

DERIVATIONS ON REAL AND COMPLEX JB^* -TRIPLES

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1. Introduction

At the regional conference held at the University of California, Irvine, in 1985 [24], Harald Upmeyer posed three basic questions regarding derivations on JB^* -triples:

- (1) Are derivations automatically bounded?
- (2) When are all bounded derivations inner?
- (3) Can bounded derivations be approximated by inner derivations?

These three questions had all been answered in the binary cases. Question 1 was answered affirmatively by Sakai [17] for C^* -algebras and by Upmeyer [23] for JB -algebras. Question 2 was answered by Sakai [18] and Kadison [12] for von Neumann algebras and by Upmeyer [23] for JW -algebras. Question 3 was answered by Upmeyer [23] for JB -algebras, and it follows trivially from the Kadison–Sakai answer to question 2 in the case of C^* -algebras.

In the ternary case, both question 1 and question 3 were answered by Barton and Friedman in [3] for complex JB^* -triples. In this paper, we consider question 2 for real and complex JBW^* -triples and question 1 and question 3 for real JB^* -triples. A real or complex JB^* -triple is said to have the *inner derivation property* if every derivation on it is inner. By pure algebra, every finite-dimensional JB^* -triple has the inner derivation property. Our main results, Theorems 2, 3 and 4 and Corollaries 2 and 3 determine which of the infinite-dimensional real or complex Cartan factors have the inner derivation property.

2. Background

We recall that a JB^* -algebra is a complete normed Jordan complex algebra (say \mathcal{A}) endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every $x \in \mathcal{A}$. Here, for every Jordan algebra \mathcal{A} , and every $x \in \mathcal{A}$, U_x denotes the operator on \mathcal{A} defined by $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$, for all $y \in \mathcal{A}$.

A JB -algebra is a complete normed Jordan real algebra (say A) satisfying the following two additional conditions for $a, b \in A$:

- (i) $\|a^2\| = \|a\|^2$.
- (ii) $\|a^2\| \leq \|a^2 + b^2\|$.

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It is due to Wright (see [25]) that the complexification of a JB-algebra is a JB*-algebra under a unique norm extending the given norm on the JB-algebra. Conversely, the self-adjoint part of a JB*-algebra is a JB-algebra under the restricted norm.

If H is a complex Hilbert space, then the real Banach space $\mathcal{H}(H)$ of all bounded hermitian operators on H is a JB-algebra with respect to the Jordan product

$$x \circ y := \frac{1}{2}(xy + yx).$$

A uniformly (respectively weakly) closed unital real subalgebra of $\mathcal{H}(H)$ is called a JC-algebra (respectively JW-algebra) on H . A norm (respectively weakly) closed (complex) Jordan*-subalgebra of a C*-algebra (respectively von Neumann algebra) is called a JC*-algebra (respectively JW*-algebra). For more details on JB-algebras and JB*-algebras we refer the reader to [9].

We recall that a (complex) JB*-triple is a complex Banach space \mathcal{J} with a continuous triple product $\{\cdot, \cdot, \cdot\} : \mathcal{J} \times \mathcal{J} \times \mathcal{J} \rightarrow \mathcal{J}$ that is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and that satisfies the following conditions.

(i) (Jordan identity) $L(a, b)\{x, y, z\} = \{L(a, b)x, y, z\} - \{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$ for all a, b, x, y, z in \mathcal{J} , where $L(a, b)x := \{a, b, x\}$.

(ii) For all $a \in \mathcal{J}$, the map $L(a, a)$ from \mathcal{J} to \mathcal{J} is a hermitian operator with non-negative spectrum.

(iii) $\|\{a, a, a\}\| = \|a\|^3$ for all a in \mathcal{J} .

It is worth mentioning that every C*-algebra is a (complex) JB*-triple with respect to $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. Also, every JB*-algebra is a JB*-triple with respect to $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. Conversely, every JB*-triple with a unitary element u (that is, $\{u, u, z\} = z$ for every z) is a unital JB*-algebra with product $a \circ b = \{a, u, b\}$, involution $a^* = \{u, a, u\}$, and unit u . We refer to [5, 15, 16] for recent surveys on the theory of JB*-triples.

Following [11], we recall that a real JB*-triple is a norm-closed real subtriple of a complex JB*-triple. Given a real JB*-triple J , there exists a unique complex JB*-triple structure on the complexification $\hat{J} = J \oplus iJ$, and a unique conjugation (that is, conjugate-linear isometry of period 2) τ on \hat{J} such that $J = \hat{J}^r := \{x \in \hat{J} : \tau(x) = x\}$. From this point of view, the real JB*-triples are real forms of complex JB*-triples.

The class of real JB*-triples includes all JB-algebras [9], all real C*-algebras [8], and all J*B-algebras [2].

A triple derivation or simply a derivation δ on a real or complex JB*-triple U is a linear operator satisfying

$$\delta\{a, b, c\} = \{\delta a, b, c\} + \{a, \delta b, c\} + \{a, b, \delta c\}$$

for all $a, b, c \in U$.

If U is a real or complex JB*-triple, we can conclude from the Jordan identity that $\delta(a, b) := L(a, b) - L(b, a)$ is a derivation, for every $a, b \in U$. An inner triple derivation δ on U is a finite sum of derivations of the form $\delta(a, b)$ ($a, b \in U$), that is,

$$\delta = \sum_{i=1}^n \delta(a_i, b_i). \quad (2.1)$$

The degree of an inner derivation is the least number of terms in a representation of the form (2.1). Any derivation that is not inner is called outer.

REMARK 1. Let E be a real JB*-triple and δ a derivation on E . Then δ can be extended to a derivation $\hat{\delta}$, on the complexification of E , defined by $\hat{\delta}(x + iy) := \delta(x) + i\delta(y)$.

It is due to Barton and Friedman [3] that every derivation on a complex JB*-triple is automatically continuous, and so, by the previous comment, every derivation on a real JB*-triple is also continuous.

3. Inner derivation property

We say that a real or complex JB*-triple U has the *inner derivation property* if every derivation on U is inner.

By [14, Chapter 8], every finite-dimensional real or complex JB*-triple has the inner derivation property. The next proposition shows that a real JB*-triple has the inner derivation property whenever its complexification satisfies this property.

PROPOSITION 1. *Let E be a real JB*-triple. Suppose that the complexification \hat{E} of E has the inner derivation property. Then E has the inner derivation property. Moreover, if M is a bound of the degree of all inner derivations of \hat{E} , then $2M$ is a bound of the degree of all inner derivations of E .*

Proof. Suppose that E is a real JB*-triple such that \hat{E} has the inner derivation property. Let δ be a derivation of E . We denote by $\hat{\delta}$ the derivation on \hat{E} , extending δ to \hat{E} . Since \hat{E} has the inner derivation property, then $\hat{\delta}$ is an inner derivation of degree n , that is,

$$\hat{\delta} = \sum_{k=1}^n \delta(a_k, b_k),$$

where $a_k, b_k \in \hat{E}$. Since $\hat{E} = E \oplus iE$, it follows that $a_k = a_{k,1} + ia_{k,2}$ and $b_k = b_{k,1} + ib_{k,2}$ for suitable $a_{k,l}, b_{k,l} \in E$, $l = 1, 2$ and $k = 1, \dots, n$.

Consider now $x \in E$. We can compute

$$\begin{aligned} \delta(a_k, b_k)x &= \delta(a_{k,1} + ia_{k,2}, b_{k,1} + ib_{k,2})x \\ &= \{a_{k,1} + ia_{k,2}, b_{k,1} + ib_{k,2}, x\} - \{b_{k,1} + ib_{k,2}, a_{k,1} + ia_{k,2}, x\} \\ &= \{a_{k,1}, b_{k,1}, x\} + \{a_{k,2}, b_{k,2}, x\} + i(\{a_{k,2}, b_{k,1}, x\} - \{a_{k,1}, b_{k,2}, x\}) \\ &\quad - \{b_{k,1}, a_{k,1}, x\} - \{b_{k,2}, a_{k,2}, x\} - i(\{b_{k,2}, a_{k,1}, x\} - \{b_{k,1}, a_{k,2}, x\}) \\ &= \delta(a_{k,1}, b_{k,1})(x) + \delta(a_{k,2}, b_{k,2})(x) \\ &\quad + i(\{a_{k,2}, b_{k,1}, x\} - \{a_{k,1}, b_{k,2}, x\} - \{b_{k,2}, a_{k,1}, x\} + \{b_{k,1}, a_{k,2}, x\}). \end{aligned}$$

Therefore

$$\begin{aligned} E \ni \delta(x) &= \hat{\delta}(x) = \sum_{k=1}^n \delta(a_{k,1} + ia_{k,2}, b_{k,1} + ib_{k,2})x \\ &= \left(\sum_{k=1}^n (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2})) \right) x \\ &\quad + i \sum_{k=1}^n (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,2}))(x) \end{aligned}$$

Since the elements $a_{k,l}, b_{k,l} \in E$, we have

$$\left(\sum_{k=1}^n (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2})) \right) (E) \subset E$$

and

$$\left(i \sum_{k=1}^n (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,1})) \right) (E) \subset iE.$$

Therefore

$$\left(\sum_{k=1}^n (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,1})) \right) (x) = 0$$

for all $x \in E$. Thus

$$\delta(x) = \hat{\delta}(x) = \sum_{k=1}^n (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2}))(x)$$

for all $x \in E$, which proves that δ is an inner derivation with degree at most $2n$. \square

From Proposition 1, it is easy to see that if E is a real JB^* -triple which does not satisfy the inner derivation property, then its complexification also does not satisfy the inner derivation property.

3.1. Reversible unital JB^* -algebras

We recall that the (complex) *type 1 Cartan factor* can be defined as the complex Banach space $\text{BL}(H, K)$ of all bounded linear operators between two complex Hilbert spaces H and K , with triple product given by

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a).$$

Next we give a brief description of the (complex) Cartan factors of type 2 and 3. Let H be a complex Hilbert space equipped with a conjugation (conjugate-linear isometry of period 2) $j : H \rightarrow H$; then for any $z \in \text{B}(H)$ we can define its transpose as $z^t := jz^*j$. The *type 2 Cartan factor* coincides with the Banach space of all t -skew symmetric elements in $\text{B}(H)$ ($z^t = -z$), and the *type 3 Cartan factor* is defined as the Banach space of all t -symmetric elements of $\text{B}(H)$ ($z^t = z$). The triple product of these Cartan factors is the restriction of the triple product in $\text{B}(H)$.

We recall that a JC^* -algebra (or a JC^* -algebra) A is said to be *reversible* if $x_1x_2 \dots x_n + x_nx_{n-1} \dots x_1 \in A$, for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in A$.

PROPOSITION 2. *Cartan factors of type 1 with $\dim H = \dim K$, Cartan factors of type 2 with $\dim H$ even, or infinite, and all Cartan factors of type 3 are reversible JW^* -algebras.*

Proof. Let C^3 be a type 3 Cartan factor. Since $x^t = x$ for all $x \in C^3$, we have $(x_1 \dots x_n + x_n \dots x_1)^t = x_n \dots x_1 + x_1 \dots x_n \in C^3$.

Let C^2 be a type 2 Cartan factor with $\dim H$ even or infinite. Then C^2 contains a

distinguished unitary

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdot & \cdot \\ -1 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & -1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

In this case we can provide a new C*-algebra structure for $B(H)$ with product $a \cdot b = au^*b$ and involution $a^\parallel = ua^*u$ in which C^2 becomes a JC*-subalgebra under $a \circ b = (a \cdot b + b \cdot a)/2$. With this Jordan product, C^2 is reversible since

$$\begin{aligned} & (x_1 u^* x_2 \dots u^* x_n + x_n u^* \dots x_2 u^* x_1)^1 \\ &= (-1)^{n+(n-1)} (x_n u^* x_{n-1} \dots x_2 u^* x_1 + x_1 u^* x_2 \dots x_{n-1} u^* x_n) \\ &= -(x_1 u^* x_2 \dots u^* x_n + x_n u^* \dots x_2 u^* x_1). \end{aligned} \quad \square$$

We recall that if \mathcal{A} is an algebra, then a *derivation* D of \mathcal{A} is a linear mapping $D : \mathcal{A} \rightarrow \mathcal{A}$ satisfying $D(ab) = D(a)b + aD(b)$, for all $a, b \in \mathcal{A}$. If \mathcal{A} is a Jordan algebra, an *inner algebra derivation* of \mathcal{A} is a finite sum of commutators of the form $[L_a, L_b]$ for some $a, b \in \mathcal{A}$, where $L_a x := a \circ x$. For an inner algebra derivation D , the *degree* of D is the least natural number n satisfying $D = \sum_{i=1}^n [L_{a_i}, L_{b_i}]$.

LEMMA 1. *Let Z be a JB*-algebra, with unit u , regarded as a complex JB*-triple. If δ is a triple derivation of Z , then $L_{\delta(u)}$ is an inner triple derivation of Z of degree 1.*

Proof. Simply note that for every triple derivation δ of Z , we have

$$\begin{aligned} \delta u &= \delta\{u, u, u\} = \{\delta u, u, u\} + \{u, \delta u, u\} + \{u, u, \delta u\} \\ &= 2\{\delta u, u, u\} + \{u, \delta u, u\} = 2\delta u \circ u + (\delta u)^* \end{aligned}$$

and hence

$$(\delta u)^* = -\delta u.$$

Now consider

$$\begin{aligned} L_{\delta u} z &= \delta u \circ z = \frac{1}{2}(\delta u \circ z - (-\delta u) \circ z) \\ &= \frac{1}{2}(\delta u \circ z - (\delta u)^* \circ z) = \frac{1}{2}(\{\delta u, u, z\} - \{u, \delta u, z\}); \end{aligned}$$

it follows that $L_{\delta u}$ is an inner triple derivation of degree 1. \square

LEMMA 2 [3, p. 263]. *Let Z be a unital JB*-algebra and D be an algebra derivation of Z that commutes with the involution of Z . Then D is a triple derivation of Z .*

Conversely, if Z is a JB-triple with a unitary element u and δ is a triple derivation of Z , then $\delta - L_{\delta u}$ is an algebra derivation of Z that commutes with the involution on Z . In particular, if δ is an inner derivation of degree 1, that is, $\delta = \delta(x, y)$, then*

$$\delta - L_{\delta(u)} = \frac{1}{2}([L_{x+x^*}, L_{y+y^*}] + [L_{-i(x-x^*)}, L_{-i(y-y^*)}]).$$

Proof. The first statement is clear. To prove the second one, let δ be a triple

derivation of Z . It is easy to check that

$$\begin{aligned}
 (\delta - L_{\delta u})(x \circ y) &= \delta\{x, u, y\} - \{\delta u, u, \{x, u, y\}\} \\
 &= \{\delta x, u, y\} + \{x, \delta u, y\} + \{x, u, \delta y\} - \{\delta u, u, \{x, u, y\}\} \\
 &= \{\delta x, u, y\} + \{x, \delta u, y\} + \{x, u, \delta y\} \\
 &\quad - \{\{\delta u, u, x\}, u, y\} + \{x, \{u, \delta u, u\}, y\} - \{x, u, \{\delta u, u, y\}\} \\
 &= \delta x \circ y + \{x, \delta u, y\} + x \circ \delta y \\
 &\quad - (\delta u \circ x) \circ y + \{x, (\delta u)^*, y\} - x \circ (\delta u \circ y)
 \end{aligned}$$

(applying $(\delta u)^* = -\delta u$)

$$\begin{aligned}
 &= (\delta - L_{\delta u})(x) \circ y + \{x, \delta u, y\} + x \circ (\delta - L_{\delta u})(y) - \{x, \delta u, y\} \\
 &= (\delta - L_{\delta u})(x) \circ y + x \circ (\delta - L_{\delta u})(y).
 \end{aligned}$$

Thus $\delta - L_{\delta u}$ is an algebra derivation.

The verification of the last formula is left to the reader. \square

By [22, Theorem 13] (see also [1, p. 255], each JW-algebra A admits a decomposition into weakly closed ideals of the form

$$A = I_{\text{fin}} \oplus I_{\infty} \oplus II_1 \oplus II_{\infty} \oplus III.$$

See [22] and [1] for the meaning of these symbols. A JW-algebra A is called *properly non-modular* if its modular part $I_{\text{fin}} \oplus II_1$ vanishes.

In 1980, Upmeyer showed that each algebra derivation on a properly non-modular JW-algebra is the sum of six commutators of the form $[L_a, L_b]$ [23, Theorem 3.8], and each algebra derivation on a reversible JW-algebra of type I_{fin} is the sum of five commutators [23, Theorem 3.9].

The proof of the following theorem is implicitly contained in [23], and we include it here for completeness.

THEOREM 1. *Let A be a reversible JW-algebra of type II_1 . Then each derivation of A is a sum of at most 140 commutators of the form $[L_a, L_b]$.*

Proof. Let A be a reversible JW-algebra of type II_1 . We denote by $\mathcal{U}(\mathcal{A})$ its complex enveloping von Neumann algebra (the smallest von Neumann algebra containing A). By [1, Theorem 8], $\mathcal{U}(\mathcal{A})^+$ (that is, $\mathcal{U}(\mathcal{A})$ with the Jordan product $w_1 \circ w_2 = (w_1 w_2 + w_2 w_1)/2$) is also of type II_1 . Thus if we follow the proof of [23, Theorem 3.10], it follows that each derivation of A has the form $D(x) = \text{ad}(w)(x) := [w, x]$ ($x \in A$), where $w = -w^* \in \mathcal{U}(\mathcal{A})$. Moreover, since $\mathcal{U}(\mathcal{A})^+$ is of type II_1 , w is the sum of ten commutators in $\mathcal{U}(\mathcal{A})$ (see [7, Theorem 2.3]), so that each derivation of A has the form

$$D = \sum_{j=1}^{10} \text{ad}([w_{1,j}, w_{2,j}]).$$

Since A is the self-adjoint part of $\mathcal{R}(\mathcal{A})$ [19], where $\mathcal{R}(\mathcal{A})$ is the real enveloping algebra of A , we have, by [20, Lemma 6.1; 21, Lemma 2.3, Theorem 2.4], $\mathcal{U}(\mathcal{A}) = \mathcal{R}(\mathcal{A}) + i\mathcal{R}(\mathcal{A})$.

Hence every element $w_{l,j}$ is the sum $w_{l,j} = u_{l,j} + iv_{l,j}$, where $u_{l,j}, v_{l,j} \in \mathcal{R}(\mathcal{A})$.

Since, for every $u_l, v_l \in \mathcal{R}(\mathcal{A})$, the equalities

$$\begin{aligned} [u_1 + iv_1, u_2 + iv_2] &= [u_1, u_2] - [v_1, v_2] + i([u_1, v_2] + [v_1, u_2]) \\ [u_1 + iu_2, x] &= [u_1, x] + i[u_2, x] \end{aligned}$$

hold for all $x \in A$, and since D maps A in A , we have

$$\sum_{j=1}^{10} [[u_{1,j}, v_{2,j}] + [v_{1,j}, u_{2,j}], x] = 0$$

for all $x \in A$. Thus

$$D = \text{ad}(w) = \sum_{j=1}^{10} \text{ad}([u_{1,j}, u_{2,j}] - [v_{1,j}, v_{2,j}]) = \sum_{j=1}^{20} \text{ad}([z_{1,j}, z_{2,j}]),$$

where $z_{i,j} \in \mathcal{R}(\mathcal{A})$ and $w = \sum_{j=1}^{20} [z_{1,j}, z_{2,j}]$.

Our next goal is to prove that every element $[z_{1,j}, z_{2,j}]$ is a finite sum of commutators of elements in A .

Let $z_{1,j}, z_{2,j} \in \mathcal{R}(\mathcal{A})$, and $l \in \{1, 2\}$. We denote by $z_{l,j}^s$ (respectively $z_{l,j}^a$) the symmetric part (respectively the skew-symmetric part) of $z_{l,j}$. Since, for every j , $[z_{1,j}^a, z_{2,j}^s]$ and $[z_{1,j}^s, z_{2,j}^a]$ are symmetric elements and $w^* = -w$, we deduce that

$$w = \sum_{j=1}^{20} [z_{1,j}^s, z_{2,j}^s] + [z_{1,j}^a, z_{2,j}^a].$$

Again, since A is the self-adjoint part of $\mathcal{R}(\mathcal{A})$, we have $z_{1,j}^s, z_{2,j}^s \in \mathcal{A}$. Therefore it is enough to show that every commutator $[z_{1,j}^a, z_{2,j}^a]$ is a finite sum of commutators of elements in A .

By [4, p. 121], $\mathcal{R}(\mathcal{A})$ is isomorphic to the matrix algebra $M_2(B)$, where B is a suitable real associative $*$ -algebra.

If we follow the proof of [23, Lemma 3.11], it follows that each commutator of skew-symmetric elements in $M_2(B)$ has the form

$$\begin{pmatrix} a & -c^* \\ c & b \end{pmatrix},$$

with

$$a + b = [a_1, a_2] + [b_1, b_2] + [c_1, c_2] + [d_1, d_2],$$

where a_j, b_j and c_j are skew-symmetric elements in B , while d_1 and d_2 are symmetric elements in B .

On the other hand, since for a, b, c, α_j and $\beta_j \in B$, with $a^* = -a$, $b^* = -b$, $\alpha_j^* = \alpha_j$ and $\beta_j^* = -\beta_j$, the following identities hold,

$$\begin{aligned} \begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} \right], \\ 2 \begin{pmatrix} a-b & 0 \\ 0 & b-a \end{pmatrix} &= \left[\begin{pmatrix} 0 & a-b \\ b-a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right], \\ \begin{pmatrix} [\alpha_1, \alpha_2] & 0 \\ 0 & [\alpha_1, \alpha_2] \end{pmatrix} &= \left[\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix} \right], \\ \begin{pmatrix} [\beta_1, \beta_2] & 0 \\ 0 & [\beta_1, \beta_2] \end{pmatrix} &= \left[\begin{pmatrix} 0 & -\beta_2 \\ \beta_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{pmatrix} \right], \\ \begin{pmatrix} a & -c^* \\ c & b \end{pmatrix} &= \begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a-b & 0 \\ 0 & b-a \end{pmatrix} + \frac{1}{2} \begin{pmatrix} a+b & 0 \\ 0 & a+b \end{pmatrix}, \end{aligned}$$

it can be concluded that each commutator $[z_{1,j}^a, z_{2,j}^a]$ is the sum of six commutators of elements in A . Therefore we have proved that

$$w = \sum_{j=1}^{140} [x_{1,j}, x_{2,j}],$$

where $x_{l,j} \in A$, for all l, j , which proves that

$$D = \sum_{j=1}^{140} \text{ad}([x_{1,j}, x_{2,j}]) = \sum_{j=1}^{140} [L_{x_{1,j}}, L_{x_{2,j}}]. \quad \square$$

Recall that a derivation on a JB-algebra is automatically continuous and that a JB-algebra has an approximate unit [9, 3.5.4]. Thus a derivation leaves each closed ideal invariant. By combining Theorem 1 with the comments preceding it, we have the following corollary.

COROLLARY 1. *Each derivation on a reversible JW-algebra is a sum of at most 151 commutators of the form $[L_a, L_b]$.*

The next theorem is the main result of this section.

THEOREM 2. *Cartan factors of type 1 with $\dim H = \dim K$, Cartan factors of type 2 with $\dim H$ even or infinite, and all Cartan factors of type 3 have the inner derivation property. Moreover, every derivation of the above Cartan factors has degree at most 153.*

Proof. By Proposition 2, such factors are unital reversible JW*-algebras. Thus it is enough to prove the statement for a unital reversible JW*-algebra Z .

It is well known that Z decomposes in the form $Z = X + iX$, where X is the symmetric part of Z , and hence X is a reversible JW-algebra.

If δ is a triple derivation of Z , then, by Lemma 2, $\delta - L_{\delta u}$ is a derivation of the JB*-algebra Z that commutes with the involution, and hence its restriction to X is a derivation of X . From the identity

$$(\delta - L_{\delta u})(z) = (\delta - L_{\delta u})(x + iy) = (\delta - L_{\delta u})|_X(x) + i(\delta - L_{\delta u})|_X(y),$$

it follows that $(\delta - L_{\delta u})|_X$ determines $(\delta - L_{\delta u})$. Now, Corollary 1 gives (except for summing the 0 commutator)

$$\begin{aligned} (\delta - L_{\delta u})(z) &= \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](x) + i \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](y) \\ &= \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](x + iy) = \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](z). \end{aligned}$$

Now, applying the identity

$$[L_a, L_b] + [L_c, L_d] = 2(\tilde{\delta} - L_{\tilde{\delta}u}),$$

for all a, b, c and d in X , where

$$\tilde{\delta} = \delta \left(\frac{a + ic}{2}, \frac{b + id}{2} \right),$$

we obtain

$$(\delta - L_{\delta u})(z) = \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](z) = 2 \sum_{j=1}^{76} (\delta(c_j, d_j) - L_{\delta(c_j, d_j)u})(z).$$

Finally, if we apply Lemma 1, it follows that

$$\delta = 2 \sum_{j=1}^{76} (\delta(c_j, d_j) - L_{\delta(c_j, d_j)u}) + L_{\delta u}$$

is an inner derivation with degree at most 153. \square

Following [13], we define a *real Cartan factor* to be a real form of a complex Cartan factor. Combining Theorem 2 and Proposition 1, we obtain the following result for real Cartan factors.

COROLLARY 2. *If E is a real form of a type 1 Cartan factor with $\dim H = \dim K$, or a real form of a Cartan factor of type 2 with $\dim H$ even or infinite, or a real form of a Cartan factor of type 3, then every derivation on E is inner with degree at most 306.*

3.2. Real or complex spin factors

In this subsection, we prove that no infinite-dimensional real spin factor satisfies the inner derivation property. Thus, by Proposition 1, it can be concluded that no complex spin factor satisfies the inner derivation property.

We recall that a *complex spin Cartan factor* is a JB*-triple that can be equipped with a complete inner produce $(\cdot|\cdot)$ and a conjugation $*$ such that the triple product satisfies

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|z^*)y^*,$$

and the norm is given by

$$\|x\|^2 := (x|x) + ((x|x)^2 - |(x|x^*)|^2)^{1/2}.$$

By a *real spin factor*, we mean any real form of a complex spin factor. By [13, Theorem 4.1], we know that every real spin factor E is an l_1 -sum

$$E = X_1 \oplus^{\ell_1} X_2,$$

where X_1 and X_2 are closed subspaces of a real Hilbert space X satisfying $X_2 = X_1^\perp$, and the triple product on E is given by

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|\bar{z})\bar{y},$$

where $(\cdot|\cdot)$ is the inner product of X and the map $x \mapsto \bar{x}$ is given by $\bar{x} = (x_1, -x_2)$ for all $x = (x_1, x_2) \in E$.

Our goal is to build a derivation that is not inner in the case of an infinite-dimensional real spin factor $E = X_1 \oplus^{\ell_1} X_2$. Without loss of generality, we can assume that X_1 is also infinite-dimensional.

First we suppose that E is separable. Let $\{e_n : n \in \mathbb{N}\}$ be a countable orthonormal basis of X_1 . Since $\bar{e}_n = e_n$, it is easy to check that $\{e_n, e_n, e_n\} = e_n$ and

$\|\delta(e_{2k-1}, e_{2k})\| \leq 2$, and hence the operator

$$\delta_0 := \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k})$$

is a well defined derivation on E . Our goal is to show that δ_0 is not inner. Suppose that δ_0 is inner; then

$$\delta_0 = \sum_{j=1}^P \delta(a_j, b_j)$$

for suitable $a_j, b_j \in E$, with $a_j = a_{j,1} + a_{j,2}$ and $b_j = b_{j,1} + b_{j,2}$, where $a_{j,i}$ and $b_{j,i}$ are in X_i ($j = 1, \dots, P$, $i = 1, 2$). Hence

$$\begin{aligned} \delta_0 &= \sum_{j=1}^P \delta(a_j, b_j) \\ &= \sum_{j=1}^P \delta(a_{j,1}, b_{j,1}) + \delta(a_{j,1}, b_{j,2}) + \delta(a_{j,2}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}). \end{aligned}$$

It is easy to check that, for all $x_1 \in X_1$,

$$\delta(a_{j,2}, b_{j,2})(x_1) = \delta(a_{j,1}, b_{j,2})(x_1) = \delta(a_{j,2}, b_{j,1})(x_1) = 0$$

and $\delta_0(X_2) = 0$. Therefore

$$\delta_0(x_1) = \sum_{j=1}^P \delta(a_{j,1}, b_{j,1})(x_1)$$

for all $x_1 \in X_1$.

Now we define K as the linear span of $\{a_{j,1}, b_{j,1} : j = 1, \dots, P\}$. Let $x_1 \in K^\perp \cap X_1$; then

$$\begin{aligned} 0 &= \sum_{j=1}^P \delta(a_{j,1}, b_{j,1})(x_1) = \delta_0(x_1) = \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k})(x_1) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} (\{e_{2k-1}, e_{2k}, x_1\} - \{e_{2k}, e_{2k-1}, x_1\}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} ((e_{2k-1}|e_{2k})x_1 + (x_1|e_{2k})e_{2k-1} - (e_{2k-1}|x_1)e_{2k} \\ &\quad - (e_{2k}|e_{2k-1})x_1 - (x_1|e_{2k-1})e_{2k} + (e_{2k}|x_1)e_{2k-1}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} ((x_1|e_{2k})e_{2k-1} - (e_{2k-1}|x_1)e_{2k}). \end{aligned}$$

Thus $(x_1|e_{2k}) = (e_{2k-1}|x_1) = 0$ for all $k \in \mathbb{N}$, and so $x_1 = 0$ since $\{e_n\}$ is a basis of X_1 . Therefore $K^\perp \cap X_1 = 0$, and hence $X_1 = K$ is finite-dimensional, which is impossible.

This proves that δ_0 is not an inner derivation. Suppose now that $\dim X_1 > \aleph_0$, and let $\{e_n\}_{\mathbb{N}}$ be a countable set of orthonormal vectors in X_1 . Let us denote by H the real separable Hilbert space generated by $\{e_n\}_{\mathbb{N}}$, and by δ_0 the derivation on E

given by

$$\delta_0 := \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k}).$$

Since $\delta_0(H) \subseteq H$, it follows that $\delta_0|_H$ is a derivation of the real spin factor H , which is not inner by the previous case. Actually, we claim that δ_0 is not an inner derivation on E . Suppose, contrary to our claim, that δ_0 is inner on E ; then

$$\delta_0 = \sum_{j=1}^P \delta(a_j, b_j)$$

with $a_j, b_j \in E$. Since

$$E = (H \oplus^{\ell_2} H^\perp) \oplus^{\ell_1} X_2,$$

the elements a_j and b_j can be expressed as $a_j = h_j + x_{j,3}$ and $b_j = k_j + y_{j,3}$, where h_j and k_j are in H and $x_{j,3}, y_{j,3} \in H^\perp \oplus^{\ell_1} X_2$ ($j = 1, \dots, P$). Thus

$$\begin{aligned} \delta_0 &= \sum_{j=1}^P \delta(a_j, b_j) \\ &= \sum_{j=1}^P \delta(h_j, k_j) + \delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j) + \delta(x_{j,3}, y_{j,3}). \end{aligned}$$

It is easy to check that

$$\delta(h_j, y_{j,3})h = -(h_j|h)\overline{y_{j,3}} - (h|h_j)y_{j,3} \in H^\perp \oplus^{\ell_1} X_2$$

$$\delta(x_{j,3}, k_j)h = (h|k_j)x_{j,3} + (k_j|h)\overline{x_{j,3}} \in H^\perp \oplus^{\ell_1} X_2$$

and

$$\delta(x_{j,3}, y_{j,3})(h) = 0$$

for all $h \in H$. From the last identity, we have

$$\delta_0(h) = \sum_{j=1}^P \delta(h_j, k_j)(h) + \sum_{j=1}^P (\delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j))(h)$$

for all $h \in H$. Since $\delta_0(H) \subseteq H$ and $\sum_{j=1}^P (\delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j))(H) \subseteq H^\perp \oplus^{\ell_1} X_2$, we have

$$\delta_0(h) = \sum_{j=1}^P \delta(h_j, k_j)(h)$$

for all $h \in H$. Therefore $\delta_0|_H$ is an inner derivation on H , which is impossible, and hence δ_0 is not an inner derivation on E .

We have thus proved the following theorem.

THEOREM 3. *Every infinite-dimensional real or complex spin factor has a derivation that is not inner, that is, none of the infinite-dimensional real or complex spin factors has the inner derivation property.*

3.3. *Non-square type 1*

As in the case of a real or complex spin factor, we are going to build an outer derivation in every real form of an infinite-dimensional and non-square ($\dim H \neq \dim K$) type 1 Cartan factor. Again, using Proposition 1, we will conclude that no complex infinite-dimensional non-square type 1 Cartan factor satisfies the inner derivation property.

By [13, Theorem 4.1], we know that the real forms of a complex type 1 Cartan factor are precisely the real Banach spaces $\text{BL}(X, Y)$ of all bounded linear operators between two real Hilbert spaces X and Y or the real Banach spaces $\text{BL}(P, Q)$ of all bounded linear operators between two Hilbert spaces P, Q over the quaternion field. Thus it is enough to prove that $\text{BL}(X, Y)$, with $+\infty = \dim(X) > \dim(Y)$, possesses an outer derivation. We will divide the proof into several steps. In a first step, we suppose that $Y = \mathbf{R}$. In this case, $\text{BL}(X, \mathbf{R})$ is isometrically isomorphic, as a real JB^* -triple, to X equipped with the triple product

$$\{x, y, z\} = \frac{1}{2}((x|y)z + (z|y)x)$$

for all $x, y, z \in X$.

Let δ be a derivation on X ; then

$$\delta\{x, y, z\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\} \quad (*)$$

for all $x, y, z \in X$. Now from the expression of the triple product, we have

$$\begin{aligned} \delta\{x, y, z\} &= \frac{1}{2}((x|y)\delta z + (z|y)\delta x), \\ \{\delta x, y, z\} &= \frac{1}{2}((\delta x|y)z + (z|y)\delta x), \\ \{x, \delta y, z\} &= \frac{1}{2}((x|\delta y)z + (z|\delta y)x), \\ \{x, y, \delta z\} &= \frac{1}{2}((x|y)\delta z + (\delta z|y)x). \end{aligned}$$

Thus it follows from (*) that

$$\frac{1}{2}(((\delta x|y) + (x|\delta y))z + ((z|\delta y) + (\delta z|y))x) = 0$$

for all $x, y, z \in X$. In particular, we have

$$(x|\delta y) = -(\delta x|y)$$

for all $x, y \in X$, that is, $\delta^* = -\delta$. Therefore every derivation on X , regarded as the real type 1 Cartan factor $\text{BL}(X, \mathbf{R})$, is a skew-symmetric operator on X . Conversely, the following holds.

LEMMA 3. *If X is a real Hilbert space, regarded as the real Cartan factor $\text{BL}(X, \mathbf{R})$, then the derivations on X coincide with the skew-symmetric operators on X .*

Proof. Suppose that T is a skew-symmetric operator on X . The identities

$$\begin{aligned} T\{x, y, z\} &= \frac{1}{2}((x|y)Tz + (z|y)Tx), \\ \{Tx, y, z\} &= \frac{1}{2}((Tx|y)z + (z|y)Tx), \\ \{x, Ty, z\} &= -\frac{1}{2}((Tx|y)z + (Tz|y)x), \\ \{x, y, Tz\} &= \frac{1}{2}((x|y)Tz + (Tz|y)x), \end{aligned}$$

show that T is a derivation on X . □

The next proposition characterizes the inner derivations on X .

PROPOSITION 3. *The inner derivations on X , regarded as the real Cartan factor $\text{BL}(X, \mathbf{R})$, coincide with the finite rank operators on X that are skew-symmetric.*

Proof. Let

$$\delta = \sum_{j=1}^P \delta(a_j, b_j)$$

be an inner derivation on X . Since

$$\delta(a_j, b_j)(x) = \frac{1}{2}((x|b_j)a_j - (x|a_j)b_j),$$

it follows that δ is a finite rank operator. The other implication follows from Lemma 3. \square

REMARK 2. Since for every infinite-dimensional Hilbert space X there exists a skew-symmetric operator T on X satisfying $T^2 = -\text{Id}$, we conclude from Lemma 3 and Proposition 3 that T is an outer derivation on X . It follows that $\text{BL}(X, \mathbf{R})$ does not satisfy the inner derivation property.

Our next goal is to build derivations on $\text{BL}(X, Y)$ from derivations on $X = \text{BL}(X, \mathbf{R})$.

LEMMA 4. *Let δ be a derivation on a real Hilbert space X (regarded as the real Cartan factor $\text{BL}(X, \mathbf{R})$), and let Y be another real Hilbert space. Then the operator*

$$\begin{aligned} \tilde{\delta} : \text{BL}(X, Y) &\longrightarrow \text{BL}(X, Y) \\ \tilde{\delta}a &= a\delta \end{aligned}$$

is a derivation on $\text{BL}(X, Y)$.

Proof. Since δ is a derivation on X , $\delta^* = -\delta$ (see Lemma 3). Given $a, b, c \in \text{BL}(X, Y)$, we have

$$\begin{aligned} &\{\tilde{\delta}a, b, c\} + \{a, \tilde{\delta}b, c\} + \{a, b, \tilde{\delta}c\} \\ &= \frac{1}{2}(a\delta b^*c + cb^*a\delta + a\delta^*b^*c + c\delta^*b^*a + ab^*c\delta + c\delta b^*a) \\ &= \frac{1}{2}(a\delta b^*c + cb^*a\delta - a\delta b^*c - c\delta b^*a + ab^*c\delta + c\delta b^*a) \\ &= \frac{1}{2}(cb^*a\delta + ab^*c\delta) = \{a, b, c\}\delta = \tilde{\delta}\{a, b, c\}, \end{aligned}$$

which proves that $\tilde{\delta}$ is a derivation. \square

At this moment, we need the following identification. Let us fix a norm one element $y_0 \in Y$. In the sequel, we will identify each $h \in X$, with the operator

$$\begin{aligned} f_h : X &\longrightarrow Y \\ f_h(x) &:= (x|h)y_0 \quad x \in X. \end{aligned}$$

In this way, X can be regarded as the subspace of $\text{BL}(X, Y)$ formed by all operators of the form f_h with $h \in X$. Using this identification, it is easy to check that if δ and $\tilde{\delta}$ are as in Lemma 4, then $\tilde{\delta}(X) \subseteq X$. In fact,

$$\begin{aligned} \tilde{\delta}(f_h)(x) &= f_h(\delta x) = (\delta x|h)y_0 \\ &= (x|\delta^*h)y_0 = (x|-\delta h)y_0 = f_{-\delta h}(x). \end{aligned}$$

The next lemma is the key tool of the main result of this subsection.

LEMMA 5. *Let δ and $\tilde{\delta}$ be as in Lemma 4, and suppose that $\tilde{\delta}$ is an inner derivation. Then δ has rank less than or equal to the hilbertian dimension of Y .*

Proof. Since $\tilde{\delta}$ is an inner derivation on $\text{BL}(X, Y)$, $\tilde{\delta}$ is the sum

$$\tilde{\delta} = \sum_{j=1}^P \delta(a_j, b_j)$$

for suitable $a_j, b_j \in \text{BL}(X, Y)$. As we have seen previously for each $h \in X$, $\tilde{\delta}f_h = f_{-\delta h} \in X$. On the other hand,

$$\begin{aligned} f_{-\delta h} &= \tilde{\delta}(f_h) = \sum_{j=1}^P \delta(a_j, b_j)(f_h) \\ &= \sum_{j=1}^P \frac{1}{2}(a_j b_j^* f_h + f_h b_j^* a_j - b_j a_j^* f_h - f_h a_j^* b_j) \\ &= \left(\sum_{j=1}^P \frac{1}{2}(a_j b_j^* - b_j a_j^*) \right) f_h + f_h \left(\sum_{j=1}^P \frac{1}{2}(b_j^* a_j - a_j^* b_j) \right) \\ &= Rf_h + f_h T, \end{aligned}$$

where

$$\begin{aligned} R &= \sum_{j=1}^P \frac{1}{2}(a_j b_j^* - b_j a_j^*) : Y \longrightarrow Y \\ T &= \sum_{j=1}^P \frac{1}{2}(b_j^* a_j - a_j^* b_j) : X \longrightarrow X \end{aligned}$$

are two skew-symmetric operators. Moreover,

$$f_h T(x) = (Tx|h)y_0 = (x| -Th)y_0 = f_{-Th}(x)$$

for all $x \in X$, so that $f_h T = f_{-Th}$, and

$$Rf_h = \tilde{\delta}f_h - f_h T = f_{-\delta h - Th} = f_{h'} \in X.$$

Therefore, for all $x, h \in X$, the equality

$$Rf_h(x) = (x|h)R(y_0) = (x|h')y_0$$

holds. Thus we have $R(y_0) = \lambda y_0$ for a suitable $\lambda \in \mathbf{R}$. Since R is a skew-symmetric operator and λ is a real eigenvalue of R , $\lambda = 0$.

In this way, since $Rf_h = 0$, we have

$$f_{-\delta h} = \tilde{\delta}(f_h) = f_h T = f_{-Th}$$

for all $h \in X$, and hence $T = \delta$.

Since each $b_j^* a_j$ and each $a_j^* b_j$ are operators that factorize through Y , they have rank at most the hilbertian dimension of Y . Therefore so does

$$\delta = T = \sum_{j=1}^P \frac{1}{2}(b_j^* a_j - a_j^* b_j).$$

□

THEOREM 4. *Let X be an infinite-dimensional real Hilbert space, and Y be a real Hilbert space with hillbertian dimension less than the hillbertian dimension of X . Then $\text{BL}(X, Y)$ does not satisfy the inner derivation property.*

Proof. We recall that, since X is infinite-dimensional, there exists a bounded linear operator T on X such that $T^2 = -\text{Id}_X$ and $T^* = -T$. Hence T has rank equal to the hillbertian dimension of X . Since $T^* = -T$, Lemma 3 states that T is a derivation on X . Moreover, by Lemma 4, the operator \tilde{T} given by $\tilde{T}a = aT$ ($a \in \text{BL}(X, Y)$) is a derivation on $\text{BL}(X, Y)$. If \tilde{T} is an inner derivation, then Lemma 5 states that T has rank at most the hillbertian dimension of Y , which is impossible, since $\dim(X) > \dim(Y)$. \square

Again combining Theorem 3 and Proposition 1, we obtain the following corollary.

COROLLARY 3. *The complex infinite-dimensional non-square type 1 Cartan factors and their real forms do not satisfy the inner derivation property.*

By virtue of the previous results, we know that there exist real and complex JB*-triples having outer derivations. Therefore it is natural to ask if any derivation can be approximated (in a convenient topology) by inner derivations. Upmeyer [23] proved that there exists a unital JB-algebra X and a derivation D on X that cannot be approximated in norm by inner algebra derivations. Let \hat{X} denote the complexification of X , and \hat{D} the complex linear extension of D to \hat{X} . Then \hat{X} is a unital JB*-algebra with unit u , and hence a JB*-triple, and \hat{D} is a triple derivation, since \hat{D} is an algebra derivation that commutes with the involution (see Lemma 2). We claim that \hat{D} cannot be approximated in norm by inner triple derivations. Otherwise, for $\varepsilon > 0$, there would exist an inner triple derivation

$$\delta = \sum_j^P \delta(e_j, f_j)$$

such that

$$\|\hat{D} - \delta\| < \varepsilon.$$

Now, by Lemma 2,

$$\begin{aligned} \delta - L_{\delta(u)} &= \sum_j^P \delta(e_j, f_j) - L_{\delta(e_j, f_j)(u)} \\ &= \frac{1}{2} \sum_j^P [L_{a_j}, L_{c_j}] + [L_{b_j}, L_{d_j}], \end{aligned}$$

where $e_j = \frac{1}{2}(a_j + ib_j)$, $f_j = \frac{1}{2}(c_j + id_j)$ with a_j, b_j, c_j, d_j in X . Therefore $\delta - L_{\delta(u)}$ is an inner derivation on X such that

$$\begin{aligned} \|D - (\delta - L_{\delta(u)})\| &= \|D - L_{D(u)} - (\delta - L_{\delta(u)})\| \\ &\leq \|\hat{D} - \delta\| + \|L_{D(u)} - L_{\delta(u)}\| \\ &\leq \|\hat{D} - \delta\| + \|L_{D(u) - \delta(u)}\| \\ &\leq \|\hat{D} - \delta\| + \|(\hat{D} - \delta)(u)\| \leq 2\varepsilon, \end{aligned}$$

which is impossible, since D cannot be approximated in norm by an inner derivation.

On the other hand, D is also a derivation on the real JB^* -triple X . If D could be approximated in norm by inner triple derivations on X , then, for every $\varepsilon > 0$, there exists

$$\delta = \sum_j^P \delta(e_j, f_j)$$

with $e_j, f_j \in X$ such that $\|D - \delta\| \leq \varepsilon$. In this case, $\delta = \sum_j^P \delta(e_j, f_j)$ is an inner derivation on \hat{X} and

$$\|(\hat{D} - \delta)\| \leq 2\varepsilon.$$

This is impossible.

Upmeyer [23], also proved that every algebra derivation on a JB -algebra can be approximated in the strong operator topology by inner derivations. In [3, Theorem 4.6], Barton and Friedman proved that the set of all inner derivations on a JB^* -triple is dense in the set of all derivations with respect to the strong operator topology. This result can be extended to real JB^* -triples.

THEOREM 5. *The set of all inner derivations on a real JB^* -triple is dense in the set of all derivations with respect to the strong operator topology.*

Proof. Let E be a real JB^* -triple and δ a derivation on E . We consider

$$\begin{aligned} \hat{\delta} : \hat{E} &\longrightarrow \hat{E} \\ \hat{\delta}(x + iy) &:= \delta(x) + i\delta(y) \end{aligned}$$

the natural extension of δ to \hat{E} . Since \hat{E} is a complex JB^* -triple, by [3, Theorem 4.6], it follows that for every $x_1, \dots, x_n \in E \subset \hat{E}$ and every $\varepsilon > 0$ that there exists an inner derivation

$$\delta_1 = \sum_{j=1}^P \delta(a_j, b_j)$$

on \hat{E} such that $\|\hat{\delta}(x_l) - \delta_1(x_l)\| \leq \varepsilon$ for all $l = 1, \dots, n$.

Since $a_j = a_{j,1} + ia_{j,2}$ and $b_j = b_{j,1} + ib_{j,2}$, where $a_{j,k}$ and $b_{j,k}$ are in E , it is easy to check that

$$\begin{aligned} \delta_1(x_l) &= \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}) \\ &\quad + i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1})))x_l. \end{aligned}$$

Since $a_{j,k}, b_{j,k}$ and x_l are elements in E , it follows that

$$\sum_{j=1}^P i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))x_l \in iE.$$

Thus

$$\begin{aligned} \|\delta(x_l) - \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}))(x_l)\| \\ \leq \|\delta(x_l) - \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}))(x_l)\| \end{aligned}$$

$$\begin{aligned}
& -i \sum_{j=1}^P (L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))(x_l) \| \\
& = \|\hat{\delta}(x_l) - \sum_{j=1}^P (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}) \\
& \quad + i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))(x_l))\| \\
& = \|\hat{\delta}(x_l) - \delta_1(x_l)\| \leq \varepsilon
\end{aligned}$$

for all $l = 1, \dots, n$. □

PROBLEM 1. If we could obtain a universal bound for the degree of all derivations in a type 2 Cartan factor with $\dim H$ odd, we could try to determine all JBW*-triples of type I satisfying the inner derivation property following the techniques contained in Ho's dissertation [10].

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