

## DERIVATIONS ON REAL AND COMPLEX JB\*-TRIPLES

TONY HO, JUAN MARTINEZ-MORENO, ANTONIO M. PERALTA  
AND BERNARD RUSSO

### 1. Introduction

At the regional conference held at the University of California, Irvine, in 1985 [24], Harald Upmeier posed three basic questions regarding derivations on JB\*-triples:

- (1) Are derivations automatically bounded?
- (2) When are all bounded derivations inner?
- (3) Can bounded derivations be approximated by inner derivations?

These three questions had all been answered in the binary cases. Question 1 was answered affirmatively by Sakai [17] for C\*-algebras and by Upmeier [23] for JB-algebras. Question 2 was answered by Sakai [18] and Kadison [12] for von Neumann algebras and by Upmeier [23] for JW-algebras. Question 3 was answered by Upmeier [23] for JB-algebras, and it follows trivially from the Kadison–Sakai answer to question 2 in the case of C\*-algebras.

In the ternary case, both question 1 and question 3 were answered by Barton and Friedman in [3] for complex JB\*-triples. In this paper, we consider question 2 for real and complex JBW\*-triples and question 1 and question 3 for real JB\*-triples. A real or complex JB\*-triple is said to have the *inner derivation property* if every derivation on it is inner. By pure algebra, every finite-dimensional JB\*-triple has the inner derivation property. Our main results, Theorems 2, 3 and 4 and Corollaries 2 and 3 determine which of the infinite-dimensional real or complex Cartan factors have the inner derivation property.

### 2. Background

We recall that a JB\*-algebra is a complete normed Jordan complex algebra (say  $\mathcal{A}$ ) endowed with a conjugate-linear algebra involution  $*$  satisfying  $\|U_x(x^*)\| = \|x\|^3$  for every  $x \in \mathcal{A}$ . Here, for every Jordan algebra  $\mathcal{A}$ , and every  $x \in \mathcal{A}$ ,  $U_x$  denotes the operator on  $\mathcal{A}$  defined by  $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$ , for all  $y \in \mathcal{A}$ .

A JB-algebra is a complete normed Jordan real algebra (say  $A$ ) satisfying the following two additional conditions for  $a, b \in A$ :

- (i)  $\|a^2\| = \|a\|^2$ .
- (ii)  $\|a^2\| \leq \|a^2 + b^2\|$ .

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It is due to Wright (see [25]) that the complexification of a JB-algebra is a JB\*-algebra under a unique norm extending the given norm on the JB-algebra. Conversely, the self-adjoint part of a JB\*-algebra is a JB-algebra under the restricted norm.

If  $H$  is a complex Hilbert space, then the real Banach space  $\mathcal{H}(H)$  of all bounded hermitian operators on  $H$  is a JB-algebra with respect to the Jordan product

$$x \circ y := \frac{1}{2}(xy + yx).$$

A uniformly (respectively weakly) closed unital real subalgebra of  $\mathcal{H}(H)$  is called a JC-algebra (respectively JW-algebra) on  $H$ . A norm (respectively weakly) closed (complex) Jordan\*-subalgebra of a  $C^*$ -algebra (respectively von Neumann algebra) is called a JC\*-algebra (respectively JW\*-algebra). For more details on JB-algebras and JB\*-algebras we refer the reader to [9].

We recall that a (complex) JB\*-triple is a complex Banach space  $\mathcal{J}$  with a continuous triple product  $\{\cdot, \cdot, \cdot\} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$  that is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and that satisfies the following conditions.

- (i) (Jordan identity)  $L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} - \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$  for all  $a, b, x, y, z$  in  $\mathcal{J}$ , where  $L(a, b)x := \{a, b, x\}$ .
- (ii) For all  $a \in \mathcal{J}$ , the map  $L(a, a)$  from  $\mathcal{J}$  to  $\mathcal{J}$  is a hermitian operator with non-negative spectrum.
- (iii)  $\|\{a, a, a\}\| = \|a\|^3$  for all  $a$  in  $\mathcal{J}$ .

It is worth mentioning that every  $C^*$ -algebra is a (complex) JB\*-triple with respect to  $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$ . Also, every JB\*-algebra is a JB\*-triple with respect to  $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$ . Conversely, every JB\*-triple with a unitary element  $u$  (that is,  $\{u, u, z\} = z$  for every  $z$ ) is a unital JB\*-algebra with product  $a \circ b = \{a, u, b\}$ , involution  $a^* = \{u, a, u\}$ , and unit  $u$ . We refer to [5, 15, 16] for recent surveys on the theory of JB\*-triples.

Following [11], we recall that a *real JB\*-triple* is a norm-closed real subtriple of a complex JB\*-triple. Given a real JB\*-triple  $J$ , there exists a unique complex JB\*-triple structure on the complexification  $\hat{J} = J \oplus iJ$ , and a unique conjugation (that is, conjugate-linear isometry of period 2)  $\tau$  on  $\hat{J}$  such that  $J = \hat{J}^r := \{x \in \hat{J} : \tau(x) = x\}$ . From this point of view, the real JB\*-triples are real forms of complex JB\*-triples.

The class of real JB\*-triples includes all JB-algebras [9], all real  $C^*$ -algebras [8], and all  $J^*B$ -algebras [2].

A *triple derivation* or simply a *derivation*  $\delta$  on a real or complex JB\*-triple  $U$  is a linear operator satisfying

$$\delta\{a, b, c\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\}$$

for all  $a, b, c \in U$ .

If  $U$  is a real or complex JB\*-triple, we can conclude from the Jordan identity that  $\delta(a, b) := L(a, b) - L(b, a)$  is a derivation, for every  $a, b \in U$ . An *inner triple derivation*  $\delta$  on  $U$  is a finite sum of derivations of the form  $\delta(a, b)$  ( $a, b \in U$ ), that is,

$$\delta = \sum_{i=1}^n \delta(a_j, b_j). \quad (2.1)$$

The *degree* of an inner derivation is the least number of terms in a representation of the form (2.1). Any derivation that is not inner is called *outer*.

REMARK 1. Let  $E$  be a real JB\*-triple and  $\delta$  a derivation on  $E$ . Then  $\delta$  can be extended to a derivation  $\hat{\delta}$ , on the complexification of  $E$ , defined by  $\hat{\delta}(x + iy) := \delta(x) + i\delta(y)$ .

It is due to Barton and Friedman [3] that every derivation on a complex JB\*-triple is automatically continuous, and so, by the previous comment, every derivation on a real JB\*-triple is also continuous.

### 3. Inner derivation property

We say that a real or complex JB\*-triple  $U$  has the *inner derivation property* if every derivation on  $U$  is inner.

By [14, Chapter 8], every finite-dimensional real or complex JB\*-triple has the inner derivation property. The next proposition shows that a real JB\*-triple has the inner derivation property whenever its complexification satisfies this property.

PROPOSITION 1. *Let  $E$  be a real JB\*-triple. Suppose that the complexification  $\hat{E}$  of  $E$  has the inner derivation property. Then  $E$  has the inner derivation property. Moreover, if  $M$  is a bound of the degree of all inner derivations of  $\hat{E}$ , then  $2M$  is a bound of the degree of all inner derivations of  $E$ .*

*Proof.* Suppose that  $E$  is a real JB\*-triple such that  $\hat{E}$  has the inner derivation property. Let  $\delta$  be a derivation of  $E$ . We denote by  $\hat{\delta}$  the derivation on  $\hat{E}$ , extending  $\delta$  to  $\hat{E}$ . Since  $\hat{E}$  has the inner derivation property, then  $\hat{\delta}$  is an inner derivation of degree  $n$ , that is,

$$\hat{\delta} = \sum_{k=1}^n \delta(a_k, b_k),$$

where  $a_k, b_k \in \hat{E}$ . Since  $\hat{E} = E \oplus iE$ , it follows that  $a_k = a_{k,1} + ia_{k,2}$  and  $b_k = b_{k,1} + ib_{k,2}$  for suitable  $a_{k,l}, b_{k,l} \in E$ ,  $l = 1, 2$  and  $k = 1, \dots, n$ .

Consider now  $x \in E$ . We can compute

$$\begin{aligned} \delta(a_k, b_k)x &= \delta(a_{k,1} + ia_{k,2}, b_{k,1} + ib_{k,2})x \\ &= \{a_{k,1} + ia_{k,2}, b_{k,1} + ib_{k,2}, x\} - \{b_{k,1} + ib_{k,2}, a_{k,1} + ia_{k,2}, x\} \\ &= \{a_{k,1}, b_{k,1}, x\} + \{a_{k,2}, b_{k,2}, x\} + i(\{a_{k,2}, b_{k,1}, x\} - \{a_{k,1}, b_{k,2}, x\}) \\ &\quad - \{b_{k,1}, a_{k,1}, x\} - \{b_{k,2}, a_{k,2}, x\} - i(\{b_{k,2}, a_{k,1}, x\} - \{b_{k,1}, a_{k,2}, x\}) \\ &= \delta(a_{k,1}, b_{k,1})(x) + \delta(a_{k,2}, b_{k,2})(x) \\ &\quad + i(\{a_{k,2}, b_{k,1}, x\} - \{a_{k,1}, b_{k,2}, x\} - \{b_{k,2}, a_{k,1}x\} + \{b_{k,1}, a_{k,2}x\}). \end{aligned}$$

Therefore

$$\begin{aligned} E \ni \delta(x) &= \hat{\delta}(x) = \sum_{k=1}^n \delta(a_{k,1} + ia_{k,2}, b_{k,1} + ib_{k,2})x \\ &= \left( \sum_{k=1}^n (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2})) \right) x \\ &\quad + i \sum_{k=1}^n (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,2}))(x) \end{aligned}$$

Since the elements  $a_{k,l}, b_{k,l} \in E$ , we have

$$\left( \sum_{k=1}^n (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2})) \right) (E) \subset E$$

and

$$\left( i \sum_{k=1}^n (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,1})) \right) (E) \subset iE.$$

Therefore

$$\left( \sum_{k=1}^n (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,1})) \right) (x) = 0$$

for all  $x \in E$ . Thus

$$\delta(x) = \hat{\delta}(x) = \sum_{k=1}^n (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2}))(x)$$

for all  $x \in E$ , which proves that  $\delta$  is an inner derivation with degree at most  $2n$ .  $\square$

From Proposition 1, it is easy to see that if  $E$  is a real JB $^*$ -triple which does not satisfy the inner derivation property, then its complexification also does not satisfy the inner derivation property.

### 3.1. Reversible unital JB $^*$ -algebras

We recall that the (complex) *type 1 Cartan factor* can be defined as the complex Banach space  $BL(H, K)$  of all bounded linear operators between two complex Hilbert spaces  $H$  and  $K$ , with triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a).$$

Next we give a brief description of the (complex) Cartan factors of type 2 and 3. Let  $H$  be a complex Hilbert space equipped with a conjugation (conjugate-linear isometry of period 2)  $j : H \longrightarrow H$ ; then for any  $z \in B(H)$  we can define its transpose as  $z^t := jz^*j$ . The *type 2 Cartan factor* coincides with the Banach space of all  $t$ -skew symmetric elements in  $B(H)$  ( $z^t = -z$ ), and the *type 3 Cartan factor* is defined as the Banach space of all  $t$ -symmetric elements of  $B(H)$  ( $z^t = z$ ). The triple product of these Cartan factors is the restriction of the triple product in  $B(H)$ .

We recall that a JC-algebra (or a JC $^*$ -algebra)  $A$  is said to be *reversible* if  $x_1 x_2 \dots x_n + x_n x_{n-1} \dots x_1 \in A$ , for all  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in A$ .

**PROPOSITION 2.** *Cartan factors of type 1 with  $\dim H = \dim K$ , Cartan factors of type 2 with  $\dim H$  even, or infinite, and all Cartan factors of type 3 are reversible JW $^*$ -algebras.*

*Proof.* Let  $C^3$  be a type 3 Cartan factor. Since  $x^t = x$  for all  $x \in C^3$ , we have  $(x_1 \dots x_n + x_n \dots x_1)^t = x_n \dots x_1 + x_1 \dots x_n \in C^3$ .

Let  $C^2$  be a type 2 Cartan factor with  $\dim H$  even or infinite. Then  $C^2$  contains a

distinguished unitary

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & \cdots \\ -1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & -1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

In this case we can provide a new  $C^*$ -algebra structure for  $B(H)$  with product  $a \cdot b = au^*b$  and involution  $a^\parallel = ua^*u$  in which  $C^2$  becomes a  $JC^*$ -subalgebra under  $a \circ b = (a \cdot b + b \cdot a)/2$ . With this Jordan product,  $C^2$  is reversible since

$$\begin{aligned} & (x_1 u^* x_2 \dots u^* x_n + x_n u^* \dots x_2 u^* x_1)^\dagger \\ &= (-1)^{n+(n-1)} (x_n u^* x_{n-1} \dots x_2 u^* x_1 + x_1 u^* x_2 \dots x_{n-1} u^* x_n) \\ &= -(x_1 u^* x_2 \dots u^* x_n + x_n u^* \dots x_2 u^* x_1). \end{aligned} \quad \square$$

We recall that if  $\mathcal{A}$  is an algebra, then a *derivation*  $D$  of  $\mathcal{A}$  is a linear mapping  $D : \mathcal{A} \rightarrow \mathcal{A}$  satisfying  $D(ab) = D(a)b + aD(b)$ , for all  $a, b \in \mathcal{A}$ . If  $\mathcal{A}$  is a Jordan algebra, an *inner algebra derivation* of  $\mathcal{A}$  is a finite sum of commutators of the form  $[L_a, L_b]$  for some  $a, b \in \mathcal{A}$ , where  $L_a x := a \circ x$ . For an inner algebra derivation  $D$ , the *degree* of  $D$  is the least natural number  $n$  satisfying  $D = \sum_{i=1}^n [L_{a_i}, L_{b_i}]$ .

**LEMMA 1.** *Let  $Z$  be a JB\*-algebra, with unit  $u$ , regarded as a complex JB\*-triple. If  $\delta$  is a triple derivation of  $Z$ , then  $L_{\delta(u)}$  is an inner triple derivation of  $Z$  of degree 1.*

*Proof.* Simply note that for every triple derivation  $\delta$  of  $Z$ , we have

$$\begin{aligned} \delta u &= \delta\{u, u, u\} = \{\delta u, u, u\} + \{u, \delta u, u\} + \{u, u, \delta u\} \\ &= 2\{\delta u, u, u\} + \{u, \delta u, u\} = 2\delta u \circ u + (\delta u)^* \end{aligned}$$

and hence

$$(\delta u)^* = -\delta u.$$

Now consider

$$\begin{aligned} L_{\delta u} z &= \delta u \circ z = \frac{1}{2}(\delta u \circ z - (-\delta u) \circ z) \\ &= \frac{1}{2}(\delta u \circ z - (\delta u)^* \circ z) = \frac{1}{2}(\{\delta u, u, z\} - \{u, \delta u, z\}); \end{aligned}$$

it follows that  $L_{\delta u}$  is an inner triple derivation of degree 1.  $\square$

**LEMMA 2** [3, p. 263]. *Let  $Z$  be a unital JB\*-algebra and  $D$  be an algebra derivation of  $Z$  that commutes with the involution of  $Z$ . Then  $D$  is a triple derivation of  $Z$ .*

*Conversely, if  $Z$  is a JB\*-triple with a unitary element  $u$  and  $\delta$  is a triple derivation of  $Z$ , then  $\delta - L_{\delta u}$  is an algebra derivation of  $Z$  that commutes with the involution on  $Z$ . In particular, if  $\delta$  is an inner derivation of degree 1, that is,  $\delta = \delta(x, y)$ , then*

$$\delta - L_{\delta(u)} = \frac{1}{2}([L_{x+x^*}, L_{y+y^*}] + [L_{-i(x-x^*)}, L_{-i(y-y^*)}]).$$

*Proof.* The first statement is clear. To prove the second one, let  $\delta$  be a triple

derivation of  $Z$ . It is easy to check that

$$\begin{aligned}
(\delta - L_{\delta u})(x \circ y) &= \delta\{x, u, y\} - \{\delta u, u, \{x, u, y\}\} \\
&= \{\delta x, u, y\} + \{x, \delta u, y\} + \{x, u, \delta y\} - \{\delta u, u, \{x, u, y\}\} \\
&= \{\delta x, u, y\} + \{x, \delta u, y\} + \{x, u, \delta y\} \\
&\quad - \{\{\delta u, u, x\}, u, y\} + \{x, \{u, \delta u, u\}, y\} - \{x, u, \{\delta u, u, y\}\} \\
&= \delta x \circ y + \{x, \delta u, y\} + x \circ \delta y \\
&\quad - (\delta u \circ x) \circ y + \{x, (\delta u)^*, y\} - x \circ (\delta u \circ y)
\end{aligned}$$

(applying  $(\delta u)^* = -\delta u$ )

$$\begin{aligned}
&= (\delta - L_{\delta u})(x) \circ y + \{x, \delta u, y\} + x \circ (\delta - L_{\delta u})(y) - \{x, \delta u, y\} \\
&= (\delta - L_{\delta u})(x) \circ y + x \circ (\delta - L_{\delta u})(y).
\end{aligned}$$

Thus  $\delta - L_{\delta u}$  is an algebra derivation.

The verification of the last formula is left to the reader.  $\square$

By [22, Theorem 13] (see also [1, p. 255], each JW-algebra  $A$  admits a decomposition into weakly closed ideals of the form

$$A = I_{\text{fin}} \oplus I_{\infty} \oplus II_1 \oplus II_{\infty} \oplus III.$$

See [22] and [1] for the meaning of these symbols. A JW-algebra  $A$  is called *properly non-modular* if its modular part  $I_{\text{fin}} \oplus II_1$  vanishes.

In 1980, Upmeier showed that each algebra derivation on a properly non-modular JW-algebra is the sum of six commutators of the form  $[L_a, L_b]$  [23, Theorem 3.8], and each algebra derivation on a reversible JW-algebra of type  $I_{\text{fin}}$  is the sum of five commutators [23, Theorem 3.9].

The proof of the following theorem is implicitly contained in [23], and we include it here for completeness.

**THEOREM 1.** *Let  $A$  be a reversible JW-algebra of type  $II_1$ . Then each derivation of  $A$  is a sum of at most 140 commutators of the form  $[L_a, L_b]$ .*

*Proof.* Let  $A$  be a reversible JW-algebra of type  $II_1$ . We denote by  $\mathcal{U}(\mathcal{A})$  its complex enveloping von Neumann algebra (the smallest von Neumann algebra containing  $A$ ). By [1, Theorem 8],  $\mathcal{U}(\mathcal{A})^+$  (that is,  $\mathcal{U}(\mathcal{A})$  with the Jordan product  $w_1 \circ w_2 = (w_1 w_2 + w_2 w_1)/2$ ) is also of type  $II_1$ . Thus if we follow the proof of [23, Theorem 3.10], it follows that each derivation of  $A$  has the form  $D(x) = \text{ad}(w)(x) := [w, x]$  ( $x \in A$ ), where  $w = -w^* \in \mathcal{U}(\mathcal{A})$ . Moreover, since  $\mathcal{U}(\mathcal{A})^+$  is of type  $II_1$ ,  $w$  is the sum of ten commutators in  $\mathcal{U}(\mathcal{A})$  (see [7, Theorem 2.3]), so that each derivation of  $A$  has the form

$$D = \sum_{j=1}^{10} \text{ad}([w_{1,j}, w_{2,j}]).$$

Since  $A$  is the self-adjoint part of  $\mathcal{R}(\mathcal{A})$  [19], where  $\mathcal{R}(\mathcal{A})$  is the real enveloping algebra of  $A$ , we have, by [20, Lemma 6.1; 21, Lemma 2.3, Theorem 2.4],  $\mathcal{U}(\mathcal{A}) = \mathcal{R}(\mathcal{A}) + i\mathcal{R}(\mathcal{A})$ .

Hence every element  $w_{l,j}$  is the sum  $w_{l,j} = u_{l,j} + iv_{l,j}$ , where  $u_{l,j}, v_{l,j} \in \mathcal{R}(\mathcal{A})$ .

Since, for every  $u_l, v_l \in \mathcal{R}(\mathcal{A})$ , the equalities

$$\begin{aligned}[u_1 + iv_1, u_2 + iv_2] &= [u_1, u_2] - [v_1, v_2] + i([u_1, v_2] + [v_1, u_2]) \\ [u_1 + iu_2, x] &= [u_1, x] + i[u_2, x]\end{aligned}$$

hold for all  $x \in A$ , and since  $D$  maps  $A$  in  $A$ , we have

$$\sum_{j=1}^{10} [[u_{1,j}, v_{2,j}] + [v_{1,j}, u_{2,j}], x] = 0$$

for all  $x \in A$ . Thus

$$D = \text{ad}(w) = \sum_{j=1}^{10} \text{ad}([u_{1,j}, u_{2,j}] - [v_{1,j}, v_{2,j}]) = \sum_{j=1}^{20} \text{ad}([z_{1,j}, z_{2,j}]),$$

where  $z_{i,j} \in \mathcal{R}(\mathcal{A})$  and  $w = \sum_{j=1}^{20} [z_{1,j}, z_{2,j}]$ .

Our next goal is to prove that every element  $[z_{1,j}, z_{2,j}]$  is a finite sum of commutators of elements in  $A$ .

Let  $z_{1,j}, z_{2,j} \in \mathcal{R}(\mathcal{A})$ , and  $l \in \{1, 2\}$ . We denote by  $z_{l,j}^s$  (respectively  $z_{l,j}^a$ ) the symmetric part (respectively the skew-symmetric part) of  $z_{l,j}$ . Since, for every  $j$ ,  $[z_{1,j}^a, z_{2,j}^s]$  and  $[z_{1,j}^s, z_{2,j}^a]$  are symmetric elements and  $w^* = -w$ , we deduce that

$$w = \sum_{j=1}^{20} [z_{1,j}^s, z_{2,j}^s] + [z_{1,j}^a, z_{2,j}^a].$$

Again, since  $A$  is the self-adjoint part of  $\mathcal{R}(\mathcal{A})$ , we have  $z_{1,j}^s, z_{2,j}^s \in \mathcal{A}$ . Therefore it is enough to show that every commutator  $[z_{1,j}^a, z_{2,j}^a]$  is a finite sum of commutators of elements in  $A$ .

By [4, p. 121],  $\mathcal{R}(\mathcal{A})$  is isomorphic to the matrix algebra  $M_2(B)$ , where  $B$  is a suitable real associative  $*$ -algebra.

If we follow the proof of [23, Lemma 3.11], it follows that each commutator of skew-symmetric elements in  $M_2(B)$  has the form

$$\begin{pmatrix} a & -c^* \\ c & b \end{pmatrix},$$

with

$$a + b = [a_1, a_2] + [b_1, b_2] + [c_1, c_2] + [d_1, d_2],$$

where  $a_j, b_j$  and  $c_j$  are skew-symmetric elements in  $B$ , while  $d_1$  and  $d_2$  are symmetric elements in  $B$ .

On the other hand, since for  $a, b, c, \alpha_j$  and  $\beta_j \in B$ , with  $a^* = -a$ ,  $b^* = -b$ ,  $\alpha_j^* = \alpha_j$  and  $\beta_j^* = -\beta_j$ , the following identities hold,

$$\begin{aligned}\begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} &= \left[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} \right], \\ 2 \begin{pmatrix} a-b & 0 \\ 0 & b-a \end{pmatrix} &= \left[ \begin{pmatrix} 0 & a-b \\ b-a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right], \\ \begin{pmatrix} [\alpha_1, \alpha_2] & 0 \\ 0 & [\alpha_1, \alpha_2] \end{pmatrix} &= \left[ \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix} \right], \\ \begin{pmatrix} [\beta_1, \beta_2] & 0 \\ 0 & [\beta_1, \beta_2] \end{pmatrix} &= \left[ \begin{pmatrix} 0 & -\beta_2 \\ \beta_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{pmatrix} \right], \\ \begin{pmatrix} a & -c^* \\ c & b \end{pmatrix} &= \begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a-b & 0 \\ 0 & b-a \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a+b & 0 \\ 0 & a+b \end{pmatrix},\end{aligned}$$

it can be concluded that each commutator  $[z_{1,j}^a, z_{2,j}^a]$  is the sum of six commutators of elements in  $A$ . Therefore we have proved that

$$w = \sum_{j=1}^{140} [x_{1,j}, x_{2,j}],$$

where  $x_{l,j} \in A$ , for all  $l, j$ , which proves that

$$D = \sum_{j=1}^{140} \text{ad}([x_{1,j}, x_{2,j}]) = \sum_{j=1}^{140} [L_{x_{1,j}}, L_{x_{2,j}}]. \quad \square$$

Recall that a derivation on a JB-algebra is automatically continuous and that a JB-algebra has an approximate unit [9, 3.5.4]. Thus a derivation leaves each closed ideal invariant. By combining Theorem 1 with the comments preceding it, we have the following corollary.

**COROLLARY 1.** *Each derivation on a reversible JW-algebra is a sum of at most 151 commutators of the form  $[L_a, L_b]$ .*

The next theorem is the main result of this section.

**THEOREM 2.** *Cartan factors of type 1 with  $\dim H = \dim K$ , Cartan factors of type 2 with  $\dim H$  even or infinite, and all Cartan factors of type 3 have the inner derivation property. Moreover, every derivation of the above Cartan factors has degree at most 153.*

*Proof.* By Proposition 2, such factors are unital reversible JW\*-algebras. Thus it is enough to prove the statement for a unital reversible JW\*-algebra  $Z$ .

It is well known that  $Z$  decomposes in the form  $Z = X + iX$ , where  $X$  is the symmetric part of  $Z$ , and hence  $X$  is a reversible JW-algebra.

If  $\delta$  is a triple derivation of  $Z$ , then, by Lemma 2,  $\delta - L_{\delta u}$  is a derivation of the JB\*-algebra  $Z$  that commutes with the involution, and hence its restriction to  $X$  is a derivation of  $X$ . From the identity

$$(\delta - L_{\delta u})(z) = (\delta - L_{\delta u})(x + iy) = (\delta - L_{\delta u})|_X(x) + i(\delta - L_{\delta u})|_X(y),$$

it follows that  $(\delta - L_{\delta u})|_X$  determines  $(\delta - L_{\delta u})$ . Now, Corollary 1 gives (except for summing the 0 commutator)

$$\begin{aligned} (\delta - L_{\delta u})(z) &= \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](x) + i \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](y) \\ &= \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](x + iy) = \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](z). \end{aligned}$$

Now, applying the identity

$$[L_a, L_b] + [L_c, L_d] = 2(\tilde{\delta} - L_{\tilde{\delta} u}),$$

for all  $a, b, c$  and  $d$  in  $X$ , where

$$\tilde{\delta} = \delta \left( \frac{a + ic}{2}, \frac{b + id}{2} \right),$$

we obtain

$$(\delta - L_{\delta u})(z) = \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](z) = 2 \sum_{j=1}^{76} (\delta(c_j, d_j) - L_{\delta(c_j, d_j)u})(z).$$

Finally, if we apply Lemma 1, it follows that

$$\delta = 2 \sum_{j=1}^{76} (\delta(c_j, d_j) - L_{\delta(c_j, d_j)u}) + L_{\delta u}$$

is an inner derivation with degree at most 153.  $\square$

Following [13], we define a *real Cartan factor* to be a real form of a complex Cartan factor. Combining Theorem 2 and Proposition 1, we obtain the following result for real Cartan factors.

**COROLLARY 2.** *If  $E$  is a real form of a type 1 Cartan factor with  $\dim H = \dim K$ , or a real form of a Cartan factor of type 2 with  $\dim H$  even or infinite, or a real form of a Cartan factor of type 3, then every derivation on  $E$  is inner with degree at most 306.*

### 3.2. Real or complex spin factors

In this subsection, we prove that no infinite-dimensional real spin factor satisfies the inner derivation property. Thus, by Proposition 1, it can be concluded that no complex spin factor satisfies the inner derivation property.

We recall that a *complex spin Cartan factor* is a JB\*-triple that can be equipped with a complete inner product  $(\cdot, \cdot)$  and a conjugation  $*$  such that the triple product satisfies

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|z^*)y^*,$$

and the norm is given by

$$\|x\|^2 := (x|x) + ((x|x)^2 - |(x|x^*)|^2)^{1/2}.$$

By a *real spin factor*, we mean any real form of a complex spin factor. By [13, Theorem 4.1], we know that every real spin factor  $E$  is an  $l_1$ -sum

$$E = X_1 \oplus^{\ell_1} X_2,$$

where  $X_1$  and  $X_2$  are closed subspaces of a real Hilbert space  $X$  satisfying  $X_2 = X_1^\perp$ , and the triple product on  $E$  is given by

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|\bar{z})\bar{y},$$

where  $(\cdot, \cdot)$  is the inner product of  $X$  and the map  $x \mapsto \bar{x}$  is given by  $\bar{x} = (x_1, -x_2)$  for all  $x = (x_1, x_2) \in E$ .

Our goal is to build a derivation that is not inner in the case of an infinite-dimensional real spin factor  $E = X_1 \oplus^{\ell_1} X_2$ . Without loss of generality, we can assume that  $X_1$  is also infinite-dimensional.

First we suppose that  $E$  is separable. Let  $\{e_n : n \in \mathbb{N}\}$  be a countable orthonormal basis of  $X_1$ . Since  $\bar{e_n} = e_n$ , it is easy to check that  $\{e_n, e_n, e_n\} = e_n$  and

$\|\delta(e_{2k-1}, e_{2k})\| \leq 2$ , and hence the operator

$$\delta_0 := \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k})$$

is a well defined derivation on  $E$ . Our goal is to show that  $\delta_0$  is not inner. Suppose that  $\delta_0$  is inner; then

$$\delta_0 = \sum_{j=1}^P \delta(a_j, b_j)$$

for suitable  $a_j, b_j \in E$ , with  $a_j = a_{j,1} + a_{j,2}$  and  $b_j = b_{j,1} + b_{j,2}$ , where  $a_{j,i}$  and  $b_{j,i}$  are in  $X_i$  ( $j = 1, \dots, P$ ,  $i = 1, 2$ ). Hence

$$\begin{aligned} \delta_0 &= \sum_{j=1}^P \delta(a_j, b_j) \\ &= \sum_{j=1}^P \delta(a_{j,1}, b_{j,1}) + \delta(a_{j,1}, b_{j,2}) + \delta(a_{j,2}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}). \end{aligned}$$

It is easy to check that, for all  $x_1 \in X_1$ ,

$$\delta(a_{j,2}, b_{j,2})(x_1) = \delta(a_{j,1}, b_{j,2})(x_1) = \delta(a_{j,2}, b_{j,1})(x_1) = 0$$

and  $\delta_0(x_2) = 0$ . Therefore

$$\delta_0(x_1) = \sum_{j=1}^P \delta(a_{j,1}, b_{j,1})(x_1)$$

for all  $x_1 \in X_1$ .

Now we define  $K$  as the linear span of  $\{a_{j,1}, b_{j,1} : j = 1, \dots, P\}$ . Let  $x_1 \in K^\perp \cap X_1$ ; then

$$\begin{aligned} 0 &= \sum_{j=1}^P \delta(a_{j,1}, b_{j,1})(x_1) = \delta_0(x_1) = \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k})(x_1) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} (\{e_{2k-1}, e_{2k}, x_1\} - \{e_{2k}, e_{2k-1}, x_1\}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} ((e_{2k-1}|e_{2k})x_1 + (x_1|e_{2k})e_{2k-1} - (e_{2k-1}|x_1)e_{2k} \\ &\quad - (e_{2k}|e_{2k-1})x_1 - (x_1|e_{2k-1})e_{2k} + (e_{2k}|x_1)e_{2k-1}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} ((x_1|e_{2k})e_{2k-1} - (e_{2k-1}|x_1)e_{2k}). \end{aligned}$$

Thus  $(x_1|e_{2k}) = (e_{2k-1}|x_1) = 0$  for all  $k \in \mathbb{N}$ , and so  $x_1 = 0$  since  $\{e_n\}$  is a basis of  $X_1$ . Therefore  $K^\perp \cap X_1 = 0$ , and hence  $X_1 = K$  is finite-dimensional, which is impossible.

This proves that  $\delta_0$  is not an inner derivation. Suppose now that  $\dim X_1 > \aleph_0$ , and let  $\{e_n\}_{\mathbb{N}}$  be a countable set of orthonormal vectors in  $X_1$ . Let us denote by  $H$  the real separable Hilbert space generated by  $\{e_n\}_{\mathbb{N}}$ , and by  $\delta_0$  the derivation on  $E$

given by

$$\delta_0 := \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k}).$$

Since  $\delta_0(H) \subseteq H$ , it follows that  $\delta_0|_H$  is a derivation of the real spin factor  $H$ , which is not inner by the previous case. Actually, we claim that  $\delta_0$  is not an inner derivation on  $E$ . Suppose, contrary to our claim, that  $\delta_0$  is inner on  $E$ ; then

$$\delta_0 = \sum_{j=1}^P \delta(a_j, b_j)$$

with  $a_j, b_j \in E$ . Since

$$E = (H \oplus^{\ell_2} H^\perp) \oplus^{\ell_1} X_2,$$

the elements  $a_j$  and  $b_j$  can be expressed as  $a_j = h_j + x_{j,3}$  and  $b_j = k_j + y_{j,3}$ , where  $h_j$  and  $k_j$  are in  $H$  and  $x_{j,3}, y_{j,3} \in H^\perp \oplus^{\ell_1} X_2$  ( $j = 1, \dots, P$ ). Thus

$$\begin{aligned} \delta_0 &= \sum_{j=1}^P \delta(a_j, b_j) \\ &= \sum_{j=1}^P \delta(h_j, k_j) + \delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j) + \delta(x_{j,3}, y_{j,3}). \end{aligned}$$

It is easy to check that

$$\delta(h_j, y_{j,3})h = -(h_j|h)\overline{y_{j,3}} - (h|h_j)y_{j,3} \in H^\perp \oplus^{\ell_1} X_2$$

$$\delta(x_{j,3}, k_j)h = (h|k_j)x_{j,3} + (k_j|h)\overline{x_{j,3}} \in H^\perp \oplus^{\ell_1} X_2$$

and

$$\delta(x_{j,3}, y_{j,3})(h) = 0$$

for all  $h \in H$ . From the last identity, we have

$$\delta_0(h) = \sum_{j=1}^P \delta(h_j, k_j)(h) + \sum_{j=1}^P (\delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j))(h)$$

for all  $h \in H$ . Since  $\delta_0(H) \subseteq H$  and  $\sum_{j=1}^P (\delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j))(h) \in H^\perp \oplus^{\ell_1} X_2$ , we have

$$\delta_0(h) = \sum_{j=1}^P \delta(h_j, k_j)(h)$$

for all  $h \in H$ . Therefore  $\delta_0|_H$  is an inner derivation on  $H$ , which is impossible, and hence  $\delta_0$  is not an inner derivation on  $E$ .

We have thus proved the following theorem.

**THEOREM 3.** *Every infinite-dimensional real or complex spin factor has a derivation that is not inner, that is, none of the infinite-dimensional real or complex spin factors has the inner derivation property.*

### 3.3. Non-square type 1

As in the case of a real or complex spin factor, we are going to build an outer derivation in every real form of an infinite-dimensional and non-square ( $\dim H \neq \dim K$ ) type 1 Cartan factor. Again, using Proposition 1, we will conclude that no complex infinite-dimensional non-square type 1 Cartan factor satisfies the inner derivation property.

By [13, Theorem 4.1], we know that the real forms of a complex type 1 Cartan factor are precisely the real Banach spaces  $BL(X, Y)$  of all bounded linear operators between two real Hilbert spaces  $X$  and  $Y$  or the real Banach spaces  $BL(P, Q)$  of all bounded linear operators between two Hilbert spaces  $P, Q$  over the quaternion field. Thus it is enough to prove that  $BL(X, Y)$ , with  $+\infty = \dim(X) > \dim(Y)$ , possesses an outer derivation. We will divide the proof into several steps. In a first step, we suppose that  $Y = \mathbf{R}$ . In this case,  $BL(X, \mathbf{R})$  is isometrically isomorphic, as a real  $JB^*$ -triple, to  $X$  equipped with the triple product

$$\{x, y, z\} = \frac{1}{2}((x|y)z + (z|y)x)$$

for all  $x, y, z \in X$ .

Let  $\delta$  be a derivation on  $X$ ; then

$$\delta\{x, y, z\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\} \quad (*)$$

for all  $x, y, z \in X$ . Now from the expression of the triple product, we have

$$\begin{aligned} \delta\{x, y, z\} &= \frac{1}{2}((x|y)\delta z + (z|y)\delta x), \\ \{\delta x, y, z\} &= \frac{1}{2}((\delta x|y)z + (z|y)\delta x), \\ \{x, \delta y, z\} &= \frac{1}{2}((x|\delta y)z + (z|\delta y)x), \\ \{x, y, \delta z\} &= \frac{1}{2}((x|y)\delta z + (\delta z|y)x). \end{aligned}$$

Thus it follows from  $(*)$  that

$$\frac{1}{2}(((\delta x|y) + (x|\delta y))z + ((z|\delta y) + (\delta z|y))x) = 0$$

for all  $x, y, z \in X$ . In particular, we have

$$(x|\delta y) = -(\delta x|y)$$

for all  $x, y \in X$ , that is,  $\delta^* = -\delta$ . Therefore every derivation on  $X$ , regarded as the real type 1 Cartan factor  $BL(X, \mathbf{R})$ , is a skew-symmetric operator on  $X$ . Conversely, the following holds.

**LEMMA 3.** *If  $X$  is a real Hilbert space, regarded as the real Cartan factor  $BL(X, \mathbf{R})$ , then the derivations on  $X$  coincide with the skew-symmetric operators on  $X$ .*

*Proof.* Suppose that  $T$  is a skew-symmetric operator on  $X$ . The identities

$$\begin{aligned} T\{x, y, z\} &= \frac{1}{2}((x|y)Tz + (z|y)Tx), \\ \{Tx, y, z\} &= \frac{1}{2}((Tx|y)z + (z|y)Tx)), \\ \{x, Ty, z\} &= -\frac{1}{2}((Tx|y)z + (Tz|y)x), \\ \{x, y, Tz\} &= \frac{1}{2}((x|y)Tz + (Tz|y)x), \end{aligned}$$

show that  $T$  is a derivation on  $X$ .  $\square$

The next proposition characterizes the inner derivations on  $X$ .

**PROPOSITION 3.** *The inner derivations on  $X$ , regarded as the real Cartan factor  $\text{BL}(X, \mathbf{R})$ , coincide with the finite rank operators on  $X$  that are skew-symmetric.*

*Proof.* Let

$$\delta = \sum_{j=1}^P \delta(a_j, b_j)$$

be an inner derivation on  $X$ . Since

$$\delta(a_j, b_j)(x) = \frac{1}{2}((x|b_j)a_j - (x|a_j)b_j),$$

it follows that  $\delta$  is a finite rank operator. The other implication follows from Lemma 3.  $\square$

**REMARK 2.** Since for every infinite-dimensional Hilbert space  $X$  there exists a skew-symmetric operator  $T$  on  $X$  satisfying  $T^2 = -\text{Id}$ , we conclude from Lemma 3 and Proposition 3 that  $T$  is an outer derivation on  $X$ . It follows that  $\text{BL}(X, \mathbf{R})$  does not satisfy the inner derivation property.

Our next goal is to build derivations on  $\text{BL}(X, Y)$  from derivations on  $X = \text{BL}(X, \mathbf{R})$ .

**LEMMA 4.** *Let  $\delta$  be a derivation on a real Hilbert space  $X$  (regarded as the real Carton factor  $\text{BL}(X, \mathbf{R})$ ), and let  $Y$  be another real Hilbert space. Then the operator*

$$\tilde{\delta} : \text{BL}(X, Y) \longrightarrow \text{BL}(X, Y)$$

$$\tilde{\delta}a = a\delta$$

is a derivation on  $\text{BL}(X, Y)$ .

*Proof.* Since  $\delta$  is a derivation on  $X$ ,  $\delta^* = -\delta$  (see Lemma 3). Given  $a, b, c \in \text{BL}(X, Y)$ , we have

$$\begin{aligned} \{\tilde{\delta}a, b, c\} + \{a, \tilde{\delta}b, c\} + \{a, b, \tilde{\delta}c\} \\ &= \frac{1}{2}(a\delta b^* c + cb^* a\delta + a\delta^* b^* c + c\delta^* b^* a + ab^* c\delta + c\delta b^* a) \\ &= \frac{1}{2}(a\delta b^* c + cb^* a\delta - a\delta b^* c - c\delta b^* a + ab^* c\delta + c\delta b^* a) \\ &= \frac{1}{2}(cb^* a\delta + ab^* c\delta) = \{a, b, c\}\delta = \tilde{\delta}\{a, b, c\}, \end{aligned}$$

which proves that  $\tilde{\delta}$  is a derivation.  $\square$

At this moment, we need the following identification. Let us fix a norm one element  $y_0 \in Y$ . In the sequel, we will identify each  $h \in X$ , with the operator

$$\begin{aligned} f_h : X &\longrightarrow Y \\ f_h(x) &:= (x|h)y_0 \quad x \in X. \end{aligned}$$

In this way,  $X$  can be regarded as the subspace of  $\text{BL}(X, Y)$  formed by all operators of the form  $f_h$  with  $h \in X$ . Using this identification, it is easy to check that if  $\delta$  and  $\tilde{\delta}$  are as in Lemma 4, then  $\tilde{\delta}(X) \subseteq X$ . In fact,

$$\begin{aligned} \tilde{\delta}(f_h)(x) &= f_h(\delta x) = (\delta x|h)y_0 \\ &= (x|\delta^* h)y_0 = (x| - \delta h)y_0 = f_{-\delta h}(x). \end{aligned}$$

The next lemma is the key tool of the main result of this subsection.

LEMMA 5. *Let  $\delta$  and  $\tilde{\delta}$  be as in Lemma 4, and suppose that  $\tilde{\delta}$  is an inner derivation. Then  $\delta$  has rank less than or equal to the hilbertian dimension of  $Y$ .*

*Proof.* Since  $\tilde{\delta}$  is an inner derivation on  $\text{BL}(X, Y)$ ,  $\tilde{\delta}$  is the sum

$$\tilde{\delta} = \sum_{j=1}^P \delta(a_j, b_j)$$

for suitable  $a_j, b_j \in \text{BL}(X, Y)$ . As we have seen previously for each  $h \in X$ ,  $\tilde{\delta}f_h = f_{-\delta h} \in X$ . On the other hand,

$$\begin{aligned} f_{-\delta h} &= \tilde{\delta}(f_h) = \sum_{j=1}^P \delta(a_j, b_j)(f_h) \\ &= \sum_{j=1}^P \frac{1}{2}(a_j b_j^* f_h + f_h b_j^* a_j - b_j a_j^* f_h - f_h a_j^* b_j) \\ &= \left( \sum_{j=1}^P \frac{1}{2}(a_j b_j^* - b_j a_j^*) \right) f_h + f_h \left( \sum_{j=1}^P \frac{1}{2}(b_j^* a_j - a_j^* b_j) \right) \\ &= Rf_h + f_h T, \end{aligned}$$

where

$$\begin{aligned} R &= \sum_{j=1}^P \frac{1}{2}(a_j b_j^* - b_j a_j^*) : Y \longrightarrow Y \\ T &= \sum_{j=1}^P \frac{1}{2}(b_j^* a_j - a_j^* b_j) : X \longrightarrow X \end{aligned}$$

are two skew-symmetric operators. Moreover,

$$f_h T(x) = (Tx|h)y_0 = (x|Th)y_0 = f_{-Th}(x)$$

for all  $x \in X$ , so that  $f_h T = f_{-Th}$ , and

$$Rf_h = \tilde{\delta}f_h - f_h T = f_{-\delta h - Th} = f_{h'} \in X.$$

Therefore, for all  $x, h \in X$ , the equality

$$Rf_h(x) = (x|h)R(y_0) = (x|h')y_0$$

holds. Thus we have  $R(y_0) = \lambda y_0$  for a suitable  $\lambda \in \mathbf{R}$ . Since  $R$  is a skew-symmetric operator and  $\lambda$  is a real eigenvalue of  $R$ ,  $\lambda = 0$ .

In this way, since  $Rf_h = 0$ , we have

$$f_{-\delta h} = \tilde{\delta}(f_h) = f_h T = f_{-Th}$$

for all  $h \in X$ , and hence  $T = \delta$ .

Since each  $b_j^* a_j$  and each  $a_j^* b_j$  are operators that factorize through  $Y$ , they have rank at most the hilbertian dimension of  $Y$ . Therefore so does

$$\delta = T = \sum_{j=1}^P \frac{1}{2}(b_j^* a_j - a_j^* b_j). \quad \square$$

**THEOREM 4.** *Let  $X$  be an infinite-dimensional real Hilbert space, and  $Y$  be a real Hilbert space with hilbertian dimension less than the hilbertian dimension of  $X$ . Then  $\text{BL}(X, Y)$  does not satisfy the inner derivation property.*

*Proof.* We recall that, since  $X$  is infinite-dimensional, there exists a bounded linear operator  $T$  on  $X$  such that  $T^2 = -\text{Id}_X$  and  $T^* = -T$ . Hence  $T$  has rank equal to the hilbertian dimension of  $X$ . Since  $T^* = -T$ , Lemma 3 states that  $T$  is a derivation on  $X$ . Moreover, by Lemma 4, the operator  $\tilde{T}$  given by  $\tilde{T}a = aT$  ( $a \in \text{BL}(X, Y)$ ) is a derivation on  $\text{BL}(X, Y)$ . If  $\tilde{T}$  is an inner derivation, then Lemma 5 states that  $T$  has rank at most the hilbertian dimension of  $Y$ , which is impossible, since  $\dim(X) > \dim(Y)$ .  $\square$

Again combining Theorem 3 and Proposition 1, we obtain the following corollary.

**COROLLARY 3.** *The complex infinite-dimensional non-square type 1 Cartan factors and their real forms do not satisfy the inner derivation property.*

By virtue of the previous results, we know that there exist real and complex JB\*-triples having outer derivations. Therefore it is natural to ask if any derivation can be approximated (in a convenient topology) by inner derivations. Upmeier [23] proved that there exists a unital JB-algebra  $X$  and a derivation  $D$  on  $X$  that cannot be approximated in norm by inner algebra derivations. Let  $\hat{X}$  denote the complexification of  $X$ , and  $\hat{D}$  the complex linear extension of  $D$  to  $\hat{X}$ . Then  $\hat{X}$  is a unital JB\*-algebra with unit  $u$ , and hence a JB\*-triple, and  $\hat{D}$  is a triple derivation, since  $\hat{D}$  is an algebra derivation that commutes with the involution (see Lemma 2). We claim that  $\hat{D}$  cannot be approximated in norm by inner triple derivations. Otherwise, for  $\varepsilon > 0$ , there would exist an inner triple derivation

$$\delta = \sum_j^P \delta(e_j, f_j)$$

such that

$$\|\hat{D} - \delta\| < \varepsilon.$$

Now, by Lemma 2,

$$\begin{aligned} \delta - L_{\delta(u)} &= \sum_j^P \delta(e_j, f_j) - L_{\delta(e_j, f_j)(u)} \\ &= \frac{1}{2} \sum_j^P [L_{aj}, L_{cj}] + [L_{bj}, L_{dj}], \end{aligned}$$

where  $e_j = \frac{1}{2}(a_j + ib_j)$ ,  $f_j = \frac{1}{2}(c_j + id_j)$  with  $a_j, b_j, c_j, d_j$  in  $X$ . Therefore  $\delta - L_{\delta(u)}$  is an inner derivation on  $X$  such that

$$\begin{aligned} \|D - (\delta - L_{\delta(u)})\| &= \|D - L_{D(u)} - (\delta - L_{\delta(u)})\| \\ &\leq \|\hat{D} - \delta\| + \|L_{D(u)} - L_{\delta(u)}\| \\ &\leq \|\hat{D} - \delta\| + \|L_{D(u) - \delta(u)}\| \\ &\leq \|\hat{D} - \delta\| + \|(\hat{D} - \delta)(u)\| \leq 2\varepsilon, \end{aligned}$$

which is impossible, since  $D$  cannot be approximated in norm by an inner derivation.

On the other hand,  $D$  is also a derivation on the real JB $^*$ -triple  $X$ . If  $D$  could be approximated in norm by inner triple derivations on  $X$ , then, for every  $\varepsilon > 0$ , there exists

$$\delta = \sum_j^P \delta(e_j, f_j)$$

with  $e_j, f_j \in X$  such that  $\|D - \delta\| \leq \varepsilon$ . In this case,  $\delta = \sum_j^P \delta(e_j, f_j)$  is an inner derivation on  $\hat{X}$  and

$$\|(\hat{D} - \delta)\| \leq 2\varepsilon.$$

This is impossible.

Upmeier [23], also proved that every algebra derivation on a JB-algebra can be approximated in the strong operator topology by inner derivations. In [3, Theorem 4.6], Barton and Friedman proved that the set of all inner derivations on a JB $^*$ -triple is dense in the set of all derivations with respect to the strong operator topology. This result can be extended to real JB $^*$ -triples.

**THEOREM 5.** *The set of all inner derivations on a real JB $^*$ -triple is dense in the set of all derivations with respect to the strong operator topology.*

*Proof.* Let  $E$  be a real JB $^*$ -triple and  $\delta$  a derivation on  $E$ . We consider

$$\begin{aligned} \hat{\delta} : \hat{E} &\longrightarrow \hat{E} \\ \hat{\delta}(x + iy) &:= \delta(x) + i\delta(y) \end{aligned}$$

the natural extension of  $\delta$  to  $\hat{E}$ . Since  $\hat{E}$  is a complex JB $^*$ -triple, by [3, Theorem 4.6], it follows that for every  $x_1, \dots, x_n \in E \subset \hat{E}$  and every  $\varepsilon > 0$  that there exists an inner derivation

$$\delta_1 = \sum_{j=1}^P \delta(a_j, b_j)$$

on  $\hat{E}$  such that  $\|\hat{\delta}(x_l) - \delta_1(x_l)\| \leq \varepsilon$  for all  $l = 1, \dots, n$ .

Since  $a_j = a_{j,1} + ia_{j,2}$  and  $b_j = b_{j,1} + ib_{j,2}$ , where  $a_{j,k}$  and  $b_{j,k}$  are in  $E$ , it is easy to check that

$$\begin{aligned} \delta_1(x_l) &= \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}) \\ &\quad + i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1})))x_l. \end{aligned}$$

Since  $a_{j,k}, b_{j,k}$  and  $x_l$  are elements in  $E$ , it follows that

$$\sum_{j=1}^P i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))x_l \in iE.$$

Thus

$$\begin{aligned} &\|\delta(x_l) - \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}))(x_l)\| \\ &\leq \|\delta(x_l) - \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}))(x_l)\| \end{aligned}$$

$$\begin{aligned}
& -i \sum_{j=1}^P (L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))(x_l)) \| \\
& = \| \hat{\delta}(x_l) - \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}) \\
& \quad + i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))(x_l)) \| \\
& = \| \hat{\delta}(x_l) - \delta_1(x_l) \| \leq \varepsilon
\end{aligned}$$

for all  $l = 1, \dots, n$ .  $\square$

PROBLEM 1. If we could obtain a universal bound for the degree of all derivations in a type 2 Cartan factor with  $\dim H$  odd, we could try to determine all JBW\*-triples of type I satisfying the inner derivation property following the techniques contained in Ho's dissertation [10].

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*Tony Ho*  
*Bernard Russo*  
*Department of Mathematics*  
*University of California*  
*Irvine*  
*CA 92697-3875*  
*USA*

[brusso@math.uci.edu](mailto:brusso@math.uci.edu)

*Juan Martínez-Moreno*  
*Antonio M. Peralta*  
*Departamento Análisis Matemático*  
*Facultad de Ciencias*  
*Universidad de Granada*  
*18071 Granada*  
*Spain*

[jmmoreno@goliat.ugr.es](mailto:jmmoreno@goliat.ugr.es)  
[aperalta@goliat.ugr.es](mailto:aperalta@goliat.ugr.es)