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Contractive projections and operator spaces

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Abstract. We define a family of Hilbertian operator spaces H_n^k, $1 \leqslant k \leqslant n$, containing the row and column Hilbert spaces R_n, C_n and show that an atomic subspace $X \subset \mathrm{B}(H)$ which is the range of a contractive projection on $\mathrm{B}(H)$ is isometrically completely contractive to an ℓ^∞-sum of the H_n^k and Cartan factors of types 1 to 4. We also give a classification up to complete isometry of w*-closed atomic JW^*-triples which have no infinite-dimensional rank 1 w*-closed ideal. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Projecteurs contractifs et espaces d'opérateurs

Résumé. *Nous définissons une famille d' espaces d'opérateurs hilbertiens H_n^k, $1 \leqslant k \leqslant n$, contenant les espaces de Hilbert en ligne et en colonne R_n, C_n et nous démontrons qu'un espace d'opérateurs atomique qui est l'image d'un projecteur contractif sur $\mathrm{B}(H)$ est isométrique et complètement contractif à une somme directe d'espaces H_n^k et de facteurs de Cartan de types 1 à 4. De plus, nous donnons une classification, à isométrie complète près, des JW^*-triples qui sont atomiques et faiblement*-fermés et ne contiennent pas de facteur de Cartan de rang 1 et de dimension infinie comme faiblement*-fermé idéal.* © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Il est entendu depuis un certain temps que l'image d'un projecteur contractif, ou d'un projecteur complètement contractif sur $\mathrm{B}(H)$ peut être identifiée, à isométrie près, avec diverses structures algébriques *voir* [3,6,8]). D'un autre côté, l'image d'un projecteur complètement contractif sur $\mathrm{B}(H)$ étant la même chose qu'un objet *injectif* dans la catégorie des espaces d'opérateurs, il est naturel de chercher une classification des espaces injectifs à une isométrie complète près, ou bien à un isomorphisme complet près. Quelques exemples qui illustrent cette idée sont [4,15,14].

Dans cette Note, nous étudions la structure des espaces d'opérateurs qui sont l'image d'un projecteur contractif sur $\mathrm{B}(H)$. Ces derniers sont appelés 1-*mixtes injectifs* dans la terminologie de [11]. Les théorèmes suivants donnent une certaine « classification » des espaces 1-*mixtes injectifs* atomiques, et en particulier, une réponse partielle au problème 3.3 de [11], et une classification à une isométrie complète près de certains JW^*-triples.

Les démonstrations des théorèmes 1 et 2 sont données en grandes lignes dans la version anglaise ci-dessous. Les détails et la démonstration du théorème 3 seront publiés ailleurs [10].

Note présentée par Gilles PISIER.

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THÉORÈME 1. – *Il existe une famille de* 1*-mixtes injectifs hilbertiens* (*constructible explicitement*) H_n^k, $1 \leqslant k \leqslant n$, *de dimension* $n < \infty$, *avec les propriétés suivantes* :

(a) H_n^k *est un sous-triple, c'est-à-dire, stable pour le produit* $ab^*c + cb^*a$, *du facteur de Cartan de type* 1 *des matrices complexes à* $\binom{n}{k}$ *lignes et* $\binom{n}{n-k+1}$ *colonnes.*

(b) *Tout* JW^**-triple irréductible de rang* 1 *et dimension* $n < \infty$ *est complètement contractif à un des espaces* H_n^k.

Un espace de Banach dual est dit *atomique* si la boule unité du prédual est l'enveloppe convexe fermée pour la norme de ses points extrémaux.

THÉORÈME 2. – *Supposons que* X *soit l'image d'un projecteur contractif sur* $\mathrm{B}(H)$.

(a) *Si* X *est atomique, alors il existe une isométrie complètement contractive de* X *sur une somme directe de facteurs de Cartan des types* 1 *à* 4 *et d' espaces* H_n^k.

(b) *De plus, si* X *est irréductible et de dimension infinie, il y a une isométrie complètement contractive de* X *sur un facteur de Cartan de type* 1 *à* 4.

Si $\mathrm{B}(H,K)$ est un facteur de Cartan de type 1, on pose $\mathrm{diag}(\mathrm{B}(H,K),\mathrm{B}(K,H)) := \{(x,x^{\mathrm{t}}) : x \in \mathrm{B}(H,K)\}$, où x^{t} signifie la transposition par rapport à des bases orthonormales fixées de H et K. Pour une dimension n fixée et chaque $j = 1,\ldots,m$, soient H_j un espace de Hilbert de dimension n et $\mathcal{B}_j = \{e_{j,1},\ldots,e_{j,n}\}$ une base orthonormale pour H_j. On pose

$$\mathrm{diag}\big(\{H_j,\mathcal{B}_j\}\big) := \left\{\left(\sum_k \alpha_k e_{1k}, \sum_k \alpha_k e_{2k}, \ldots, \sum_k \alpha_k e_{mn}\right) : \alpha_k \in \mathbb{C},\ 1 \leqslant k \leqslant m\right\}.$$

THÉORÈME 3. – *Tout* JW^**-triple irréductible atomique et faiblement***-fermé, qui n'est pas de dimension infinie et de rang* 1, *est complètement isométrique soit à un facteur de Cartan de type* 1–4, *soit à un des espaces* $\mathrm{diag}(\mathrm{B}(H,K),\mathrm{B}(K,H))$, *soit enfin à* $\mathrm{diag}(H_n^{k_1},\ldots,H_n^{k_m})$, *où* $k_1 > k_2 > \cdots > k_m$.

1. Introduction

It has been known for some time that the range of a contractive projection, or of a completely contractive projection, on $\mathrm{B}(H)$ can be identified, up to a Banach isometry, with various algebraic structures (*see* [3], [6], and [8]).

Since the injectives in the category of operator spaces are precisely those subspaces of $\mathrm{B}(H)$ which are the range of a completely bounded projection, it is natural to ask for a classification of injectives up to completely bounded isomorphism or complete isometry. Some examples in the literature which illustrate this are [4], [15], and [14].

In this paper, we investigate the structure of operator spaces which are the range of a contractive projection on $\mathrm{B}(H)$. These are known as 1-mixed injectives in operator space parlance. We define a family of Hilbertian operator spaces H_n^k, $1 \leqslant k \leqslant n$, generalizing the row and column Hilbert spaces R_n, C_n and show that an atomic operator space $X \subset \mathrm{B}(H)$ which is the range of a contractive projection on $\mathrm{B}(H)$ is isometric and completely contractive to an ℓ^∞-sum of these spaces and Cartan factors of types 1 to 4. Such morphisms are said to be *complete semi-isometries* and they preserve 1-mixed injectivity [11]. Hence we have a kind of "classification" of atomic 1-mixed injectives, partially answering Problem 3.3 in [11]. We also classify up to complete isometry all atomic w*-closed JW^*-triples which have no infinite-dimensional w*-closed ideals of rank 1.

2. Main results and reduction

An *operator space* is defined to be a subspace of $\mathrm{B}(H)$ (*see* [7], [2], [12], [13]). A JC^*-triple (resp. JW^*-triple) is a subspace of a C^*-algebra which is closed under the *triple product* $\{x\ y\ z\} =$

$(1/2)(xy^*z + zy^*x)$ (resp. and is a Banach dual space). A JC^*-triple is called *ternary* if it is closed under the product ab^*c. A Banach space is *irreducible* if it has no M-summand, and *atomic* if it has a predual whose unit ball is the closed convex hull of its extreme points. We shall outline the proofs of Theorems 1 and 2. Details and the proof of Theorem 3 will be published elsewhere [10].

THEOREM 1.– *There is a family of Hilbertian* 1*-mixed injective operator spaces* (*which can be explicitly constructed*) H_n^k, $1 \leqslant k \leqslant n$, *of dimension* $n < \infty$, *with the following properties*:

(a) H_n^k *is a subtriple of the Cartan factor of type* 1 *consisting of all* $\binom{n}{k}$ *by* $\binom{n}{n-k+1}$ *complex matrices.*

(b) *Every irreducible* JW^**-triple of rank* 1 *and dimension* n *is isometrically completely contractive to some* H_n^k.

THEOREM 2.– *Assume* X *is the range of a contractive projection on* $\mathrm{B}(H)$.

(a) *If* X *is atomic, then there is a completely contractive isometry from* X *onto to a direct sum of Cartan factors of types* 1 *to* 4 *and the spaces* H_n^k.

(b) *Assume* X *is infinite-dimensional, atomic, and irreducible. Then there is a completely contractive isometry from* X *onto a Cartan factor of type* 1–4.

If $\mathrm{B}(H,K)$ is a Cartan factor of type 1, then $\operatorname{diag}(\mathrm{B}(H,K),\mathrm{B}(K,H)) := \{(x, x^{\mathrm{t}}) : x \in \mathrm{B}(H,K)\}$, where the transpose is respect to fixed orthonormal bases for H and K. For a fixed dimension n and each $j = 1,\dots,m$, let H_j be a Hilbert space of dimension n with a specified orthonormal basis $\mathcal{B}_j = \{e_{j,1},\dots,e_{j,n}\}$. Then

$$\operatorname{diag}\bigl(\{H_j,\mathcal{B}_j\}\bigr) := \left\{\left(\sum_k \alpha_k e_{1k}, \sum_k \alpha_k e_{2k},\dots, \sum_k \alpha_k e_{mn}\right) : \alpha_k \in \mathbb{C}, 1 \leqslant k \leqslant m\right\}.$$

THEOREM 3.– *Every atomic irreducible and w*-closed* JW^**-triple, not infinite-dimensional of rank* 1, *is completely isometric to either a Cartan factor of type* 1–4 *or to one of the spaces* $\operatorname{diag}(\mathrm{B}(H,K),\mathrm{B}(K,H))$ *or* $\operatorname{diag}(H_n^{k_1},\dots,H_n^{k_m})$, *where* $k_1 > k_2 > \cdots > k_m$.

By [8], Theorem 2, if X is the range of a contractive projection, there is a linear isometry $\mathcal{E}_0$ from X onto a JW^*-subtriple $Y \subset A := \mathrm{B}(H)^{**}$ of the form $\mathcal{E}_0(x) = pxq$ for suitable projections p, q in the von Neumann algebra A. It is clear that $\mathcal{E}_0$ is completely contractive and that Y is atomic. By the structure theorem for atomic JW^*-triples in [5], $Y = \bigoplus_\alpha Y_\alpha$ is the ℓ^∞ direct sum of a family of subtriples Y_α, each of which is triple isomorphic to a Cartan factor of one of the types 1–4. All triple isomorphisms are isometries as shown in [9]. Theorems 1 and 2 therefore follow from the following proposition, whose proof will be sketched in the following sections in the case of finite dimensions.

PROPOSITION 1.– *For every atomic irreducible* JW^**-subtriple* Y *of a von Neumann algebra* A, *there is a completely contractive triple isomorphism to some* H_n^k *or to a Cartan factor of type* 1, 2, 3 *or* 4.

3. Cartan factors of types 2, 3 and 4

JW^*-subtriples of a von Neumann algebra A which are triple isomorphic to a Cartan factor of type 3 or 4 have a unital Jordan *-algebra structure. Each possesses a partial isometry v defining an isotope C^*-algebra $A_2(v)$ (the Peirce-2 space of A) in which we can compute the C^*-algebra generated by Y, which is shown to be *-isomorphic and thus completely isometric to the C^*-envelope of a Cartan factor. The product in the isotope C^*-algebra is given by $a \cdot b = av^*b$ and the involution by $a^\sharp = va^*v$. The C^*-algebra $A_2(v)$ with this product and involution will be denoted by $A_2(v)^{(v)}$. This passage to isotopes is easily seen to be a complete isometry if $vv^*Yv^*v = Y$. A JW^*-triple which is isomorphic to a Cartan factor of type 2 is not an algebra. However, it has a similar partial isometry outside of Y which gives an isotope in which we can compute the C^*-algebra generated by Y as long as $\dim Y > 6$ (in the case $\dim Y \leqslant 6$, Y is also a factor of type 1 or 4, so this is covered elsewhere).

Type 4. – A *spin system* is a subset $\mathcal{S} = \{1, s_1, \ldots, s_k\}$ of self-adjoint elements of $\mathrm{B}(H)$ containing the unit and satisfying $s_i s_j + s_j s_i = \delta_{ij} 2$. A *spin factor*, or Cartan factor of type 4, is a closed self-adjoint subspace of $\mathrm{B}(H)$ which is the closed linear span of a spin system.

Suppose now that the Y in the statement of Proposition 1 is triple isomorphic to a Cartan factor of type 4. Then Y contains a spin grid of partial isometries $\{u_j, \tilde{u}_j : j \in J\}$ or $\{u_j, \tilde{u}_j : j \in J\} \cup \{u_0\}$, depending on whether the dimension of Y is even or odd. We refer to [5] for grid properties.

LEMMA 2. – *Let $v = i(u_1 + \tilde{u}_1)$. A spin system in the C^*-algebra $A_2(v)^{(v)}$ which linearly spans $Y = Y_2(v)$ is given by $\{v,\ s_j = u_j + \tilde{u}_j,\ t_k = i(u_k - \tilde{u}_k) : j \in J \setminus \{1\},\ k \in J\}$ together with u_0 if the grid is odd dimensional.*

Since any spin factor is well-known and easily seen to be determined up to complete isometry by the cardinality of the underlying spin system, this proves Proposition 1 in the type 4 case.

Type 3. – Suppose ψ is a triple isomorphism from Y onto $S_n(\mathbb{C})$ ($n \times n$ symmetric matrices). Y is spanned by an Hermitian grid (*cf.* [5]) $\{u_{ij} : i, j \in I,\ u_{ij} = u_{ji}\}$ of partial isometries, $\psi(u_{ij}) = E_{ij} + E_{ji}$ if $i \neq j$ and $\psi(u_{ii}) = E_{ii}$, where $\{E_{ij}\}$ is the canonical system of matrix units for $M_n(\mathbb{C})$.

Now define $v = \sum u_{ii}$, $e_{ij} = u_{ii} \cdot u_{ij}$, and $e_{ii} = u_{ii}$, where we use $a \cdot b$ to denote the associative product in $A_2(v)^{(v)}$. The following lemma shows that ψ is a complete isometry.

LEMMA 3. – *The collection $\{e_{ij}\}$ forms a system of matrix units in $A_2(v)^{(v)}$ such that $u_{ij} = e_{ij} + e_{ji}$. Thus ψ extends to a *-isomorphism $\widetilde{\psi} : \mathrm{sp}_{\mathbb{C}}\{e_{ij}\} \to \mathrm{sp}_{\mathbb{C}}\{E_{ij}\} = M_n(\mathbb{C})$ satisfying $\widetilde{\psi}(e_{ij}) = E_{ij}$.*

Type 2. – Let $\{u_{ij}\}$ be a symplectic grid (*cf.* [5]) of partial isometries in Y. Let $\psi : Y \to A_n(\mathbb{C})$ (the anti-symmetric matrices) be the triple isomorphism determined by $\psi(u_{ij}) = U_{ij}$, where $\{U_{ij}\} = E_{ij} - E_{ji}$ denotes the canonical basis for $A_n(\mathbb{C})$. The following lemma shows ψ is a complete isometry.

LEMMA 4. – *Assume $n > 5$.*

(a) *For any indices i, j, k, ℓ, m, we have $u_{ik} u_{k\ell}^* u_{i\ell} = u_{ij} u_{jm}^* u_{im}$, and for $1 \leqslant i \leqslant n$, the elements e_{ii} unambiguously defined by $e_{ii} = u_{ij} u_{jm}^* u_{im}$ are non-zero orthogonal partial isometries in A.*

Let $v = \sum e_{kk}$ and let $\widetilde{Y}$ be the ternary envelope of Y and define, for $i \neq j$, $e_{ij} = e_{ii} e_{ii}^* u_{ij} e_{jj}^* e_{jj}$ (product in A). Then:

(b) *the Peirce projection $P_2(v) = vv^* \cdot v^* v$ of A fixes each element of Y, and is a completely isometric ternary isomorphism of $\widetilde{Y}$ into $A_2(v)^{(v)}$;*

(c) *$u_{ij} = e_{ij} - e_{ji}$ and ψ extends to a *-isomorphism $\widetilde{\psi} : \mathrm{sp}_{\mathbb{C}}\{e_{ij}\} \to \mathrm{sp}_{\mathbb{C}}\{E_{ij}\} = M_n(\mathbb{C})$ satisfying $\widetilde{\psi}(e_{ij}) = E_{ij}$, where $\{E_{ij}\}$ is the canonical system of matrix units for $M_n(\mathbb{C})$.*

4. Cartan factors of type 1

Let $\{u_{ij} : i \in I,\ j \in J\}$ be a rectangular grid for Y. We shall assume throughout this section that Y is triple isomorphic to $M_{m,n}(\mathbb{C})$, that is, $|I| = m$ and $|J| = n$ (*see* [5]). This case turns out to be more complicated than the others, especially in the rank 1 case, that is, if $m = 1$ or $n = 1$.

4.1. *The case of rank* > 1

LEMMA 5. – *Assume that $u_{i\ell}^* u_{ik} = 0$ (resp. $u_{i\ell} u_{ik}^* = 0$) for some fixed values of i, k, ℓ with $k \neq \ell$. Then*:

(a) *for all i, k, ℓ with $k \neq \ell$, $u_{i\ell}^* u_{ik} = 0$ (resp. $u_{i\ell} u_{ik}^* = 0$) and for all i, j, k with $i \neq j$, $u_{ik} u_{jk}^* = 0$ (resp. $u_{ik}^* u_{jk} = 0$);*

(b) *Y is ternary isomorphic and completely isometric to $M_{n,m}(\mathbb{C})$ (resp. $M_{m,n}(\mathbb{C})$).*

Let us now assume that Y is triple isomorphic to a Cartan factor of type 1 and rank at least 2, and without loss of generality by Lemma 5 that $u_{ik} u_{ij}^* \neq 0$ and $u_{ik}^* u_{ij} \neq 0$ for all i, j, k. Under this assumption we obtain the following:

PROPOSITION 6. – *Suppose that Y is of rank at least* 2.

(a) *Let $p = \sum_{i=1}^{m} \prod_{k=1}^{n} u_{ik} u_{ik}^{*}$, which is a sum of non-zero orthogonal projections. Then the map $Y \ni x \mapsto \psi(x) = px \in pY$ is a completely contractive triple isomorphism.*

(b) *Y is completely contractively isometric to either $M_{m,n}(\mathbb{C})$ or $M_{n,m}(\mathbb{C})$.*

4.2. *The case of rank* 1. – Let us denote a rectangular grid of rank 1 for Y by $\{u_1, \ldots, u_n\}$. By the grid properties and the identity $\|yy^*y\| = \|y\|^3$, Y is a Hilbert space with orthonormal basis $\{u_j\}_{j=1}^{n}$ (*see* [5], p. 306).

We shall denote, for $J = \{j_1, \ldots, j_i\} \subset \{1, 2, \ldots, n\}$, $u_{j_1}^* u_{j_1} u_{j_2}^* u_{j_2} \cdots u_{j_i}^* u_{j_i}$ by $(u^*u)_J$. By commutativity of the projections $u_k^* u_k$ we may and shall assume that $1 \leqslant j_1 < \cdots < j_i \leqslant n$. Similarly, $(uu^*)_J$ will denote $u_{j_1} u_{j_1}^* u_{j_2} u_{j_2}^* \cdots u_{j_i} u_{j_i}^*$. Define i_R to be the largest i such that $(uu^*)_J \neq 0$ for any J with $|J| = i$ and i_L to be the largest i such that $(u^*u)_J \neq 0$ for any J with $|J| = i$. The numbers i_R and i_L are well-defined and give a measure of how the JC^*-triple Y sits in its ternary envelope.

LEMMA 7. – *Define projections $p_R = \sum_{|J|=i_R} (uu^*)_J$ and $p_L = \sum_{|J|=i_L} (u^*u)_J$ and linear maps $\psi_R(x) = p_R x$ and $\psi_L(x) = x p_L$ of Y into A.*

(a) *The maps ψ_R and ψ_L are completely contractive triple isomorphisms of Y into A.*

(b) *In general, $i_R + i_L \geq n + 1$. Let $\psi = \psi_L$ and let $w_j = \psi(u_j) = u_j p$. Let i'_L and i'_R denote the corresponding indices for $w_1, \ldots, w_n$. Then $i'_L + i'_R = n + 1$.*

Thus, we may assume in the sequel that Y is of rank 1 and dimension n given by a rectangular rank 1 grid $\{u_1, \ldots, u_n\}$ and that $i_R + i_L = n + 1$. If $i_L = 1$ or if $i_R = 1$, then Y is completely isometric to the type 1 Cartan factors R_n or C_n by Lemma 5. Otherwise, we shall show that Y is completely contractively isometric to one of the spaces $H_n^{i_R}$ in Theorem 1. This will be achieved by constructing, from the given grid $\{u_j\}$, a rectangular grid whose linear span is a ternary JW^*-triple containing Y and which is ternary isomorphic to a Cartan factor of type 1, namely the $\binom{n}{i_L}$ by $\binom{n}{i_R}$ complex matrices.

Here is the construction. We define some elements which are indexed by an arbitrary pair of subsets I, J of $\{1, \ldots, n\}$ satisfying $|I| = i_R - 1$, $|J| = i_L - 1$. Note that the number of possible sets I is $\binom{n}{i_R - 1}$ $(= \binom{n}{i_L})$ and the number of such J is $\binom{n}{i_L - 1}$ $(= \binom{n}{i_R})$. Moreover, if $|I \cap J| = s \geq 0$, then $|(I \cup J)^{\mathrm{c}}| = s + 1$. Hence we may write $I = \{i_1, \ldots, i_k, d_1, \ldots, d_s\}$, $J = \{j_1, \ldots, j_\ell, d_1, \ldots, d_s\}$, where $I \cap J = \{d_1, \ldots, d_s\}$. Let us write $(I \cup J)^{\mathrm{c}} = \{c_1, \ldots, c_{s+1}\}$, and let us agree that the elements are ordered as follows: $c_1 < c_2 < \cdots < c_{s+1}$ and $d_1 < d_2 < \cdots < d_s$. Then we define

$$u_{IJ} = u_{I,J} = (uu^*)_{I \smallsetminus J} u_{c_1} u_{d_1}^* u_{c_2} u_{d_2}^* \cdots u_{c_s} u_{d_s}^* u_{c_{s+1}} (u^*u)_{J \smallsetminus I}. \tag{1}$$

In the special case of (1) where $I \cap J = \varnothing$, we have $s = 0$ and $u_{I,J}$ has the form $u_{I,J} = (uu^*)_I u_c (u^*u)_J$, where since $i_R + i_L = n + 1$, $I \cup J \cup \{c\} = \{1, \ldots, n\}$. We call such an element a "one", and denote it by $u_{I,c,J}$.

LEMMA 8. – *For any $c \in \{1, \ldots, n\}$, $u_c = \sum_{I,J} u_{I,J} = \sum_{I,J} u_{I,c,J}$, where the sum is over all disjoint I, J satisfying $|I| = i_R - 1$, $|J| = i_L - 1$ and not containing c.*

We next assign a *signature* to each "one" as follows: let the elements of I be $i_1 < i_2 < \cdots < i_p$ (where $p = i_R - 1$) and the elements of J be $j_1 < j_2 < \cdots < j_q$ (where $q = i_L - 1$)). Then $\varepsilon(I, k, J)$ is defined to be the signature of the permutation taking the n-tuple $(i_1, \ldots, i_p, k, j_1, \ldots, j_q)$ onto $(1, 2, \ldots, n)$. Using this, and by proving a unique decomposition of $u_{I,J}$ into "ones", one can define a signature $\varepsilon(I, J)$ for all elements $u_{I,J}$ in such a way that the following proposition holds.

PROPOSITION 9. – *The family $\{\varepsilon(IJ) u_{I,J}\}$ forms a rectangular grid which satisfies*

$$\varepsilon(IJ) u_{IJ} \left[\varepsilon(IJ') u_{IJ'}\right]^* \varepsilon(I'J') u_{I'J'} = \varepsilon(I'J) u_{I'J}. \tag{2}$$

Thus by Lemma 8, Y is completely isometric to a subtriple, which we call $H_n^{i_R}$, of a Cartan factor of type 1. This completes the proof of Proposition 1 and hence of Theorems 1 and 2.

Note that the spaces $H_n^{i_R}$ can be explicitly constructed. Simply index rows (resp. columns) by combinations J (resp. I) of $\{1,\dots,n\}$ of length i_R-1 (resp. i_L-1). Then define an orthonormal basis $\{U_i\}$ for $H_n^{i_R}$ by the requirement that U_i equals the sum of all elements $\varepsilon_{I,J}E_{J,I}$, where $I\cap J=\varnothing$ and $(I\cup J)^{\mathrm{c}}=\{i\}$. We now give some examples and conclude with some questions about the spaces H_n^k.

Example 1. – Suppose that $Y=\mathrm{sp}_{\mathbb{C}}\{u_1,u_2,u_3\}$ and $i_R=i_L=2$. Our Hilbert space H_3^2 is the 3×3 anti-symmetric matrices:

$$H_3^2=\left\{\begin{bmatrix}0 & -a & b\\ a & 0 & -c\\ -b & c & 0\end{bmatrix} : a,\, b,\, c\in\mathbb{C}\right\}.$$

Example 2. – If $n=4$ and $i_R=3$ we obtain the following Hilbertian subspace of 4×6 complex matrices.

$$H_4^3=\left\{\begin{bmatrix}0 & 0 & 0 & d & -c & b\\ 0 & -d & c & 0 & 0 & -a\\ d & 0 & -b & 0 & a & 0\\ -c & b & 0 & -a & 0 & 0\end{bmatrix} : a,\, b,\, c,\, d\in\mathbb{C}\right\}.$$

Problem 1. – Is H_n^k completely semi-isometric to R_n or C_n?

Problem 2. – What is the completely bounded Banach–Mazur distance $\mathrm{d_{cb}}(H_n^k,R_n)$?

Remark 1. – A positive answer to Problem 1 would completely solve Problem 3.3 in [11] in the affirmative. A negative answer to Problem 1 would imply a negative answer to the problem of Oikhberg and Rosenthal.

Remark 2. – After completing this paper, the authors discovered that the spaces H_n^k appear, in a slightly different form, in [1] in their solution to the contractive projection problem on the compact operators on a separable Hilbert space. Their methods and proofs are different from ours. In the special case that the projection is weak*-weak* continuous and H is separable, Theorem 2 can be derived from their results.

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