http://dx.doi.org/10.1090/conm/213/02860

Contemporary Mathematics
Volume 213, 1998

Holomorphic Composition Operators in Several Complex Variables

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Abstract: A survey of known results and open problems concerning boundedness, compactness, and trace ideal membership of composition operators over the Bergman and Hardy spaces in several complex variables, with special attention to strongly pseudoconvex domains and bounded symmetric domains.

This expository paper attempts to give the state of the art of holomorphic composition operators in several complex variables insofar as it is concerned with certain basic problems for operators associated with a symbol. Although it is exclusively concerned with composition operators (which is a relatively new concept), also mentioned are Toeplitz and Hankel operators (which are each much older and have much wider scope and applicability). These three types of operators are at the center of the study of certain aspects of contemporary operator theory in function spaces and the relations between them cannot be ignored.

As the author is relatively new to several complex variables, the exposition will be influenced strongly by his affinity and bias for a special type of domain (a bounded symmetric domain) and a special approach to their study (Jordan algebras).

Another tool, besides Jordan theory, that is dear to the author is harmonic analysis on Euclidean spaces, for which the book of Stein [43] is an excellent source of information. Of greater impact is *multiparameter* harmonic analysis, as utilized in the context of the unit polydisk in several complex variables. I quote from the preface of the book by Stein:

> The subject of harmonic analysis has undergone a vast development in the last 25 years which has transformed the whole field. This has had not only a profound effect on Fourier analysis itself, but has had a major influence in such areas as partial differential equations, *several complex variables*, and *analysis on symmetric spaces* (emphasis mine).

The unit polydisk and the unit ball are examples of bounded symmetric domains, and a bounded symmetric domain is of course a special case of a symmetric space. The above quotation indicates that any study of operator theory on function spaces over bounded symmetric domains in several complex variables must rely heavily on harmonic analysis.

1991 Mathematics Subject Classification. Primary 47B10, 47B38; Secondary 32M15, 32M35

Although there are a large number of works on operator theory over symmetric domains, most of them (including the author's) have not used the Jordan theory. Notable exceptions are the study of Toeplitz operators by Upmeier [44],[47], and the study of Hankel operators by Arazy [1]. An underlying theme of this paper is to suggest that the two tools: *multiparameter harmonic analysis*, and *Jordan algebra theory*, should be exploited in order to obtain results for composition operators over the unit polydisk and more generally, over bounded symmetric domains.

This paper contains four main sections. Section 1 gives the background on the types of function spaces, domains, and operators of interest, and poses the problems to be discussed in later sections. It includes a rather heavy dose of Jordan theory. In Section 2, the known results for what I call the classical domains (unit disk, unit ball, unit polydisk) are given. All of these results are developed fully in the recent monograph [12] so they are only mentioned briefly here without much discussion. Sections 3 and 4 are concerned with two topics in which the author had a hand in developing, namely, weak compactness of composition operators on the Hardy spaces over a strongly pseudoconvex domain; and trace ideal criteria for composition operators on the Bergman spaces of a bounded symmetric domain. Some suggestions for further study are summarized in Section 5.

This paper follows closely the format of my talk at the Rocky Mountain Mathematics Consortium Conference to which this volume is devoted. Details and discussion are not always given, but references are provided for the reader (this includes the author) who wishes to learn more about a particular topic.

Acknowledgements. The author thanks Don Sarason for introducing him to composition operators, and Song-Ying Li for encouraging him to study them. The author also thanks the editors for the invitation to contribute this paper.

1 Preliminaries

1.1 Function Spaces

1.1.1 Bergman and Hardy spaces

Let Ω be a domain in $\mathbf{C}^n$. The *Bergman space* is the set of all holomorphic functions on Ω which are p-integrable with respect to Lebesgue volume measure dV on $\mathbf{C}^n = \mathbf{R}^{2n}$:

$$A^p(\Omega) \subset L^p(\Omega, dV) \qquad 0 < p < \infty.$$

$A^p(\Omega)$ is a closed subspace of $L^p(\Omega, dV)$. When $n = 1$, we use the notation dA for dV. The Hardy space $\mathcal{H}^p(\Omega)$, $0 < p < \infty$, as well as the embedding $\mathcal{H}^p(\Omega) \subset L^p(\partial\Omega)$ is a little more complicated. We begin with three familiar case. For any function f and $r > 0$, let $f_r(z) = f(rz)$.

- $\Omega =$ the unit disk: for a holomorphic function f on the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\} \subset \mathbf{C}$ and $0 < p < \infty$, $f \in \mathcal{H}^p(\Delta)$ if

$$||f||^p_{\mathcal{H}^p} = \sup_{0<r<1} \int_0^{2\pi} |f_r(e^{i\theta})|^p \, d\theta/2\pi < \infty.$$

- Ω = the unit ball: for a holomorphic function f on the unit ball $B = \{z = (z_1, \cdots, z_n) \in \mathbf{C}^n : \sum |z_j|^2 < 1\} \subset \mathbf{C}^n$ and $0 < p < \infty$, $f \in \mathcal{H}^p(B)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0<r<1} \int_{\partial B} |f_r(\zeta)|^p \, d\sigma(\zeta) < \infty.$$

- Ω = the unit polydisk: for a holomorphic function f on the unit polydisk $\Delta^n \subset \mathbf{C}^n$ and $0 < p < \infty$, $f \in \mathcal{H}^p(\Delta^n)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0<r<1} \int_{\mathbf{T}^n} |f_r(e^{i\theta})|^p \, d\theta_1 \cdots d\theta_n/(2\pi)^n < \infty.$$

In the above, $d\theta/2\pi$ denotes normalized Lebesgue measure on the unit circle $\mathbf{T}$, σ denotes a unique rotation invariant measure on the unit sphere $S = \partial B$, and in the case of the unit polydisk, $r = (r_1, \cdots, r_n)$ and $\theta = (\theta_1, \cdots, \theta_n)$, with obvious meanings for $0 < r < 1$ and $e^{i\theta}$ in this case.

In each of the above cases, any $\mathcal{H}^p$ function f has nontangential boundary values $f^* \in L^p(\partial\Omega)$ almost everywhere, and the map

$$\mathcal{H}^p \ni f \mapsto f^* \in L^p(\partial\Omega)$$

is norm preserving. In fact, for any bounded domain in $\mathbf{C}^n$ with C^2-boundary, f^* exists, see [26, Ch. 8]. Moreover, in the two cases considered below, that is, bounded symmetric domains and strongly pseudoconvex domains, the embedding $\mathcal{H}^p(\Omega) \subset L^p(\partial\Omega)$ is an isometry.

1.2 Domains of Interest

We shall limit our attention in this paper to two types of domains, namely, bounded symmetric domains and strongly pseudoconvex domains. Before giving the precise definitions, we show how the Hardy spaces are defined in each case. Then we shall say a little more about each type of domain.

A bounded symmetric domain can be defined as a domain in $\mathbf{C}^n$ which is the open unit ball of a Banach space structure on $\mathbf{C}^n$ and which also carries a certain type of algebraic structure known as a Jordan triple system. This approach is followed in finite and infinite dimensions in [25],[34], and [24]. In the infinite dimensional case, $\mathbf{C}^n$ is replaced by any complex Banach space. In finite dimensions, the bounded symmetric domains have been completely classified, first using Lie theory [3], and afterwards using Jordan theory [25]. The latter method extends to the infinite dimensional case [24],[45],[46].

The underlying Banach spaces of all bounded symmetric domains are obtained by taking ℓ^∞ direct sums of spaces in the following list. We shall not specify the norms in the last three cases.

- $M_{m,n}(\mathbf{C})$: rectangular m by n complex matrices with the operator norm
- $S_n(\mathbf{C})$: symmetric n by n complex matrices with the operator norm
- $A_n(\mathbf{C})$: anti-symmetric n by n complex matrices with the operator norm
- Spin_n: the complex "spin factor" of dimension n

- I_{16}: the "exceptional" complex Jordan triple system of dimension 16
- I_{27}: the "exceptional" complex Jordan algebra of dimension 27

In particular, we obtain the unit disk, unit ball, and unit polydisk, from $M_{1,1}$, $M_{1,n}$, and $M_{1,1} \times M_{1,1} \times \cdots \times M_{1,1}$, respectively.

An important tool in the above mentioned classification, which remains true in infinite dimensions, is that the connected component G of the group $\operatorname{Aut}\Omega$ of all holomorphic automorphisms of Ω is a (Banach) Lie group, and as such has a Cartan decomposition $G = KP$, where K is the compact subgroup of all linear isometries (called "rotations") and P is the set (not a group) of automorphisms (called "translations") analogous to the Möbius transformations on the unit disk. Indeed, in the case of the unit disk, the Cartan decomposition of an automorphism σ of Δ has the form

$$\sigma(z) = e^{i\theta}\varphi_a(z), \qquad z \in \Delta,$$

where $a \in \Delta$ and

$$\varphi_a(z) = \frac{z+a}{1+\bar{a}z}. \tag{1}$$

For any bounded symmetric domain, there is a unique probability measure σ on the Silov boundary $\partial^*\Omega$, which is invariant under the action of K. For a holomorphic function f on the bounded symmetric domain Ω, Ω is the open unit ball for a norm on $\mathbf{C}^n$, so the following definition makes sense. For $0 < p < \infty$, $f \in \mathcal{H}^p(\Omega)$ if

$$\|f\|^p_{\mathcal{H}^p} = \sup_{0<r<1} \int_{\partial^*\Omega} |f_r(\zeta)|^p \, d\sigma(\zeta) < \infty.$$

A strongly pseudoconvex domain Ω is given by a defining function $\rho : \mathbf{C}^n \to (0,\infty)$ with certain properties which will be mentioned in section 1.2.2: $\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\}$. With Ω_ϵ defined by $\{\rho < -\epsilon\}$, the conditions on ρ guarantee the existence of a surface area probability measure σ_ϵ on $\partial\Omega_\epsilon$ so the the following definition makes sense. For $0 < p < \infty$, $f \in \mathcal{H}^p(\Omega)$ if

$$\|f\|^p_{\mathcal{H}^p} = \sup_{\epsilon>0} \int_{\partial\Omega_\epsilon} |f(z)|^p \, d\sigma_\epsilon(z) < \infty.$$

The unit ball is an example of a bounded symmetric domain and of a strongly pseudoconvex domain, the defining function given by $\rho(z) = |z|^2 - 1$.

1.2.1 Bounded Symmetric Domains

We now take a closer look at bounded symmetric domains. Because it doesn't cost anything, we will work occasionally in arbitrary dimensions, specializing to $\mathbf{C}^n$ when convenient or necessary.

An open connected subset Ω of a complex Banach space E is said to be a bounded symmetric domain if for each $a \in \Omega$ there exists a unique element $\sigma = \sigma_a \in \operatorname{Aut}\Omega$ such that $\sigma(a) = a$ and $\sigma^2 = \mathrm{Id}$. It is known that this is equivalent to Ω being homogeneous, that is, the group $\operatorname{Aut}\Omega$ acts transitively on Ω, and symmetric at a single point.

For example, in the case of the unit disk Δ, $\sigma_a = \varphi_a \circ s \circ \varphi_a$ is involutive and fixes a, where $s(z) = -z$ and φ_a is the element of $\operatorname{Aut}\Omega$ given by (1), that is, with $\varphi_a(0) = a$ and $\varphi_a = {\varphi_a}^{-1}$.

A great number of finite and infinite dimensional examples of bounded symmetric domains are provided by the algebraic structure known as a JC^*-triple. These are defined as the open unit ball, in the operator norm, denoted $\operatorname{ball}(M) = \{z \in M : \|z\| < 1\}$, of those norm closed subspaces M of $\mathcal{L}(H, K)$ which are stable for the triple product

$$\{abc\} := (ab^*c + cb^*a)/2.$$

Since $\{abc\}$ is the quadratization and linearization of $\{aaa\} = aa^*a$, it is enough to consider subspaces stable under the map $a \mapsto aa^*a$. These objects were first studied in [20] under the name "J^*-algebra". With hindsight, the "J" in JC^*-triple stands for *Jordan* as in Jordan algebra, and the "C" stands for "concrete". The following formula (from [20]), valid for JC^*-triples, is a far reaching generalization of the Möbius automorphisms of the unit disk:

$$\varphi_a(b) = (1 - aa^*)^{-1/2}(b + a)(I + a^*b)(1 - a^*a)^{1/2}.$$

For $a \in \operatorname{ball}(M)$, $\varphi_a \in \operatorname{Aut}(\operatorname{ball}(M))$ satisfies, $\varphi_a(0) = a$ and $\varphi_a = {\varphi_a}^{-1}$.

This work of Harris [20] was an outgrowth of his thesis, which contained a version of the Schwarz lemma in normed linear spaces [19].

I want to mention two watershed results from the generalization of Harris's work carried out by the Tübingen school of infinite dimensional holomorphy. The first is "Kaup's Riemann Mapping Theorem", from [24], which states that every bounded symmetric domain in a complex Banach space is holomorphically equivalent to the open unit ball of a Banach space carrying the structure of a JB^*-triple. The latter are the appropriate abstract generalization of JC^*-triple, the "B" standing for Banach, as in Banach Jordan algebra.

The second is Upmeier's structure theorem for the Toeplitz C^*-algebra, that is, the C^*-algebra $\mathcal{A}$ generated by all Toeplitz operators with continuous symbol, defined on the Hardy space $\mathcal{H}^2(\Omega)$ of a bounded symmetric domain. In [44], see also the book by Upmeier [47], it is shown that this C^*-algebra is solvable in the sense that there is a sequence of norm closed ideals I_j, $j = 0, 1, \ldots, r + 1$, where r is the rank of the bounded symmetric domain, such that

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r+1} = \mathcal{A},$$

and $I_{k+1}/I_k = C(S_k) \otimes \mathcal{K}$. Here $\mathcal{K}$ is the compact operators, and S_k certain compact Hausdorff spaces defined using the Jordan triple structure associated with the bounded symmetric domain in question. Moreover, the spectrum of $\mathcal{A}$, that is, the space of irreducible strongly continuous representations on Hilbert space are in one-to-one correspondence with the set of tripotents of that Jordan structure, which are the appropriate analog of the idempotents in binary algebras.

For a survey and history of some of the topics in this section, see [38].

1.2.2 Strongly pseudoconvex domains

We now take a closer look at strongly pseudoconvex domains. A domain $\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\} \subset \mathbf{C}^n$ with C^2 boundary is said to be *pseudoconvex* if there is a defining function ρ which satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(P) w_j \overline{w_k} \geq 0$$

for all $P \in \partial\Omega$ and for all $w \in \mathbf{C}^n$ which satisfy $\sum_j \partial\rho/\partial z_j(P) w_j = 0$.

You get a strongly pseudoconvex domain if this last expression is positive for every non-zero w (which implies that it is not less than $C|w|^2$ for some constant $C > 0$ and all $(P, w) \in \partial\Omega \times \mathbf{C}^n$).

The following domain is pseudoconvex:

$$\Omega = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^4 < 1\}.$$

Its defining function is $\rho(z) = |z_1|^2 + |z_2|^4 - 1$. The domain in $\mathbf{C}^2$ with defining function $\rho(z) = |z_1|^2 + |z_2|^2 + 1 - 2|z_1| - r^2$ is strongly pseudoconvex if $0 < r < 1$.

The pseudoconvex domains coincide with domains of holomorphy, that is, those domains Ω which admit a holomorphic function which cannot be continued analytically to any domain Ω' properly containing Ω. The proof of this fact in one direction uses Hörmander's solution to the $\overline{\partial}$-problem on pseudoconvex domains (See for example [37, p226]).

For the material in this subsection, as well as "22 equivalent characterizations of the principal domains that are studied in several complex variables", see the book by Krantz, [26].

1.3 Operators of Interest

1.3.1 Toeplitz and Hankel operators

The Bergman space $A^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega)$ and its orthogonal projection (the Bergman projection) is given as an integral operator with kernel $K(z, w)$ (the Bergman kernel). We shall denote this projection by P,

$$Pf(z) = \int_\Omega f(w) K(z, w)\, dV(w) \quad , \quad f \in L^2(\Omega).$$

Similarly, the Hardy space $\mathcal{H}^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\partial\Omega)$ and its orthogonal projection (the Szegö projection) is given as an integral operator with kernel $S(z, w)$ (the Szegö kernel). We shall denote this projection by S,

$$Sf(z) = \int_{\partial\Omega} f(w) S(z, w)\, d\sigma(w), \quad f \in L^2(\partial\Omega).$$

Although our main interest is in composition operators, we first define the other two types of operators which are of interest in the field of operator theory in function spaces. All three operators of interest are defined both for the Bergman spaces and for the Hardy spaces, as well as many other holomorphic functions spaces.

Let $f : \Omega \to \mathbf{C}$ and define formally the following:

Toeplitz $T_f : A^2 \to A^2 \;;\quad T_f g = P(fg), \quad g \in A^2,\ fg \in L^2$

Hankel $H_f : A^2 \to {A^2}^\perp \;;\quad H_f g = (I - P)(fg), \quad g \in A^2,\ fg \in L^2$

Small Hankel $h_f : A^2 \to A^2$; $h_f g = P(f\overline{g}), \quad g \in A^2,\ f\overline{g} \in L^2$

We make several remarks in connection with these definitions. The definitions above are for the operators on the Bergman space. There is a corresponding Hardy space operator in each case; simply replace A^2 by $\mathcal{H}^2$ and P by S. All of these operators are densely defined, and the small Hankel operators are conjugate linear. The small Hankel operator is essentially the same as the Hankel operator only in the case of $\mathcal{H}^2(\Delta)$, because $\mathcal{H}^2(\Delta)^\perp$ is one dimension away from $\overline{\mathcal{H}^2(\Delta)}$. It is sometimes convenient to consider these operators as acting from L^2 into L^2.

1.3.2 The Bergman and Szegö projections

The Bergman and Szegö projections are important tools in the study of operator theory in function spaces, and indeed are instrumental in the very definition of Toeplitz and Hankel operators.

Let's give some explicit formulas for the Bergman kernel in the cases of interest to us. Similar formulas hold for the Szegö projections in each case.

unit ball In this case,

$$K(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \overline{w})^{n+1}}$$

strongly pseudoconvex domain In this case, there is no explicit formula, but an asymptotic expansion due to C. Fefferman [18] allows one to transfer techniques for the unit ball to this setting.

unit polydisk In this case,

$$K(z,w) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - z_j \overline{w_j})^2}$$

bounded symmetric domain In this case, the Bergman kernel can be expressed in terms of the Jordan algebraic structure associated with bounded symmetric domains as follows:

$$K(z,w) = c \det B(z,w)^{-1}$$

where $B(x,y)$ is the "Bergman operator" (see the next subsection). This description of the Bergman kernel can be found in [34] and [14].

1.3.3 The Bergman operator of a JB^*-triple

Recall that except for the two exceptional ones, all finite dimensional bounded symmetric domains, as well as a great number of infinite dimensional ones, occur as the open unit balls of norm closed subspaces M of $\mathcal{L}(H,K)$ which are stable for the triple product

$$\{abc\} := (ab^*c + cb^*a)/2. \tag{2}$$

It is not difficult to check the following properties of the symmetrized triple product (2).

1. $\{xyz\}$ is jointly continuous, symmetric and bilinear in x, z, and conjugate linear in y

2. Let $\delta(x)$ be the map $y \mapsto \{xxy\}$. Then $\exp[it\delta(x)]$ is a surjective isometry of M for each $t \in \mathbf{R}$, $\sigma_{\mathcal{L}(M)}(\delta(x)) \geq 0$, and $i\delta(x)$ is a derivation of the Jordan triple product, that is, with $\delta = \delta(x)$,

$$\delta\{abc\} = \{\delta a, b, c\} - \{a, \delta b, c\} + \{a, b, \delta c\} \tag{3}$$

3. $\|\{xxx\}\| = \|x\|^3$ (or $\|\delta(x)\| = \|x\|^2$)

Note that (3) is the "Leibniz rule", and that $\delta(x) = L_{xx^*} + R_{x^*x}$, where L_a for example denotes left multiplication by a

The above three properties becomes the definition of a JB^*-triple, that is, of an abstract JC^*-triple. More precisely, if $A, \{\cdots\}, \|\cdot\|$ is a complex Banach space equipped with a triple product which satisfies the above three properties, we call A a JB^*-triple. The analogy with this concept and that of the theory of C^*-algebras, B^*-algebras, and W^*-algebras cannot be resisted. In fact, a theory parallel to the theory of operator algebras on Hilbert space has been developed, culminating in a Gelfand-Naimark Theorem for JB^*-triples, [17], see also [38]. What made this ternary product theory difficult to develop is the complete lack of a global order structure, which was a key tool in the case of a binary product.

We can now define the Bergman operator of a JB^*-triple A. For $x, y \in A$:

$$B(x, y) = I - 2D(x, y) + Q(x)Q(y),$$

where $D(x, y)$ is the linear operator $z \mapsto \{xyz\}$ and $Q(a)$ is the conjugate linear operator $z \mapsto \{aza\}$. In the case of JC^*-triples, $B(x, y)z = (I - xy^*)z(I - y^*x)$, or $B(x, y) = L_{I-xy^*}R_{I-y^*x}$.

As a reality check, if A is the complex numbers, where $\{xyz\} = xz\overline{y}$, then $B(x, y) = (1 - x\overline{y})^2$ so that $K(x, y) = \det B(x, y)^{-1} = (1 - x\overline{y})^{-2}$.

1.3.4 Composition operators

For a function $\varphi : \Omega \to \Omega$, the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi.$$

We will always be interested in the case that the symbol φ is a holomorphic mapping (transformation) and the operator C_φ will always be acting on spaces of holomorphic functions.

Composition operators are not only of great intrinsic interest in pure operator theory. They appear in the following contexts:

- commutants of multiplication operators
- dynamical systems
- deBranges-Bieberbach conjecture
- ergodic transformations
- isometries of Banach function spaces
- homomorphisms of algebras

The main references for the theory of composition operators are the three recent monographs [12], [41],[42], and the two survey articles [11],[48].

1.4 Problems of Interest

For holomorphic composition operators on the Hardy space $\mathcal{H}^p(\Omega)$ or the Bergman space $A^p(\Omega)$ we shall be interested in the following natural questions.

1. For $0 < p \leq \infty$, for which symbols φ is C_φ bounded?
2. For $0 < p \leq \infty$, for which symbols φ is C_φ compact?
3. For $p = 2$, for which symbols φ does C_φ belong to some Schatten-von Neumann class $\mathcal{S}_q$, $0 < q < \infty$? (A compact operator T on a Hilbert space belongs to $\mathcal{S}_q$ if its sequence of singular numbers, that is, the eigenvalues of $(T^*T)^{1/2}$, belongs to the sequence space ℓ^q)
4. For $p = 1$, for which symbols φ is C_φ weakly compact? (An operator T on a Banach space is weakly compact if it is bounded and maps bounded sets to relatively weakly compact sets)

For a given domain, the above list implies that there are eight questions of interest, 4 for the Hardy space and 4 for the Bergman space. As will be discussed in the rest of this survey, in the case of the Bergman space, all of these problems have been solved except for the boundedness and compactness criteria in the case of a strongly pseudoconvex domain, see 1.4.2. In particular, all problems are solved for the Bergman space of a bounded symmetric domain and therefore for the Bergman space of the unit ball.

On the contrary, all 4 problems are completely open in the case of the Hardy space of a bounded symmetric domain, and two of them (3. and 4.) are open in the particular case of the unit polydisk. In addition, Problem 3 for the Hardy space is still open in the case of the unit ball, or more generally, a strongly pseudoconvex domain.

The above discussion is summarized in the following two tables, whose entries are the appropriate literature references for the solution of the problem associated with the entry. The author apologizes if there are some other references that should have been included here which have been overlooked.

Although these tables demonstrate that serious progress haas been made on the specified problems, many of the results present analytic conditions, such as Carleson measure criteria, integrability criteria, growth of reproducing kernels, etc. in terms of the symbol, for the operator to be of a specific type. However, it remains to obtain more explicit geometric criteria, power series criteria, angular derivative criteria, etc. for many of these settings.

1.4.1 Problems on the Hardy space

	bounded	compact	Schatten	weak compact
unit disk	Littlewood26	Shapiro & Taylor73	Luecking87 Luecking & Zhu92	Sarason92
unit ball	MacCluer & Shapiro86 MacCluer85 Cima-Wogen87 Wogen88	MacCluer85	OPEN	(Li-Russo95)
unit polydisk	Jafari90	Jafari90	OPEN	OPEN
SPCD	Li-Russo95	Li-Russo95	OPEN	Li-Russo95
BSD	OPEN	OPEN	OPEN	OPEN

1.4.2 Problems on the Bergman space

	bounded	compact	Schatten	weak compact
unit disk	MacCluer & Shapiro86	MacCluer & Shapiro86	Luecking87 Luecking & Zhu92	(Chen96)
unit ball	MacCluer & Shapiro86 MacCluer85 MacCluer & Mercer93	MacCluer85	(Li95)	(Chen96)
unit polydisk	Jafari90	Jafari90	(Zhu88, Li-Russo96)	(Chen96)
SPCD	OPEN	OPEN	Li95	Chen96
BSD	Jafari92	Jafari92	Zhu88, Li-Russo96	Chen96

2 The State of affairs for the Classical domains

The material in subsections 2.1.1 to 2.1.3 and 2.2.1 to 2.2.3 are from [12], to which the reader is referred for discussion, precise statements and proofs.

2.1 Operators on the Hardy and Bergman spaces of the Unit Disk

This seems like a good place to introduce the weighted Bergman spaces $A^p_\alpha(\Delta) \subset L^p(\Delta, (1-|z|^2)^\alpha dA(z))$ with $0 < p < \infty$ and $\alpha > -1$:

$$A^p_\alpha(\Delta) = \{f \text{ holomorphic on } \Delta : \int_\Delta |f(z)|^p (1-|z|^2)^\alpha \, dA(z) < \infty\}.$$

2.1.1 Automatic Continuity

[12, Corollary 3.7,p.123] As a consequence of the Littlewood subordination principle, C_φ is automatically bounded on $\mathcal{H}^p(\Delta)$ for $p \geq 1$.

[12, Exercise 3.1.3,p.127] Discussion of when C_φ is automatically bounded on the weighted Bergman space ${A^2}_\alpha(\Delta)$, $\alpha > -1$

2.1.2 Boundedness and Compactness Criteria

For a function $\varphi : \Delta \to \Delta$, let μ be the pull-back measure defined as follows: $\mu(E) = \sigma(\varphi^{*-1}(E))$, where $E \subset \overline{\Delta}$ and σ denotes the normalized Lebesgue measure on $\mathbf{T}$: $\sigma = d\theta/2\pi$. In the following two results, $\mathcal{S}(\zeta, h)$ denotes the Carleson region defined below in subsection 2.2.1 for the unit ball in $\mathbf{C}^n$.

[12, Theorem 3.12,p.129] C_φ is bounded on $\mathcal{H}^p(\Delta)$ $0 < p < \infty$ if and only if $\mu(\mathcal{S}(\zeta, h)) = \mathrm{O}(h)$ for all $\zeta \in \mathbf{T}$ and $0 < h < 1$.

[12, Theorem 3.12,p.129] C_φ is compact on $\mathcal{H}^p(\Delta)$ $0 < p < \infty$ if and only if $\mu(\mathcal{S}(\zeta, h)) = \mathrm{o}(h)$ as $h \to 0$ uniformly for $\zeta \in \mathbf{T}$.

[12, Exercise 3.2.6,p.142] The analog of the previous two results holds for the weighted Bergman spaces ${A^p}_\alpha(\Delta)$.

The boundedness or compactness of C_φ on $\mathcal{H}^p(\Delta)$ or on $A^p(\Delta)$ is thus independent of p. Moreover, in the case of the Hardy space, the 'big O' Carleson measure condition automatically holds.

2.1.3 More on compactness

Compactness can be described in terms of *angular derivatives* as well as the *Nevanlinna counting function* $N_\varphi(z)$. We refer to [12] for the definitions. However, see the next subsection.

[12, Corollary 3.14,p.132] If C_φ is compact on $\mathcal{H}^p(\Delta)$ or ${A^p}_\alpha(\Delta)$, $0 < p < \infty$ then φ has no finite angular derivative on $\mathbf{T}$.

[12, Theorem 3.20,p.139] C_φ is compact on $\mathcal{H}^2(\Delta)$ if and only if

$$\limsup_{|w| \to 1} \frac{N_\varphi(w)}{-\log|w|} = 0$$

[12, Theorem 3.22,p.141] C_φ is compact on ${A^p}_\alpha(\Delta)$ if and only if φ has no finite angular derivative on $\mathbf{T}$.

2.1.4 Trace Ideal Criteria

There are two main results that deal with the membership of a composition operator on the unit disk in a Schatten-von Neumann class $\mathcal{S}_p$. They appear in [35] and [36]. For the case of the Hardy space, this problem has not yet been extended beyond the unit disk.

For the unit disc in **C**, D. Luecking [35] initiated a systematic study of trace ideal criteria ($0 < p < \infty$) for Toeplitz operators T_μ with measures as symbols on some standard Hilbert spaces of holomorphic functions. His conditions are expressed in terms of a dyadic hyperbolic decomposition of the unit disc. By an appropriate choice of measure and weight, his result applies to composition operators on the Hardy space and the weighted Bergman spaces. See section 4.1 for a discussion of T_μ on the weighted Bergman spaces of bounded symmetric domains.

In order to describe the main result of [35] we need to introduce some notation from that paper. For $\alpha < 1$, H_α will denote the space of analytic functions $f(z) = \sum_0^\infty a_n z^n$ on Δ satisfying $\sum (n+1)^\alpha |a_n|^2 < \infty$. The parameter values $\alpha = 0$ and $\alpha = -1$ correspond, respectively, to the Hardy space H^2 and the Bergman space A^2. The norm is chosen so that the reproducing kernel $k_w^\alpha(z) = k^\alpha(z,w)$ for H_α has the form $k^\alpha(z,w) = (1 - z\overline{w})^{\alpha-1}$. For $\alpha < 0$, H_α is the weighted Bergman space $A^2_{-1-\alpha}$.

The dyadic hyperbolic decomposition is basically a decomposition of Δ into disjoint sets of roughly equal (hyperbolic) size. A dyadic arc is an arc $I \subset \partial\Delta$ of the form

$$I = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi(k+1)}{2^n} \right\}, \qquad k = 0, 1, \ldots, 2^n - 1, \quad n = 0, 1, \ldots.$$

Given an arc I, let $\ell(I)$ denote its length and let $S(I)$ denote the corresponding Carleson "square": $S(I) = \{z \in \Delta : z/|z| \in I, 1 - |z| \leq \ell(I)/2\pi\}$. Also, let $R(I)$ denote the half of $S(I)$ nearest the origin, that is $R(I) = \{z \in S(I) : \ell(I)/4\pi < 1 - |z| \leq \ell(I)/2\pi\}$. The family $\{R(I)\}$ where I runs over all dyadic arcs is pairwise disjoint and covers Δ. Fix an enumeration of this family and call it $\{R_i\}$.

[35] Let $p > 0$ and $\alpha < 1$ be such that $p\alpha < 1$, and let μ be a measure on Δ. In order that the Toeplitz operator

$$T_\mu f(w) = \int f(z)(1 - \overline{z}w)^{\alpha-1}\, d\mu(z), \quad f \in H_\alpha$$

belong to the Schatten class $\mathcal{S}_p(H_\alpha)$ it suffices that

$$\sum [|\mu|(R_i)\ell(R_i)^{\alpha-1}]^p < \infty.$$

In this case $\|T_\mu\|^p_{\mathcal{S}_p} \leq C \sum [|\mu|(R_i)\ell(R_i)^{\alpha-1}]^p$. If μ is a positive measure, this condition is also necessary, and $\sum [|\mu|(R_i)\ell(R_i)^{\alpha-1}]^p \leq C\|T_\mu\|^p_{\mathcal{S}_p}$.

We next describe the results of [36], They involve the Nevanlinna counting function in both the Hardy space and the weighted Bergman spaces. For the latter, it is necessary to recall the generalized Nevanlinna counting function of [40], that is, for $\beta \geq 1$,

$$N_{\varphi,\beta}(w) = \sum_{z \in \varphi^{-1}(w)} \left(\log \frac{1}{|z|} \right)^\beta.$$

[36, Theorem 1] For a composition operator C_φ on $\mathcal{H}^2(\Delta)$, and $0 < p < \infty$,

$$C_\varphi \in \mathcal{S}_p(\mathcal{H}^2) \Leftrightarrow \frac{N_\varphi(w)}{-\log|w|} \in L^{p/2}(d\lambda(w))$$

where $d\lambda(z) = (1-|z|^2)^{-2} dA(z)$

[36, Theorem 3] The composition operator C_φ belongs to $\mathcal{S}_p(A^2_\alpha)$ if and only if

$$N_{\varphi,\alpha+2}(z)\left(\log\frac{1}{|z|}\right)^{-\alpha-2} \in L^{p/2}(d\lambda(z)).$$

The special case of the Hilbert Schmidt composition operators is easier and can be found in [12, Exercise 3.3.2,p. 149].

2.1.5 Weak compactness

For $1 < p < \infty$, $\mathcal{H}^p(\Omega)$ and $A^p(\Omega)$ are reflexive Banach spaces, so any bounded operator is automatically weakly compact. For $p = 1$ and $\Omega = \Delta$, we have the following result of Sarason, which uses real variable Hardy spaces and Fefferman duality, as expressed by $VMO(\Delta)^* = H_1(\Delta)$ and $H_1(\Delta)^* = BMO(\Delta)$.

Since C_φ is automatically bounded on $H_1(\Delta)$, so is its adjoint on $BMO(\Delta)$, and it becomes important to ask when C_φ is itself the adjoint of some operator on $VMO(\Delta)$, as expressed in the following diagram:

$$\begin{array}{ccc} BMO(\Delta) & \overset{C_\varphi^*}{\longleftarrow} & BMO(\Delta) \\ H_1(\Delta) & \overset{C_\varphi}{\longrightarrow} & H_1(\Delta) \\ VMO(\Delta) & \longleftarrow & VMO(\Delta) \end{array}$$

This is the case if $C^*_\varphi(BMO(\Delta)) \subset VMO(\Delta)$, which is one of the ingredients in the proof of

[39] If C_φ is weakly compact on $\mathcal{H}^1(\Delta)$, then C_φ is compact.

This result has been extended to higher dimensional Hardy and Bergman spaces ([31],[6]), see subsections 3.2 and 3.3.

2.2 Operators on the Hardy and Bergman spaces of the unit Ball

2.2.1 Boundedness and Compactness Criteria

The Carleson measure conditions on the unit ball are the same as the corresponding conditions on the unit disk noted above.

For a function $\varphi : B_N \to B_N$, let μ be the pull-back measure defined as follows: $\mu(E) = \sigma_N(\varphi^{*-1}(E))$, where $E \subset \overline{B_N}$ and σ_N denotes the normalized surface measure on $S := \partial B_N$. In the following two results, φ is a holomorphic map from Ω into Ω, and $\mathcal{S}(\zeta, h)$ denotes the Carleson region defined as follows:

$$\mathcal{S}(\zeta,h) = \{z \in \overline{B_N} : |1 - \langle z|\zeta\rangle| < h\}$$

[12, Theorem 3.35,p.161] C_φ is bounded on $\mathcal{H}^p(B_N)$ $0 < p < \infty$ if and only if $\mu(\mathcal{S}(\zeta,h)) = \mathrm{O}(h^N)$ for all $\zeta \in S$ and $0 < h < 1$.

[12, Theorem 3.35,p.161] C_φ is compact on $\mathcal{H}^p(B_N)$ $0 < p < \infty$ if and only if $\mu(\mathcal{S}(\zeta,h)) = \mathrm{o}(h^N)$ as $h \to 0$ uniformly for $\zeta \in S$.

[12, Theorem 3.37,p.164] The analog of the previous two results holds for the weighted Bergman spaces ${A^p}_\alpha(B_N)$. In this case, $\mu(E) = V_\alpha(\varphi^{-1}(E))$ for $E \subset B_N$, where $dV_\alpha(z) = (1-|z|^2)^\alpha dV_N$.

2.2.2 Some results on boundedness and unboundedness

[12, Theorem 3.39,p.166] If $\sup_{z\in B_N} \|\varphi'(z)\| < \infty$ and $\sup\{\mathrm{card}\{\varphi^{*-1}(\zeta)\} : \zeta \in \partial B_N\} = \infty$, then C_φ is unbounded on $\mathcal{H}^p(B_N)$.

[12, Theorem 3.41,p.167] If $\sup_{z\in B_N} \|\varphi'(z)\|/|J_\varphi(z)| < \infty$, then C_φ is bounded on $\mathcal{H}^p(B_N)$.

[12, Theorem 6.5,p.227] $C_\varphi(\mathcal{H}^p(B_N) \subset {A^p}_{N-2}(B_N)$ and C_φ is automatically bounded as an operator from $\mathcal{H}^p(B_N)$ to ${A^p}_{N-2}(B_N)$.

2.2.3 Angular derivative

Once again, we refer to [12] for the definition of angular derivative for the unit ball.

[12, Theorem 3.43,p.171] If C_φ is compact on $\mathcal{H}^p(B_N)$ or ${A^p}_\alpha(B_N)$, $0 < p < \infty$ then φ has no finite angular derivative on ∂B_N.

[12, Exercise 6.1.2,p.228] If φ has no finite angular derivative on S, then $C_\varphi : \mathcal{H}^p(B_N) \to {A^p}_{N-2}(B_N)$ is compact.

Some Geometric conditions for boundedness, compactness, and unboundedness for composition operators on the unit ball can be found in [12, Chapter 6], to which we refer for discussion, statements, and proofs.

2.3 Operators on the Hardy and Bergman spaces of the unit Polydisk

For the unit polydisk Δ^n, the notions of Carleson measure, compact Carleson measure, α-Carleson measure, compact α-Carleson measure, $\alpha > -1$, were introduced and studied, along with angular derivatives, in [21],[22]. Let m_n denote n-dimensional normalized Lebesgue measure on the torus $\mathbf{T}^n$.

[21] Let $\varphi : \Delta^n \to \Delta^n$ be holomorphic, $1 < p < \infty$, $\alpha > -1$.

- C_φ is bounded on $\mathcal{H}^p(\Delta^n)$ (resp. ${A^p}_\alpha(\Delta^n)$) if and only if $\mu(E) = m_n(\varphi^{*-1}(E))$ is a Carleson measure (resp. an α-Carleson measure).
- C_φ is compact on $\mathcal{H}^p(\Delta^n)$ (resp. ${A^p}_\alpha(\Delta^n)$) if and only if $\mu_\alpha(E) = V_\alpha(\varphi^{*-1}(E))$ is a compact Carleson measure (resp. a compact α-Carleson measure).

3 Weak Compactness

3.1 Fefferman duality in several complex variables

It had been considered a part of the folklore for some time that the result of C. Fefferman identifying the dual of $H^1(\mathbf{R}^n)$ as $BMO(\mathbf{R}^n)$ [43], can be extended (in suitable form) to the unit ball in $\mathbf{C}^n$. In order to understand Fefferman duality better, Coifman showed that the atomic structure of $H^1(\mathbf{R})$ was equivalent to the duality theorem. Fefferman duality and atomic decomposition (of $\mathcal{H}^p$ $0 < p \leq 1$) are now known to hold in the following cases:

- The unit disk: Fefferman, Coifman ([15],[8])
- The unit ball: Coifman-Rochberg-Weiss ([10])
- The unit polydisk: Chang-Fefferman ([4])
- Strongly pseudoconvex domains: Krantz-Li ([28])

The unit ball in $\mathbf{C}^n$ and the unit polydisk are examples of bounded symmetric domains, so this is an area of investigation which is wide open for bounded symmetric domains.

3.1.1 Atomic decomposition and duality for the unit ball

We state here the atomic decomposition and "Fefferman duality" for the case of the unit ball in $\mathbf{C}^n$. Let $B = B_N$ be the unit ball in $\mathbf{C}^n$, and let σ denote Lebesgue area measure on $S = \partial B$. Recall that the Hardy space $\mathcal{H}^1(B)$ consists of all holomorphic functions $F : B \to \mathbf{C}$ satisfying

$$\|F\|_1 = \sup_{0<r<1} \int_{\partial B} |F(rz)|\, d\sigma(z) < \infty.$$

An *atom* is a function $a : \partial B \to \mathbf{C}$ which is supported on a sphere S (with respect to the metric $d(z,\zeta) = |1 - \langle z|\zeta\rangle|^{1/2}$) and satisfying

$$|a(\zeta)| \leq \frac{1}{|S|} \quad , \quad \int_{\partial B} a\, d\sigma = 0.$$

A *holomorphic atom* A is the image of an atom a under the Szegö projection:

$$A(z) = S(a)(z) = c \int_{\partial B} \frac{a(\zeta)}{(1 - \langle z|\zeta\rangle)^n}\, d\sigma(\zeta).$$

The space $BMO(B)$ is defined as the space of functions $b : \partial B :\to \mathbf{C}$ such that

$$\|b\|_{BMO} = \sup_S \frac{1}{|S|} \int_S |b(y) - m_S(b)|\, d\sigma < \infty$$

where S is sphere and $m_S(b) = \int_S b\, d\sigma/|S|$. The main results from [10], for our purposes, are

[10] Atomic decomposition Every $F \in \mathcal{H}^1(B)$ can be written $F = \sum_i \lambda_i A_i$, where the A_i are holomorphic atoms and $\sum |\lambda_i| \leq C\|F\|_1$.

[10] Duality $\mathcal{H}^1(B)^* = BMOA(B)$, where $BMOA$ consists of the holomorphic functions in $BMO(B)$.

As noted above, these two results have each been formulated and proved in the context of strongly pseudoconvex domains [28], as well as for the unit polydisk [4]. In each case, nontrivial modifications are required. In the case of the polydisk, the methods of multiparameter harmonic analysis [5] are used.

3.2 Weak Compactness on Hardy spaces

In 2.1.5 above, we discussed weak compactness of composition operators on the Hardy space of the unit disk. In this and the following section, we discuss this problem in higher dimensions. For $\Omega=$ the unit ball or a strongly pseudoconvex domain, real variable Hardy spaces exist and Fefferman duality holds, [28]:

$$H^1(\partial\Omega)^* = BMO(\partial\Omega) \quad , \quad H^1(\partial\Omega) = VMO(\partial\Omega)^*.$$

The analog of the theorem on Sarason on weak compactness, as well as the standard criteria for boundedness and compactness on Hardy spaces was extended the to setting of a strongly pseudoconvex domain in [31].

[31, Theorem 1] Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ with smooth boundary. The following are equivalent for a composition operator C_φ which is bounded on $\mathcal{H}^2(\Omega)$:

1. $C_\varphi^* : L^\infty(\partial\Omega) \to VMO(\partial\Omega)$ is bounded, that is, $C_\varphi : H^1(\partial\Omega) \to L^1(\partial\Omega)$ is bounded and $C_\varphi^*(L^\infty(\partial\Omega)) \subset VMO(\partial\Omega)$.
2. $C_\varphi : H^1(\partial\Omega) \to L^1(\partial\Omega)$ is compact.
3. $C_\varphi : \mathcal{H}^1(\Omega) \to \mathcal{H}^1(\Omega)$ is compact.
4. $C_\varphi : \mathcal{H}^p(\Omega) \to \mathcal{H}^p(\Omega)$ is compact for some $0 < p < \infty$.
5. $C_\varphi : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact for some $1 < p < \infty$.
6. The pull-back measure dV_φ is a vanishing Carleson measure.
7. $C_\varphi : \mathcal{H}^p(\Omega) \to \mathcal{H}^p(\Omega)$ is compact for all $0 < p < \infty$.

The diagram for the equivalence of 1. and 2. which corresponds to the diagram in subsection 2.1.5 is

$$\begin{array}{ccc} BMO(\partial\Omega) & \xleftarrow{C_\varphi^*} & L^\infty(\partial\Omega) \\ H^1(\partial\Omega) & \xrightarrow{C_\varphi} & L^1(\partial\Omega) \\ VMO(\partial\Omega) & \longleftarrow & \end{array}$$

Note that the bottom row of this diagram is incomplete since L^1 has no predual in this case. For $n = 1$ and $\Omega = \Delta$, if $C_\varphi : H^1(\partial\Omega) \to L^1(\partial\Omega)$ is compact, then it follows that $C_\varphi(H^1(\partial\Omega)) \subset H^1(\partial\Omega)$. In higher dimensions, it is more convenient to state the result in the form given above.

3.3 Weak Compactness on Bergman spaces

Motivated by the results of Sarason and Li-Russo, Lang Chen, in his 1994 thesis considered the weak compactness of composition operators on the Bergman space $A^1(\Omega)$ of both strongly pseudoconvex domains and bounded symmetric domains.

In the following, we denote by $\mathcal{B}(\Omega)$ and $\mathcal{B}_0(\Omega)$ the Bloch and little Bloch spaces defined for a bounded strongly pseudoconvex domain Ω. According to [27],[29],and [6], $\mathcal{B}_0(\Omega)^* = A^1(\Omega)$ and $A^1(\Omega)^* = \mathcal{B}(\Omega)$.

For a bounded symmetric domain Ω, the dual and predual of $A^1(\Omega)$ were determined in [49]. There, certain Bloch type spaces $\tilde{\mathcal{B}}^s(\Omega)$ and $\tilde{\mathcal{B}}_0^s(\Omega)$ were defined in terms of a differential operator depending on a (Jordan theoretic) real parameter s and were shown to satisfy $\tilde{\mathcal{B}}^s_0(\Omega)^* = A^1(\Omega)$ and $A^1(\Omega)^* = \tilde{\mathcal{B}}^s(\Omega)$.

In each of the above cases, the Bergman projection was shown to map $L^\infty(\Omega)$ (resp. $C_0(\Omega)$) onto the Bloch space $\mathcal{B}(\Omega)$ in the case of a strongly pseudoconvex domain, and $\tilde{\mathcal{B}}^s(\Omega)$ in the case of a bounded symmetric domain (resp. the little Bloch space $\mathcal{B}_0(\Omega)$ and $\tilde{\mathcal{B}}^s_0(\Omega)$).

[6, Theorem 4.1,Theorem 4.5] Let Ω be a bounded strongly pseudoconvex domain (resp. a bounded symmetric domain) in $\mathbf{C}^n$ with smooth boundary. The following are equivalent:

1. $C_\varphi : A^1(\Omega) \to A^1(\Omega)$ is compact.
2. $C_\varphi : A^1(\Omega) \to A^1(\Omega)$ is weakly compact.
3. $C^*_\varphi : \mathcal{B}(\Omega) \to \mathcal{B}_0(\Omega)$ is bounded (resp $C^*_\varphi : \tilde{\mathcal{B}}^s(\Omega) \to \tilde{\mathcal{B}}^s_0(\Omega)$ is bounded).

4 Trace Ideal Criteria

4.1 Known Results

We have already mentioned in subsection 2.1.4 the known results in the case of the unit disk. A thorough study of boundedness, compactness, and trace ideal criteria for Toeplitz operators on weighted Bergman spaces of bounded symmetric domains was done in [51]. Again, by appropriate choice of the symbol, the theory can be made to cover composition operators, namely $C^*_\varphi C_\varphi = T_{V^\alpha_\varphi}$. The theorem of Zhu which is of interest to us is stated below. The *Toeplitz operator* with symbol a measure μ is defined by

$$T_\mu f(z) = \int_\Omega K^\alpha(z,w) f(w)\, d\mu(w),$$

where $K^{(\alpha)} := K(z,w)^{1-\alpha}$ is the Bergman reproducing kernel for the weighted Bergman space $A^2{}_\alpha(\Omega) \subset L^2(\Omega, dV^\alpha)$, and $dV^\alpha = C_\alpha^{-1} K(z,z)^\alpha dV(z)$ is normalized weighted Lebesgue measure. The *Berezin symbol* of the measure μ is defined by

$$\tilde{\mu}_\alpha(z) = \int_\Omega |k_z(w)|^{2(1-\alpha)}\, d\mu(w),$$

and $k_z(w) = K(w,z)/K(z,z)^{1/2}$.

[51, Theorem C] For a finite positive Borel measure μ on a bounded symmetric domain Ω, and $p \geq 1$, the following are equivalent:

1. The Toeplitz operator T_μ belongs to $\mathcal{S}_p(A^2{}_\alpha(\Omega))$
2. The Berezin transform $\tilde{\mu}_\alpha$ belongs to $L^p(\Omega, K(z,z)dV(z))$

4.2 More recent results

We need the following definitions in the statements of the results of this section. For a composition operator with symbol φ, the Berezin transform B_φ^α of φ is defined to be the Berezin symbol of the pullback measure $V_\varphi^\alpha(E) = V^\alpha(\varphi^{-1}(E))$ for $E \subset \Omega$. More precisely,

$$(B_\varphi^\alpha(z))^2 = \int_\Omega K^{(\alpha)}(z,z)^{-1}|K^{(\alpha)}(z,w)|^2\, dV_\varphi^\alpha(z).$$

Another useful transform can be defined using the Bergman metric is

$$b_\varphi^\alpha(z,r) = V_\varphi^\alpha(E(z,r))|E(z,r)|^{\alpha-1}$$

where $\beta(z,w)$ is the distance function on Ω arising from the Bergman metric, $E(z,r) = \{w \in \Omega : \beta(z,w) < r\}$ is the ball in that metric, and $|E| = \int_E dV$.

We let λ denote the measure $C_\alpha K^{(\alpha)}(z,z)dV^\alpha(z) = K(z,z)dV(z)$.

[30, Theorem 1.1] Let Ω be a smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$, and let $2n/(n+1) < p < \infty$. Then $C_\varphi \in \mathcal{S}_p(A^2(\Omega))$ if and only if $B_\varphi \in L^p(\Omega, d\lambda)$.

[32, Theorem 1.1] Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and suppose $\alpha < \alpha_\Omega$.

1. For $0 < p < \infty$, $C_\varphi \in \mathcal{S}_{2p}(A^2{}_\alpha(\Omega))$ if and only if $b_\varphi^\alpha \in L^p(\Omega, d\lambda)$.
2. For $2(1-\alpha_\Omega)/(1-\alpha) < p < \infty$, $C_\varphi \in \mathcal{S}_p(A^2{}_\alpha(\Omega))$ if and only if $B_\varphi^\alpha \in L^p(\Omega, d\lambda)$.

A very fertile area of proposed investigation would be to establish trace ideal criteria for composition operators in Hardy spaces of bounded symmetric domains and of strongly pseudoconvex domains. Nothing has been done about this even in the context of the unit ball or the unit polydisk.

For the sake of completeness, we mention the following companion results on boundedness and compactness.

[23, Theorem 3.1] Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and suppose $\alpha < \alpha_\Omega$. Then for $1 \le p < \infty$,

1. C_φ is bounded on $A^p{}_\alpha(\Omega))$ if and only if $b_\varphi^\alpha \in L^\infty(\Omega)$
2. C_φ is compact on $A^p{}_\alpha(\Omega))$ if and only if $b_\varphi^\alpha \in C_0(\Omega)$

[23, Theorem 3.2] Let Ω be a bounded symmetric domain in $\mathbf{C}^n$, and suppose $\alpha < \alpha_\Omega$. Then for $1 \le p < \infty$,

1. C_φ is bounded on $A^p{}_\alpha(\Omega))$ if and only if $B_\varphi^\alpha \in L^\infty(\Omega)$
2. C_φ is compact on $A^p{}_\alpha(\Omega))$ if and only if $B_\varphi^\alpha \in C_0(\Omega)$

4.3 Techniques for bounded symmetric domains

The following are some of the techniques used in the proofs of the above results on bounded symmetric domains. I refer to the indicated references for a statement and discussion of these techniques.

[2] Asymptotic properties of the Bergman kernel and metric

[13] Forelli-Rudin type inequalities for the quantity

$$I_{\alpha,c}(z) = \int_\Omega K(w,w)^\alpha |K(z,w)|^{1-\alpha+c}\, dV(w)$$

[9] Atomic decomposition of Bergman spaces

5 Epilogue

The following four projects for further study are suggested by the discussion in this paper. The first three involve Schatten class operators on the Hardy spaces, and their execution would complete the table in 1.4.1, while the fourth involves boundedness and compactness on the Bergman spaces and would complete the table in 1.4.2. The author welcomes hearing of any progress which has been made on any of these problems.

1. Study the boundedness, compactness, trace ideal criteria, and weak compactness of holomorphic composition operators on the appropriate Hardy spaces of a bounded symmetric domain. Here the main tool would be the Jordan algebra approach as set forth in [14] and as used in the study of Hankel operators, for example in [1].

2. Study the trace ideal criteria, and the weak compactness of holomorphic composition operators on the appropriate Hardy spaces of the unit polydisk. Here the main tool would be the multiparameter harmonic analysis, [4], [5], as used in the study of the small Hankel operator in [33].

3. Study the trace ideal criteria for holomorphic composition operators on the Hardy space $\mathcal{H}^2(\Omega)$ of a strongly pseudoconvex domain, or of the unit ball. Here the main tool might be the analysis on the Heisenberg group, for example as in [16], and the duality of Bergman spaces, for example as in [50].

4. Study the boundedness and compactness for holomorphic composition operators on the Bergman spaces $\mathcal{H}^p(\Omega)$ of a strongly pseudoconvex domain, using the Carleson measure conditions, as in [7].

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