two lemmas come from pages 51–52 of Garnett (1981).

LEMMA 2.2 For $0 < p < \infty$, if $f \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$ then $g = f \circ \tau \in H^p(\triangle^2)$ and $\|g\|_{H^p(\triangle^2)} \leq \pi^{-2}\|f\|_{H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)}$.
Proof: Fubini's theorem implies

$$\begin{aligned}
\int_{T^2} |(f\circ\tau)(e^{i\theta_1}, e^{i\theta_2})|^p d\theta_1 d\theta_2 \ &= \ \int_T \int_T |f(\tau_1(e^{i\theta_1}), \tau_2(e^{i\theta_2}))|^p d\theta_2 d\theta_1 \\
&= \ \int_T \|f(\tau_1(e^{i\theta_1}), \tau_2(\cdot))\|^p_{H^p(\triangle)} d\theta_1 \\
&\leq \ (1/\pi)^p \int_T \|f(\tau_1(e^{i\theta_1}), \cdot)\|^p_{H^p(\mathbf{R}^2_+)} d\theta_1 \\
&= \ (1/\pi)^p \int_T (\int_{\mathbf{R}} |f(\tau_1(e^{i\theta_1}), x_2)|^p dx_2) d\theta_1 \\
&= \ (1/\pi)^p \int_{\mathbf{R}} \int_T |f(\tau_1(e^{i\theta_1}), x_2)|^p d\theta_1 dx_2 \\
&= \ (1/\pi)^p \int_{\mathbf{R}} \|f(\tau_1(\cdot), x_2)\|^p_{H^p(\triangle)} dx_2 \\
&\leq \ (1/\pi)^{2p} \int_{\mathbf{R}} \|f(\cdot, x_2)\|^p_{H^p(\mathbf{R}^2_+)} dx_2 \\
&= \ (1/\pi)^{2p} \int_{\mathbf{R}} \int_{\mathbf{R}} |f(x_1, x_2)|^p dx_1 dx_2
\end{aligned}$$

LEMMA 2.3 For $0 < p < \infty$, if $g \in H^p(\triangle^2)$ then

$$F(z) = (\pi(z_1+i)^2)^{-1/p}(\pi(z_2+i)^2)^{-1/p} g(\tau^{-1}(z)) \in H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$$

and $\|g\|_{H^p(\triangle^2)} = \|F\|_{H^p(\mathbf{R}^2_+ \times \mathbf{R}^2_+)}$.

Proof: We use the fact that if you fix one variable in an H^p-function, the resulting function of the other variables is also an H^p-function.

Identifying functions with their boundary values, by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbf{R}^2} |F(x)|^p dx &= \int_{\mathbf{R}} (\pi(x_2+1)^2)^{-1} \int_{\mathbf{R}} |(\pi(x_1+1)^2)^{-1/p} g(\tau_1^{-1}(x_1), \tau_2^{-1}(x_2)|^p dx_1 dx_2 \\
&= \int_{\mathbf{R}} (\pi(x_2+1)^2)^{-1} \int_T |g(w_1, \tau_2^{-1}(x_2)|^p dw_1 dx_2 \\
&= \int_T \int_{\mathbf{R}} (\pi(x_2+1)^2)^{-1/p} g(w_1, \tau_2^{-1}(x_2))|^p dx_2 dw_1 \\
&= \int_T \int_T |g(w_1, w_2)|^p dw_2 dw_1 \\
&= \int_{T^2} |g(w)|^p dw.
\end{aligned}$$

DEFINITION 2.4 We define $BMO(T^2)$ to be the space of integrable functions g defined on T^2 such that $g \circ \tau^{-1} \in BMO(\mathbf{R}^2)$, with norm

$$\|g\|_{BMO(T^2)} := \|g \circ \tau^{-1}\|_{BMO(\mathbf{R}^2)},$$

where $\tau = (\tau_1, \tau_2)$ is the Cayley transform from $\triangle^2$ onto $\mathbf{R}^2_+ \times \mathbf{R}^2_+$.

For notational simplicity, we shall denote the norms in $BMO(T^2)$ and $BMO(\mathbf{R}^2)$ by $\|\cdot\|_*$.

LEMMA 2.5 Let g be analytic on $\mathbf{R}^2_+ \times \mathbf{R}^2_+$. Then $g \in H^1(\mathbf{R}^2)$ if and only if

$$G(w) = (\prod_{j=1}^{2}(1+w_j)^2)^{-1} g(\tau(w)) \in H^1(T^2).$$

Moreover, $4\|G\|_{H^1(T^2)} = \|g\|_{H^1(\mathbf{R}^2)}$.

Proof:

$$\begin{aligned}
\int_{T^2} |G(w)|\, dw &= \int_{T^2} |g(\tau(w))[(1+w_1)(1+w_2)]^{-2}|\, dw \\
&= \frac{1}{4} \int_{\mathbf{R}^2} |g(z)|\, dz.
\end{aligned}$$

THEOREM 2.6 $BMO(T^2)$ is the dual of $H^1(\triangle^2)$.

Proof: Let $g \in BMO(T^2)$. We shall prove that

$$\left| \int gf \right| \le \|g\|_* \|f\|_{H^1}, \quad f \in H^1(T^2).$$

By definition, $g \circ \tau^{-1} \in BMO(\mathbf{R}^2)$. For any $f \in H^1(\triangle^2)$ we have from Lemma 2.3 that

$$F(z) = f(\tau^{-1}(z))[\pi(z_1 + i)(z_2 + i)]^{-2} \in H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)$$

and $\|F\|_{H^1(\mathbf{R}^2)} = \|f\|_{H^1(T^2)}$.

Thus

$$\begin{aligned} \|g\|_* \|f\|_{H^1(\triangle^2)} &= \|g \circ \tau^{-1}\|_* \|F\|_{H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)} \\ &\geq \left| \int_{\mathbf{R}^2} g(\tau^{-1}(x)) f(\tau^{-1}(x)) [\pi(x_1 + i)(x_2 + i)]^{-2} dx \right| \\ &= c \left| \int_{\mathbf{R}} (x_2 + i)^{-2} \int_T g(w_1, x_2) f(w_1, x_2) \frac{(1 + w_1)^2(-2i)}{(2i)^2(1 + w_1)^2} dw_1 dx_2 \right| \\ &= c \left| \int_{\mathbf{R}} (x_2 + i)^{-2} \int_T g(w_1, x_2) f(w_1, x_2) dw_1 dx_2 \right| \\ &= c \left| \int_T \int_{\mathbf{R}} (x_2 + i)^{-2} g(w_1, x_2) f(w_1, x_2) dx_2 dw_1 \right| \\ &= c \left| \int_T \int_T g(w_1, w_2) f(w_1, w_2) dw_2 dw_1 \right| \\ &= c \left| \int_{T^2} g(w) f(w) dw \right|. \end{aligned}$$

This proves that $g \in (H^1(\triangle^2))^*$.

To prove the converse, note first that by the Hahn-Banach and Riesz representation theorems, every bounded functional on $(H^1(\triangle^2))$ is given by some bounded measurable function. So, let $g \in (H^1(\triangle^2))^*$. We have to prove that $g \in BMO(T^2)$, that is, $g \circ \tau^{-1} \in BMO(\mathbf{R}^2)$.

If $F \in H^1(\mathbf{R}^2)$, we know from Lemma 2.2 that $F \circ \tau \in H^1(\triangle^2)$, and

$$\|F \circ \tau\|_{H^1(\triangle^2)} \leq (1/\pi) \|f\|_{H^1(\mathbf{R}^2)}.$$

Moreover,

$$\left| \int_{\mathbf{R}^2} (g \circ \tau^{-1})(x) F(x) dx \right| = \left| \int_{T^2} g(w) F(\tau(w)) \frac{-4}{(1 + w_1)^2(1 + w_2)^2} dw \right|.$$

Put $f(w) = F(\tau(w)) \frac{-4}{(1+w_1)^2(1+w_2)^2}$. By Lemma 2.5, $f \in H^1(\triangle^2)$ and $\|f\|_{H^1(\triangle^2)} = 4\|F\|_{H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)}$. Hence we obtain

$$\begin{aligned} \left| \int_{\mathbf{R}^2} (g \circ \tau^{-1})(x) F(x) dx \right| &= \left| \int g f \right| \\ &\leq \|g\|_* \|f\|_{H^1(\triangle^2)} \\ &\leq \|g\|_* \|F\|_{H^1(\mathbf{R}^2_+ \times \mathbf{R}^2_+)}. \end{aligned}$$

Note that the integral $\int_{\mathbf{R}^2} (g \circ \tau^{-1})(x) F(x) dx$ exists by Remark 1.14.

This implies

$$g \circ \tau^{-1} \in (H^1(\mathbf{R}^2))^* = BMO(\mathbf{R}^2).$$

Thus $g \in BMO(T^2)$.

2.2 Boundedness of Hankel operators

Here we shall prove the analog of Nehari's theorem (Theorems 1.1 and 1.6) for the bidisc. Before defining the Hankel operator formally, we give a proposition.

PROPOSITION 2.7 For $g \in H^2$, the following two properties are equivalent.

1. $g \in BMO(T^2)$.

2. $g = S(f)$, for some $f \in L^\infty(T^2)$.

Proof: First we prove "(2) implies (1)".

Suppose $g \in H^2$ is such that $g = S(f)$ for some $f \in H^\infty$ and let $k \in H^2$ with $\|k\|_1 = 1$. Then

$$\langle g, k\rangle = \langle S(f), k\rangle = \langle f, S(k)\rangle = \langle f, k\rangle,$$

so we have

$$|\langle g, k\rangle| \leq \int_{T^2} |f\overline{k}| \leq \|f\|_\infty \|k\|_1 = \|f\|_\infty,$$

that is, $g \in (H^1)^*$. Hence, $g \in BMO(T^2)$ (and $\|g\|_* \leq \|f\|_\infty$).

Now we show that "(1) implies (2)".

Suppose $g \in BMO(T^2)$. Since H^2 is dense in H^1, l_g defined by

$$l_g(u) = \langle u, g\rangle = \int_{T^2} u\overline{g} \quad u \in H^2$$

extends to a bounded linear functional on H^1. By the Hahn-Banach Theorem , we can extend it further to a bounded linear functional l'_g on $L^1(T^2)$.

By the Riesz Representation Theorem, there exists f in $L^\infty(T^2)$ such that

$$l'_g(u) = \int_{T^2} \overline{f} u, \quad \forall u \in L^1(T^2).$$

For all $u \in H^2$,

$$\langle u, g\rangle = l_g(u) = l'_g(u) = \langle u, f\rangle = \langle S(u), f\rangle = \langle u, S(f)\rangle.$$

Since H^2 is dense in H^1, we see that

$$\langle u, g\rangle = \langle u, S(f)\rangle, \forall u \in H^1.$$

If $u \in L^2$ then

$$\langle u, g\rangle = \langle u, S(g)\rangle = \langle S(u), g\rangle = \langle S(u), S(f)\rangle = \langle u, S^2(f)\rangle = \langle u, S(f)\rangle.$$

Hence $g = S(f)$, because L^2 is dense in L^1.

DEFINITION 2.8 For fixed $f \in H^2(\triangle^2)$, the *Hankel operator* K_f on $H^2(\triangle^2)$ is the conjugate-linear densely defined operator given by

$$K_f(u) = S(f\overline{u}), \quad u \in H^2(\triangle^2).$$

where S is the Szegö projection.

Thus,

$$K_f(u)(z) = \int_{T^2} S(z,w) f(w)\overline{u(w)}\,dw.$$

THEOREM 2.9 For $g \in H^2$, the following are equivalent.

1. K_g maps H^2 continuously into H^2
2. $g \in BMO(T^2)$.
3. $g = S(f)$, for some $f \in L^\infty(T^2)$.

Moreover, $\|K_g\| \approx \|g\|_*$

Proof: Owing to Proposition 2.7, we only need to prove (1) is equivalent to (2).

Suppose $g \in BMO(T^2)$ and $k \in H^2$. Then for $f \in H^\infty(\subset H^2)$, we obtain

$$\langle K_g f, k\rangle = \langle S(g\overline{f}), k\rangle = \langle g\overline{f}, S(k)\rangle = \langle g\overline{f}, k\rangle = \langle g, fk\rangle.$$

Since $fk \in H^1$ and $\|fk\|_{H^1} \leq \|f\|_{H^2}\|k\|_{H^2}$, we have

$$|\langle K_g f, k\rangle| \leq \|g\|_* \|f\|_{H^2}\|k\|_{H^2} \quad \forall f \in H^\infty.$$

Since H^∞ is dense in H^2, $\|K_g\| \leq \|g\|_*$, thereby proving that K_g is continuous.

To prove the converse, suppose K_g is continuous.

For $f \in H^1$, by Theorem 1.17 there exists $F_j, G_j \in H^2$ such that

$$f = \sum_{j=1}^{\infty} F_j G_j,$$

and

$$\sum_{j=1}^{\infty} \|F_j\|_{H^2}\|G_j\|_{H^2} \leq c\|f\|_{H^1}.$$

for some absolute constant c.

Note that

$$\begin{aligned}\sum_{j=1}^{\infty}\langle K_g F_j, G_j\rangle &= \sum_{j=1}^{\infty}\langle S(g\overline{F_j}), G_j\rangle = \sum_{j=1}^{\infty}\langle g\overline{F_j}, S(G_j)\rangle \\ &= \sum_{j=1}^{\infty}\langle g\overline{F_j}, G_j\rangle = \sum_{j=1}^{\infty}\langle g, F_j G_j\rangle = \langle g, f\rangle.\end{aligned}$$

Since K_g is bounded, we obtain

$$|\langle g, f\rangle| \leq \|K_g\| \sum_{j=1}^{\infty} \|F_j\|_{H^2}\|G_j\|_{H^2} \leq c\|K_g\|\|f\|_{H^1}.$$

Thus $\|g\|_* \leq c\|K_g\|$.

2.3 Duality of VMO and H^1

A space VMO was introduced in the classical setting by Sarason (1975). It is somewhat different from ours; we adopt a modified version for which the analog of theorem 2.6 holds. By using the Cayley transform, we shall prove that the dual of $VMO(\triangle^2)$ is $H^1(\triangle^2)$.

DEFINITION 2.10 The space $VMO(T^2)$ is the subspace of $BMO(T^2)$ which is the $BMO(T^2)$-closure of the analytic polynomials.

To prove that

$$VMO(T^2)^* = H^1(\triangle^2),$$

we need the following lemmas.

LEMMA 2.11 The Szegö projection $S : L^2(T^2) \to H^2(\triangle^2)$ restricts to a bounded map of $L^\infty(T^2)$ into $BMO(T^2)$.
Proof: If $f \in L^\infty$, then $Sf \in H^2$, so to prove that $Sf \in BMO(T^2)$ we must show that it determines a bounded linear functional on H^1. To this end, note that for $g \in H^2$, $\langle S(f), g\rangle = \langle f, S(g)\rangle = \langle f, g\rangle$ so $|\langle S(f), g\rangle| \le \|g\|_{H^1}\|f\|_\infty$ and thus

$$\|Sf\|_* = \sup_{\|g\|_{H^1}\le 1, g\in H^2} |\langle S(f), g\rangle| \le \|f\|_\infty.$$

LEMMA 2.12 $S(L^\infty) = BMO$.
Proof: For $f \in BMO(T^2)$, there is a linear functional l_f on $H^1(\triangle^2)$, defined by

$$l_f(u) = \int fu, \text{ for all analytic polynomials } u \in H^2(\triangle^2).$$

Since $H^1(\triangle^2) \subset L^1(T^2)$, we have from Hahn-Banach Theorem that there exists a bounded linear functional l on $L^1(T^2)$ with $l|_{H^1} = l_f$. Then by Riesz Representation Theorem there exists $g \in L^\infty(T^2)$ such that l defined by

$$l(u) = \langle g, u\rangle = \int gu, \ u \in L^1(T^2)$$

is a bounded linear functional on $L^1(T^2)$ satisfying, for all analytic polynomials $u \in H^2(\triangle^2)$,

$$\int fu = l_f(u) = l(u) = \langle g, u\rangle = \langle g, S(u)\rangle = \langle S(g), u\rangle.$$

So $\langle f - S(g), u\rangle = 0, \forall$ analytic polynomials $u \in H^2(\triangle^2)$ and thus $f - S(g) = 0$, i.e., $f = S(g)$.

LEMMA 2.13 The Szegö projection $S : C(T^2) \to VMO(T^2)$ is bounded with dense range.
Proof: Since $VMO(T^2)$ is the $BMO(T^2)$-closure of the analytic polynomials and the space of polynomials in $z, \bar{z}$ is dense in $C(T^2)$, we get our result from Lemma 2.11.

The analog of the following theorem was proved by Coifman and Weiss (see Coifman-Weiss (1977)) when Ω is the unit ball in $\mathbf{C}^n$ and by Krantz and Li (Krantz-Li (1992)), if Ω is a smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$ or a smoothly bounded pseudoconvex domain of finite type in $\mathbf{C}^2$.

THEOREM 2.14

$$VMO(T^2)^* = H^1(\triangle^2).$$

Proof: Since $H^1(\triangle^2)^* = BMO(T^2)$, it follows by restriction that

$$H^1(\triangle^2) \subseteq VMO(T^2)^*.$$

For the converse, take $l \in VMO(T^2)^*$. Then we define a linear functional L on $C(T^2)$ as follows:

$$L(f) = l(S(f)),\ f \in C(T^2).$$

Lemma 2.13 implies that L is a bounded linear functional on $C(T^2)$. By another Riesz Representation Theorem, there is a finite complex Borel measure μ on T^2 such that

$$L(f) = \int_{T^2} f\, d\mu \qquad f \in C(T^2).$$

Thus $\forall f \in C(T^2) \bigcap H^2(\triangle^2)^\perp$ we have

$$\int_{T^2} f\, d\mu = l(S(f)) = l(0) = 0.$$

So, if we take $f = z^\alpha \overline{z}^\beta$, then $S(f) = 0$ if there exists $j \in \{1, 2\}$ such that $\alpha_j - \beta_j < 0$. Thus, by the F. and M. Riesz Theorem (p. 201 of Rudin (1962)) $d\mu = hdm$, for some conjugate holomorphic function h, where m is Lebesgue measure on T^2.

Now for $g \in C(T^2)$, we have

$$\begin{aligned} l(S(g)) &= \int_{T^2} gh\, dm = \int_{T^2} g\overline{\overline{h}}\, dm \\ &= \langle g, \overline{h}\rangle = \langle g, S(\overline{h})\rangle \\ &= \langle S(g), \overline{h}\rangle = \int_{T^2} S(g) h\, dm. \end{aligned}$$

By Lemma 2.13,

$$l(\varphi) = \int \varphi h \quad \forall \varphi \in VMO,$$

that is, $l = l_h$, and

$$VMO(T^2)^* \subset H^1(\triangle^2).$$

2.4 Compactness of Hankel operators

The Hartman theorem holds if the unit disk is replaced by the unit ball in $\mathbf{C}^n$ (see Theorem 1.7 of Coifman, Rochberg, and Weiss). In this section we show that Hartman's theorem holds on the bidisc.

THEOREM 2.15 For $g \in H^2(\triangle^2)$, the Hankel operator

$$K_g : H^2(\triangle^2) \to H^2(\triangle^2)$$

is compact if and only if $g \in VMO(T^2)$.

Proof: Suppose K_g is compact. We shall use the following two facts, which are easily verified:

- For $f \in H^2(\triangle^2)$, $0 < r < 1$, set $(T_r f)(z) = f(rz)$. The operator norm of T_r as a map from $H^2(\triangle^2)$ into $L^2(T^2)$ is one.
- For $f \in H^2(\triangle^2)$,

$$K_{T_r g} f = T_r K_g T_r f.$$

Using these two facts and the compactness of K_g, it follows that

$$\lim_{r \to 1} \|K_g - K_{T_r g}\| = \lim_{r \to 1} \|K_{g - T_r g}\| = 0.$$

By Theorem 2.9, we have $\lim_{r \to 1} \|g - T_r g\|_* = 0$. Since $T_r g$ is continuous, it belongs to $VMO(T^2)$. Hence the limit $g \in VMO(T^2)$.

Suppose $g \in VMO(T^2)$. By Lemma 2.13 there exists a sequence $\{f_n\}$ in $C(T^2)$ such that

$$S(f_n) \to g \text{ in } BMO.$$

Since the space of the polynomials in z and $\overline{z}$ is dense in $C(T^2)$, for each f_n there exists a sequence of such polynomials $\{p_m^{(n)}\}$ such that

$$p_m^{(n)} \to f_n \text{ uniformly on } T^2.$$

For $\epsilon > 0$, there exists a positive integer $N = N(\epsilon)$ such that

$$\|g - S(f_n)\|_* < \epsilon, \ \forall n \geq N,$$

thus

$$\|g - S(f_N)\|_* < \epsilon$$

and by Theorem 2.9

$$\|K_g - K_{S(f_N)}\| < c\epsilon. \tag{1}$$

For this N, there also exists an integer $M = M(\epsilon, N(\epsilon)) > 0$ such that

$$\|f_N - p_m^{(N)}\|_{C(T^2)} < \epsilon, \forall m \geq M. \tag{2}$$

From (2) and Lemma 2.11, we obtain

$$\|S(f_N) - S(p_m^{(N)})\|_* < c\epsilon, \text{ for some constant } c. \tag{3}$$

Equation (3) and Theorem 2.9 imply

$$\|K_{S(f_N)} - K_{S(p_m^{(N)})}\|_* < cc'\epsilon, \text{ for some constant } c'. \tag{4}$$

Since

$$K_g - K_{S(p_m^{(N)})} = K_g - K_{S(f_N)} + K_{S(f_N)} - K_{S(p_m^{(N)})},$$

we have from equations (1) and (4) that

$$\|K_g - K_{S(p_m^{(N)})}\| \leq \|K_g - K_{S(f_N)}\| + \|K_{S(f_N)} - K_{S(p_m^{(N)})}\| < C\epsilon.$$

Since $S(p_m^{(N)})$ is an analytic polynomial, $K_{S(p_m^{(N)})}$ has finite dimensional range and hence $K_{S(p_m^{(N)})}$ is compact.

Since $K_{S(p_m^{(N)})} \to K_g$ and the norm limit of compact operators is compact, K_g is compact.

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