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Automatic continuity of derivations on C*-algebras and JB*-triples

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ABSTRACT

We introduce the notion of Banach Jordan triple modules and determine the precise conditions under which every derivation from a JB*-triple E into a Banach (Jordan) triple E-module is continuous. In particular, every derivation from a real or complex JB*-triple into its dual space is automatically continuous, motivating the study (which we have carried out elsewhere) of weakly amenable JB*-triples. Specializing to C*-algebras leads to a unified treatment of derivations and Jordan derivations into modules, shedding light on a celebrated theorem of Barry Johnson.

1. Introduction

Results on automatic continuity of linear operators defined on Banach algebras comprise a fruitful area of research intensively developed during the last sixty years. The monographs [44,11,13] review most of the main achievements obtained during the last fifty years.

A linear mapping D from a Banach algebra A to a Banach A-bimodule is said to be a *derivation* if $D(ab) = D(a)b + aD(b)$, for every a, b in A. The pioneering work of W.G. Bade and P.C. Curtis (see [2]) studies the automatic continuity of a module homomorphism between bi-modules over $C(K)$-spaces.

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Some techniques developed in the just quoted paper were exploited by J.R. Ringrose to prove that every (associative) derivation from a C*-algebra A to a Banach A-bimodule M is continuous (compare [39]). The case in which $M = A$ was previously treated by S. Sakai [41] by way of spectral theory in A $(= M)$.

We consider the class of Banach (Jordan) triple modules, a class which includes, besides Banach modules over Banach algebras and Banach Jordan modules over Banach Jordan algebras, the dual space of every real or complex JB*-triple. In this setting, a conjugate linear (resp., linear) mapping δ from a complex (resp., real) Jordan triple E to a triple E-module is called a *derivation* if

$$\delta\{a,b,c\} = \big\{\delta(a),b,c\big\} + \big\{a,\delta(b),c\big\} + \big\{a,b,\delta(c)\big\}, \tag{1}$$

for every $a,b,c \in E$, where $\{.,.,.\}$ denotes the triple product in E on the left side of (1), and the module actions on the right side of (1).

We determine (in Theorem 8) the precise conditions in order that a derivation from a complex JB*-triple, E, into a Banach (Jordan) triple E-module is continuous. We subsequently show that every derivation from a real or complex JB*-triple into its dual space is automatically continuous, a fact which has significance for the recent study by the present authors (and a third author) of ternary weak amenability [25].

From one point of view (another is through infinite dimensional holomorphy) the theory of JB*-triples may be viewed as parallel to the theory of C*-algebras. The analog of the theorem of Sakai mentioned above, namely, the automatic continuity of a derivation from a JB*-triple into itself, that is, a *linear* map satisfying the derivation property (1), was proved by T.J. Barton and Y. Friedman [3] in the complex case and extended to the real case in [24]. Among the consequences of our main results, we obtain a completely different proof for the automatic continuity results obtained in the just quoted papers [3] and [24] as well as the automatic continuity result of Ringrose.

A *Jordan derivation* from a Banach algebra A into a Banach A-module is a linear map D satisfying $D(a^2) = aD(a) + D(a)a$, $(a \in A)$, or equivalently, $D(ab+ba) = aD(b) + D(b)a + D(a)b + bD(a)$, $(a,b \in A)$. Sinclair proved that a bounded Jordan derivation from a semisimple Banach algebra to itself is a derivation [42, Theorem 3.3], although this result fails for derivations of semisimple Banach algebras into a Banach bi-module. Nevertheless, a celebrated result of B.E. Johnson states that every bounded Jordan derivation from a C*-algebra A to a Banach A-bimodule is an associative derivation (cf. [29]). We are also able to remove the continuity assumption in the result of Johnson, a result which wasn't explicitly stated in the literature.

In Section 2 of this paper we recall the definition and basic properties of Jordan triples, define Jordan triple modules and submodules, and introduce and study a basic tool in our paper: the *quadratic annihilator* of a submodule. In Section 3 we prove the automatic continuity results by relating triple derivations to triple module homomorphisms and using the well known technique of separating spaces. The final section contains the consequences of our main results, both for JB*-triples and for C*-algebras.

All of our results, excepting Theorem 8, are valid for real or complex JB*-triples. It should be noted however that in Section 4 we use the fact that Theorem 8 is valid for the self-adjoint part of a C*-algebra, considered as a (reduced) real JB*-triple (see Proposition 13).

2. Jordan triple modules

2.1. Jordan triples

A complex (resp., real) *Jordan triple* is a complex (resp., real) vector space E equipped with a triple product

$$E \times E \times E \to E,$$

$$(xyz) \mapsto \{x,y,z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear (resp., linear) in the middle one satisfying the so-called *"Jordan Identity"*:

$$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{c,\{b,a,d\},e\} + \{c,d,\{a,b,e\}\} \tag{2}$$

for all a,b,c,d,e in E. When E is a Banach space and the triple product of E is continuous, we say that E is a *Banach Jordan triple*. Given x, y in E, the symbol $L(x, y)$ will denote the operator defined by $L(x, y)z := \{x, y, z\}$.

Some of the basic facts about Jordan algebras, triples, and modules that we refer to are in [25, 2.1, 2.2], which the reader is encouraged to review. A summary of the basic facts about the important subclass of JB*-triples (defined below) can be found in [40] and some of the references therein, such as [30,17,18,45,46].

An element e in a Jordan triple E is called a *tripotent* if $\{e,e,e\} = e$. Each tripotent e in E induces two decompositions of E (called *Peirce decompositions*) in the form:

$$E = E_0(e) \oplus E_1(e) \oplus E_2(e) = E^1(e) \oplus E^{-1}(e) \oplus E^0(e)$$

where $E_k(e) = \{x \in E\colon\ L(e,e)x = \frac{k}{2}x\}$ is the $k/2$-eigenspace of $L(e,e)$, for $k = 0,1,2$ and $E^k(e)$ is the k-eigenspace of the operator $Q(e)x = \{e,x,e\}$ for $k = 1,-1,0$. The projection onto $E_k(e)$, which is contractive, is denoted by $P_k(e)$ for $k = 0,1,2$. The following *Peirce rules* are satisfied:

(*a*) $E_2(e) = E^1(e) \oplus E^{-1}(e)$ and $E^0(e) = E_1(e) \oplus E_0(e)$,
(*b*) $\{E^i(e), E^j(e), E^k(e)\} \subseteq E^{ijk}(e)$ if $ijk \neq 0$,
(*c*) $\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e)$, where $i, j, k = 0,1,2$ and $E_l(e) = 0$ for $l \neq 0,1,2$,
(*d*) $\{E_0(e), E_2(e), E\} = \{E_2(e), E_0(e), E\} = 0$.

A (*complex*) *JB*-triple* is a complex Jordan Banach triple E satisfying the following axioms:

(*JB**1) For each a in E the map $L(a,a)$ is a hermitian operator on E with non-negative spectrum.
(*JB**2) $\|\{a,a,a\}\| = \|a\|^3$ for all a in A.

Every C*-algebra (resp., every JB*-algebra) is a JB*-triple with respect to the product $\{a,b,c\} = \frac{1}{2}\,(ab^*c + cb^*a)$ (resp., $\{a,b,c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$).

We recall that a *real* JB**-triple* is a norm-closed real subtriple of a complex JB*-triple (compare [27]). The class of real JB*-triples includes all complex JB*-triples, all real and complex C*- and JB*-algebras and all JB-algebras.

A complex (resp., real) *JBW*-triple* is a complex (resp., real) JB*-triple which is also a dual Banach space. It is a non-trivial result that the second dual of a JB*-triple E is a JBW*-triple.

A tripotent e in a real or complex JB*-triple E is called *minimal* if $E^1(e) = \mathbb{R}e$. In the complex setting this is equivalent to say that $E_2(e) = \mathbb{C}e$, because $E^{-1}(e) = iE^1(e)$, whereas in the real situation the dimensions of $E^1(e)$ and $E^{-1}(e)$ need not be correlated.

2.2. Jordan triple modules

Motivated by the theory of modules over a Jordan algebra due to Jacobson [28], we introduce Jordan triple modules. Let E be a complex (resp. real) Jordan triple. A *Jordan triple E-module* (also called *triple E-module*) is a vector space X equipped with three mappings

$$\{.,.,.\}_1 : X \times E \times E \to X, \qquad \{.,.,.\}_2 : E \times X \times E \to X,$$
$$\text{and} \quad \{.,.,.\}_3 : E \times E \times X \to X$$

satisfying the following axioms:

(*JTM*1) $\{x,a,b\}_1$ is linear in a and x and conjugate linear in b (resp., trilinear), $\{a,b,x\}_3$ is linear in b and x and conjugate linear in a (resp., trilinear) and $\{a,x,b\}_2$ is conjugate linear in a,b,x (resp., trilinear).

(*JTM*2) $\{x,b,a\}_1 = \{a,b,x\}_3$, and $\{a,x,b\}_2 = \{b,x,a\}_2$ for every $a,b \in E$ and $x \in X$.

(*JTM*3) Denoting by $\{.,.,.\}$ any of the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$, the Jordan identity (2) holds whenever one of the elements a,b,c,d,e is in X and the rest are in E.

When the products $\{.,.,.\}_1$, $\{.,.,.\}_2$ and $\{.,.,.\}_3$ are (jointly) continuous we shall say that X is a *Banach (Jordan) triple E-module*. Hereafter, the triple products $\{.,.,.\}_j$, $j=1,2,3$, which occur in the definition of Jordan triple module will be denoted simply by $\{.,.,.\}$ whenever the meaning is clear from the context.

A subspace S of a triple E-module X is said to be a *Jordan triple submodule* or a *triple submodule* if $\{E,E,S\} \subseteq S$ and $\{E,S,E\} \subseteq S$. In particular, every triple ideal J of E (i.e. $\{E,E,J\} \subseteq J$ and $\{E,J,E\} \subseteq J$) is a Jordan triple E-submodule of E.

It is obvious that every real or complex Jordan triple E is a *real* triple E-module. It is problematical whether every complex Jordan triple E is a complex triple E-module for a suitable triple product. We shall see later that triple modules have a priori a different behavior than bi-modules over associative algebras and Jordan modules over Jordan algebras (see Remark 12).

It is a bit laborious to check that the dual space, E^*, of a complex (resp., real) Jordan Banach triple E is a complex (resp., real) triple E-module with respect to the products:

$$\{a,b,\varphi\}(x) = \{\varphi,b,a\}(x) := \varphi\{b,a,x\}, \quad \text{and,} \quad \{a,\varphi,b\}(x) := \overline{\varphi\{a,x,b\}},$$

for all $\varphi \in E^*$, $a,b,x \in E$.

Given a triple E-module X over a Jordan triple E, the space $E \oplus X$ can be equipped with a structure of real Jordan triple with respect to the product $\{a_1+x_1,a_2+x_2,a_3+x_3\}_s = \{a_1,a_2,a_3\} + \{x_1,a_2,a_3\} + \{a_1,x_2,a_3\} + \{a_1,a_2,x_3\}$. Consistent with the terminology in [28, §II.5], $E \oplus X$ will be called the *triple split null extension* of E and X. In order to simplify notation, we shall write $\{.,.,.\}_s = \{.,.,.\}$ when no confusion arises.

As noted above, our definition of Jordan triple module is motivated by the theory of modules over a Jordan algebra due to Jacobson [28]. Subsequently, we noticed that Jordan triple modules over a commutative ring were defined in [33] in a form more suitable to a purely algebraic setting. Our definition is more suitable for the applications to C*-algebras.

2.3. Quadratic annihilator

Given an element a in a Jordan triple E, we shall denote by $Q(a)$ the conjugate linear operator on E defined by $Q(a)(b) := \{a,b,a\}$. The following formula is always satisfied

$$Q(a)Q(b)Q(a) = Q\big(Q(a)b\big) \quad (a,b \in E),$$

and remains true for $Q(\cdot)$ acting on a triple E-module X:

$$\big\{a,\big\{b,\{a,x,a\},b\big\},a\big\} = \big\{\{a,b,a\},x,\{a,b,a\}\big\}, \quad x \in X. \tag{3}$$

For each submodule S of a triple E-module X, we define its *quadratic annihilator*, $\mathrm{Ann}_E(S)$, as the set $\{a \in E\colon\ Q(a)(S) = \{a,S,a\} = 0\}$. Since S is a triple submodule of X, it follows by (3) that

$$\{a,E,a\} \subset \mathrm{Ann}_E(S), \quad \forall a \in \mathrm{Ann}_E(S), \quad \text{and} \tag{4}$$

$$\big\{b,\mathrm{Ann}_E(S),b\big\} \subseteq \mathrm{Ann}_E(S), \quad \forall b \in E. \tag{5}$$

Consequently, $\mathrm{Ann}_E(S)$ is an inner ideal of E (that is, a linear subspace J with $\{J,E,J\}\subset J$) whenever it is a linear subspace of E. Further, $\mathrm{Ann}_E(S)$ is a triple ideal of E whenever E is a JB*-triple and $\mathrm{Ann}_E(S)$ is a linear subspace of E since for JB*-triples, (5) implies $\{E,\mathrm{Ann}_E(S),E\}\subset \mathrm{Ann}_E(S)$.

Let E be a Jordan triple. Two elements a and b in E are said to be *orthogonal* (written $a\perp b$) if $L(a,b)=L(b,a)=0$.

Given an element a in a Jordan triple E, we denote $a^{[1]}=a$, $a^{[3]}=\{a,a,a\}$ and $a^{[2n+1]}:=\{a,a^{[2n-1]},a\}$ ($\forall n\in\mathbb{N}$). Jordan triples are power associative, that is, $\{a^{[k]},a^{[l]},a^{[m]}\}=a^{[k+l+m]}$ ($k,l,m\in 2\mathbb{N}+1$). The element a is called *nilpotent* if $a^{[2n+1]}=0$ for some n.

Let a be an element in a real (resp., complex) JB*-triple E. The symbol E_a will denote the JB*-subtriple generated by the element a.

Let X be a triple module over a Jordan triple E. We shall say that X has the property of *lifting orthogonality* (LOP for short) if

$$\{a,b,x\}=0,\quad \text{for every } x\in X,\ a,b\in E \text{ with } a\perp b.$$

It is easy to see that for every Jordan Banach triple E, the Jordan triple E-module E^* satisfies LOP. When a Jordan triple E is regarded as a real triple E-module with its natural products, then E also has LOP (see Corollary 5, which is where this concept is used). However, not every triple module has this property. Let A be a C*-algebra regarded as a complex JB*-triple with respect to $\{a,b,c\}:=\frac{1}{2}(ab^*c+cb^*a)$. The vector space $X=A$ is a real triple A-module with respect to the products $\{a,b,x\}_3:=\frac{1}{2}(abx+xba)$ and $\{a,x,b\}_2:=\frac{1}{2}(axb+bxa)$. Two elements a and b in a C*-algebra A are orthogonal if and only if $ab^*=b^*a=0$ (compare [21, p. 18], see also [8, Lemma 1]). Every real JB*-triple E is a real form of a complex JB*-triple, more concretely, there exists a unique complex JB*-triple structure on the complexification $\widehat{E}=E\oplus iE$, and a unique conjugation (i.e., conjugate-linear isometry of period 2) τ on $\widehat{E}$ such that $E=\widehat{E}^\tau:=\{x\in\widehat{E}\colon \tau(x)=x\}$ (cf. [27, Proposition 2.2]). Therefore, elements $a,b\in E$ are orthogonal in E if and only if they are orthogonal in $\widehat{E}$. In particular, results on orthogonality for elements of complex JB*-triples can be also used for the real case. On the other hand, it is not hard to find a C*-algebra A containing two orthogonal elements a,b with $\{a,b,x\}_3\neq 0$, for some $x\in A$.

We now give an example of the quadratic annihilator. Let J be a triple ideal of a real or complex JB*-triple E regarded as a Jordan triple E-submodule. We clearly have

$$\mathrm{Ann}_E(J):=\bigl\{a\in E\colon Q(a)(J)=0\bigr\}\supseteq J^\perp:=\{a\in E\colon a\perp J\}.$$

Suppose now that $a\in\mathrm{Ann}_E(J)$. Replacing J with its weak*-closure in E^{**}, we may assume that E is a JBW*-triple, J is a weak*-closed triple ideal and $Q(a)(J)=0$. By [26, Theorem 4.2(4)], there exists a weak*-closed triple ideal K in E such that $E=J\oplus K$ and $J\perp K$. Writing $a=a_1+a_2$ with $a_1\in J$ and $a_2\in K$, we deduce, by orthogonality, that $a_1^{[3]}=Q(a)(a_1)\in Q(a)(J)=0$, and hence $a=a_2\perp J$. It should be remarked that [26, Theorem 4.2(4)] is established only for complex JBW*-triples. However, every real JBW*-triple E can be regarded as a real form of a complex JBW*-triple B with respect to a weak*-conjugation τ on B (cf. [35]). Furthermore, a subset $J\subseteq E$ is a weak*-closed ideal of E if and only if $J\oplus iJ$ is a weak*-closed ideal of B. This argument shows that [26, Theorem 4.2(4)] remains valid for real JBW*-triples. We state the above example as:

Remark 1. Let E be a JB*-triple (resp., a real JB*-triple). For each triple ideal J in E we have $\mathrm{Ann}_E(J)=J^\perp$ is a norm-closed triple ideal of E. $\square$

3. Triple derivations and triple module homomorphisms

3.1. Triple derivations

Separating spaces have been revealed as a useful tool in results of automatic continuity. This tool has been applied by many authors in the study of automatic continuity of binary and ternary homomorphisms, derivations and module homomorphisms (see, for example, [37,2,48,9,43,44,11,12,23,14], among others).

Let $T : X \to Y$ be a linear mapping between two normed spaces. Following [38, p. 70], the *separating space*, $\sigma_Y(T)$, of T in Y is defined as the set of all z in Y for which there exists a sequence $(x_n) \subseteq X$ with $x_n \to 0$ and $T(x_n) \to z$. The *separating space*, $\sigma_X(T)$, of T in X is defined by $\sigma_X(T) := T^{-1}(\sigma_Y(T))$.

A straightforward application of the closed graph theorem shows that a linear mapping T between two Banach spaces X and Y is continuous if and only if $\sigma_Y(T) = \{0\}$. It is known that $\sigma_X(T)$ and $\sigma_Y(T)$ are closed linear subspaces of X and Y, respectively. For each bounded linear operator R from Y to another Banach space Z, the composition RT is continuous if, and only if, $\sigma_Y(T) \subseteq \ker(R)$. Further, there exists a constant $M > 0$ (which does not depend on R nor Z) such that $\|RT\| \leqslant M\|R\|$, whenever RT is continuous (compare [44, Lemma 1.3]).

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. We recall that a conjugate linear (resp., linear) mapping $\delta : E \to X$ is said to be a *derivation* if it satisfies (1).

Note that derivations on complex JB*-triples to themselves are linear mappings but that a derivation from a complex JB*-triple into a complex triple module is conjugate linear by this definition. This is not inconsistent, since as we have noted earlier, it is not clear that a complex JB*-triple E can be made into a complex triple E-module.

Lemma 2. *Let $\delta : E \to X$ be a triple derivation from a Jordan Banach triple to a Banach (Jordan) triple E-module. Then $\sigma_X(\delta)$ is a norm-closed triple E-submodule of X and $\sigma_E(\delta)$ is a norm-closed subtriple of E.*

Proof. Given a, b in E and $x \in \sigma_X(\delta)$, there exists a sequence (c_n) in E with $(c_n) \to 0$ and $\delta(c_n) \to x$ in norm. The sequence $(\{a, b, c_n\})$ (resp., $(\{a, c_n, b\})$) tends to zero in norm and $\delta\{a, b, c_n\} = \{\delta a, b, c_n\} + \{a, \delta b, c_n\} + \{a, b, \delta(c_n)\} \to \{a, b, x\}$ (resp., $\delta\{a, c_n, b\} \to \{a, x, b\}$), which proves the first statement.

If $a, b, c \in \sigma_E(\delta)$, then $\delta(a), \delta(b), \delta(c) \in \sigma_X(\delta)$ and hence, by the first statement, $\delta\{a, b, c\} \in \sigma_X(\delta)$, as required. □

Let $\delta : E \to X$ be a triple derivation from a Jordan Banach triple E to a Banach triple E-module. Since $\sigma_X(\delta)$ is a norm-closed triple E-submodule of X, $\mathrm{Ann}_E(\sigma_X(\delta))$ is a norm-closed inner ideal of E whenever it is a linear subspace of E (actually, in such a case, it is a triple ideal when E is a real or complex JB*-triple).

Let us take a in E. Since δ is in particular a conjugate linear mapping, $\sigma_X(\delta) \subseteq \ker(Q(a))$ if, and only if, $Q(a)\delta$ is a continuous linear mapping from E to X, and we deduce that

$$\mathrm{Ann}_E\big(\sigma_X(\delta)\big) = \big\{a \in E\colon\ Q(a)\delta \text{ is continuous}\big\}.$$

Moreover, for each a in E, $\delta Q(a) = Q(a)\delta + 2Q(a, \delta a)$, and it follows that $Q(a)\delta$ is continuous if, and only if, $\delta Q(a)$ is.

3.2. Triple module homomorphisms

Let X and Y be two triple E-modules over a real or complex Jordan triple E. A linear mapping $T : X \to Y$ is said to be a *triple E-module homomorphism* if the identities

$$T\{a, b, x\} = \big\{a, b, T(x)\big\} \quad \text{and} \quad T\{a, x, b\} = \big\{a, T(x), b\big\},$$

hold for every $a, b \in E$ and $x \in X$.

As above,

$$\mathrm{Ann}_E\big(\sigma_Y(T)\big) = \big\{a \in E\colon\ Q(a)T \text{ is continuous}\big\},$$

and since a triple E-module homomorphism $T : X \to Y$ commutes with $Q(a)$ (acting on X), we have

$$\mathrm{Ann}_E\big(\sigma_Y(T)\big) = \big\{a \in E\colon\ TQ(a) \text{ is continuous}\big\},$$

where $Q(a)$ acts on Y.

The argument applied in the proof of Lemma 2 is also valid to prove the following result.

Lemma 3. *Let E be a Jordan Banach triple and let $T : X \to Y$ be a triple E-module homomorphism between two Banach spaces which are triple E-modules with continuous module operations. Then $\sigma_Y(T)$ and $\sigma_X(T)$ are norm-closed triple E-submodules of Y and X, respectively.* □

The following lemma provides a key tool needed in our main result.

Lemma 4. *Let E be a Jordan Banach triple, X a Banach triple E-module satisfying LOP, Y a Banach space which is a triple E-module with continuous module operations and $T : X \to Y$ a triple module homomorphism. Then for every sequence (a_n) of mutually orthogonal non-zero elements in E, we have*:

(*a*) *$Q(a_n)^2T$ is continuous for all but a finite number of n*;
(*b*) *$a_n^{[3]}$ belongs to $\mathrm{Ann}_E(\sigma_Y(T))$ for all but a finite number of n*;
(*c*) *the set*

$$\left\{\frac{\|Q(a_n^{[3]})T\|}{\|a_n\|^6}\colon\ Q\big(a_n^{[3]}\big)T \text{ is continuous}\right\}$$

is bounded.

Proof. Suppose that the statement (a) of the lemma is false. Passing to a subsequence, we may assume that $Q(a_n)^2T$ is an unbounded operator for every natural n. In this case we can find a sequence (x_n) in X satisfying $\|x_n\| \leqslant 2^{-n}\|a_n\|^{-2}$, and $\|Q(a_n)^2T(x_n)\| > n\ K_n$, where K_n is the norm of the bounded conjugate linear operator $Q(a_n) : Y \to Y$, $Q(a_n)y = \{a_n, y, a_n\}$. Since $Q(a_n)^2T$ is discontinuous $K_n = \|Q(a_n)\| \neq 0$, for every n. (Note that $\|Q(a)\| \leqslant M\|a\|^2$ for some constant M.)

The series $\sum_{k=1}^{\infty} Q(a_k)(x_k)$ defines an element z in the Banach triple module X. For $n \neq k$, the LOP and the identity

$$\big\{x, a_n, \{a_k, a_n, a_k\}\big\} + \big\{a_k, \{a_n, x, a_n\}, a_k\big\} = \big\{\{x, a_n, a_k\}, a_n, a_k\big\} + \big\{a_k, a_n, \{x, a_n, a_k\}\big\}$$

shows that $\{a_k, \{a_n, x, a_n\}, a_k\} = 0$. That is, $Q(a_k)Q(a_n) = 0$ for $k \neq n$ and the same argument shows that for any $b \in E$,

$$Q(a_k, b)Q(a_n) = 0 \quad \text{for } n \neq k. \tag{6}$$

Hence, for each natural n, we have

$$\begin{aligned} K_n\big\|T(z)\big\| &\geqslant \big\|Q(a_n)T(z)\big\| = \big\|TQ(a_n)(z)\big\| \\ &= \big\|TQ(a_n)^2(x_n)\big\| = \big\|Q(a_n)^2T(x_n)\big\| > K_n n, \end{aligned}$$

which is impossible. This proves (a).

Since $Q(a_n)^2T$ is continuous for all but a finite number of n and the module operations are continuous on Y, it follows that $Q(a_n)Q(a_n)^2T = Q(a_n)^3T = Q(a_n^{[3]})T$ is continuous (and hence, $a_n^{[3]} \in \mathrm{Ann}_E(\sigma_Y(T))$) for all but a finite number of n. This proves (b).

In order to prove (c) we may assume that $Q(a_n)^2 T$ is continuous for every natural n. Arguing by reduction to the absurd, we assume that $\{\frac{\|Q(a_n^{[3]})T\|}{\|a_n\|^6}\colon n \in \mathbb{N}\}$ is unbounded. There is no loss of generality in assuming that $\|a_n\| = 1$, for every n. By the Cantor diagonal process we may find a doubly indexed family $(a_{p,q})_{p,q\in\mathbb{N}}$ of mutually different elements from (a_n) and a doubly indexed family $(x_{p,q})$ in the unit sphere of X such that $\|Q(a_{p,q}^{[3]})T(x_{p,q})\| > 4^{2q}qp$. Let $b_p := \sum_{q=1}^{+\infty} 2^{-q} a_{p,q} \in E$. We observe that $a_{p,q} \perp a_{l,m}$ for every $(p,q) \neq (l,m)$. It is therefore clear that (b_p) is a sequence of mutually orthogonal elements in E. Having in mind that X satisfies LOP, we deduce from (3) and (6) that $Q(b_p)^2 Q(a_{p,q})(x) = 4^{-2q} Q(a_{p,q}^{[3]})(x)$, for every x in X. Thus,

$$\begin{aligned}\|Q(b_p)^2 T Q(a_{p,q})(x_{p,q})\| &= \|T Q(b_p)^2 Q(a_{p,q})(x_{p,q})\| \\ &= 4^{-2q}\|T Q(a_{p,q}^{[3]})(x_{p,q})\| = 4^{-2q}\|Q(a_{p,q}^{[3]})T(x_{p,q})\| > qp,\end{aligned}$$

for every p,q in $\mathbb{N}$, which shows that $Q(b_p)^2 T$ is unbounded for every $p \in \mathbb{N}$. This contradicts the first statement of the lemma and proves (c). □

Let E be a complex (resp., real) Jordan triple and let X be a triple E-module. It is not hard to see that for every derivation $\delta : E \to X$ the mapping

$$\begin{aligned}\Theta_\delta : E &\to E \oplus X, \\ a &\mapsto a + \delta(a)\end{aligned}$$

is a real linear Jordan triple monomorphism from the real Jordan triple E to the triple split null extension $(E \oplus X, \{.,.,.\}_s)$. (We observe that, in this case, E is regarded as a real Jordan triple whenever it is a complex Jordan triple.)

When X is a Jordan Banach triple E-module over a real or complex JB*-triple E, we define a norm, $\|.\|_0$, on the triple split null extension of E and X by the assignment $a + x \mapsto \|a + x\|_0 := \|a\| + \|x\|$. The real Jordan triple $E \oplus X$ becomes a real Jordan Banach triple. It is not hard to see that, in this setting, a derivation δ is continuous if, and only if, the triple monomorphism Θ_δ is. Moreover, the separating spaces $\sigma_X(\delta)$ and $\sigma_{E\oplus X}(\Theta_\delta)$ and their quadratic annihilators are linked by the following identities

$$\sigma_{E\oplus X}(\Theta_\delta) = \{0\} \times \sigma_X(\delta) \quad \big(\text{and hence, } \mathrm{Ann}_E\big(\sigma_{E\oplus X}(\Theta_\delta)\big) = \mathrm{Ann}_E\big(\sigma_X(\delta)\big)\big). \tag{7}$$

The linear space $E \oplus X$ and is made into a real triple E-module for the new products

$$\begin{aligned}\{a, b, c + x\}' = \{c + x, b, a\}' &:= \Theta_\delta\big(\{a, b, c\}\big) = \big\{\Theta_\delta(a), \Theta_\delta(b), \Theta_\delta(c)\big\}, \\ \{a, b + x, c\}' &:= \Theta_\delta\big(\{a, b, c\}\big)\end{aligned}$$

$(a, b, c \in E, x \in X)$. Clearly the $\|.\|_0$-closure, $\overline{\Theta_\delta(E)}$, of $\Theta_\delta(E)$ is a Jordan Banach real triple E-module with respect to the product $\{.,.,.\}'$, and satisfies the LOP. Under this point of view, the mapping $\Theta_\delta : E \to (\overline{\Theta_\delta(E)}, \{.,.,.\}')$ is a triple E-module homomorphism. The following result derives from the previous Lemma 4, since $Q(a)\Theta_\delta = Q(a) \oplus Q(a)\delta$ and every JB*-triple E satisfies LOP as a real triple E-module.

Corollary 5. *Let E be a complex (resp., real) JB*-triple, X a Banach space which is a triple E-module with continuous module operations and let $\delta : E \to X$ be a triple derivation. Then for every sequence (a_n) of mutually orthogonal non-zero elements in E, $Q(a_n)^2\delta$ is continuous for all but a finite number of n. It follows that $a_n^{[3]}$*

belongs to $\mathrm{Ann}_E(\sigma_X(\delta))$ *for all but a finite number of n. Moreover, the set*

$$\left\{\frac{\|Q(a_n^{[3]})\delta\|}{\|a_n\|^6}:\ Q\left(a_n^{[3]}\right)\delta \text{ is continuous}\right\}$$

is bounded. □

Let E be a real or complex JB*-triple. We shall say that E is *algebraic* if all singly-generated subtriples of E are finite dimensional. If in fact there exists $m \in \mathbb{N}$ such that all single-generated subtriples of E have dimension $\leqslant m$, then E is said to be of *bounded degree*, and the minimum such an m will be called the *degree* of E.

Corollary 6. *Let E be a complex (resp., real) JB*-triple, X a Banach triple E-module and let* $\delta : E \to X$ *be a triple derivation. Suppose that* $\mathrm{Ann}_E(\sigma_X(\delta))$ *is a norm-closed triple ideal of E. Then every element in* $E/\mathrm{Ann}_E(\sigma_X(\delta))$ *has finite triple spectrum, in other words, the JB*-triple* $E/\mathrm{Ann}_E(\sigma_X(\delta))$ *is isomorphic to a Hilbert space or, equivalently, it is algebraic of bounded degree.*

Proof. Let $\bar{a}$ be an element in the JB*-triple $F = E/\mathrm{Ann}_E(\sigma_X(\delta))$. Let I_a denote the intersection of E_a with $\mathrm{Ann}_E(\sigma_X(\delta))$. It is clear that I_a is a norm-closed triple ideal of E_a. Moreover, the subtriple $F_{\bar{a}}$ is JB*-triple isomorphic to the quotient of E_a with I_a.

E_a is JB*-triple isomorphic (and hence isometric) to $C_0(L) = C_0(L, \mathbb{C})$ (resp., $C_0(L) = C_0(L, \mathbb{R})$) for some locally compact Hausdorff space $L \subseteq (0, \|a\|]$ (called the triple spectrum of a) such that $L \cup \{0\}$ is compact (compare [30, Lemma 1.14] and [31, Proposition 3.5]). We shall identify E_a with $C_0(L)$. It is known (compare [16, Proposition 3.10]) that $E_a/I_a \cong C_0(\Lambda)$ where

$$\Lambda = \left\{t \in L:\ b(t) = 0,\ \text{for every } b \in I_a\right\}.$$

We claim that the set Λ is finite. Otherwise, there exists an infinite sequence (t_n) in Λ. We find a sequence (f_n) of mutually orthogonal elements in $C_0(L)$ such that $f_n(t_n) \neq 0$ and hence $f_n \notin I_a$ and $f_n^{[3]} \notin I_a$. Since orthogonality is a "local" concept, (compare Lemma 1 in [8], whose proof remains valid for real JB*-triples), (f_n) is a sequence of mutually orthogonal elements in E and $(f_n^{[3]}) \notin \mathrm{Ann}_E(\sigma_X(\delta))$, we have a contradiction to Corollary 5. It follows that $E_a/I_a \cong F_{\bar{a}}$ is finite dimensional. The final statement is derived from [6, §4] and [4, §3, Theorems 3.1 and 3.8]. □

3.3. Automatic continuity results

Our main result (Theorem 8) will be proved in two steps, the first being the following proposition.

Proposition 7. *Let E be a complex (resp., real) JB*-triple, X a Banach triple E-module, and let* $\delta : E \to X$ *be a triple derivation. Assume that* $\mathrm{Ann}_E(\sigma_X(\delta))$ *is a (norm-closed) linear subspace of E and that in the triple split null extension* $E \oplus X$,

$$\left\{\mathrm{Ann}_E\left(\sigma_X(\delta)\right), \mathrm{Ann}_E\left(\sigma_X(\delta)\right), \sigma_X(\delta)\right\} = 0. \tag{8}$$

Then $\delta|_{\mathrm{Ann}_E(\sigma_X(\delta))} : \mathrm{Ann}_E(\sigma_X(\delta)) \to X$ *is continuous.*

Proof. By Lemma 2, $\sigma_X(\delta)$ is a triple E-submodule of X. By assumptions the set $\mathrm{Ann}_E(\sigma_X(\delta))$ is a norm-closed subspace of E, then, as we commented before, it is a norm-closed triple ideal of E.

Fix two arbitrary elements a, b in $\mathrm{Ann}_E(\sigma_X(\delta))$. Since $a + b \in \mathrm{Ann}_E(\sigma_X(\delta))$, for every x in $\sigma_X(\delta)$, we have

$$2\{a, x, b\} = \{a + b, x, a + b\} - \{a, x, a\} - \{b, x, b\} = 0.$$

Hence, in addition to our assumption (8), we also have

$$\{a, x, b\} = 0, \quad \text{for every } x \in \sigma_X(\delta),\ a, b \in \operatorname{Ann}_E\big(\sigma_X(\delta)\big),$$

that is,

$$\big\{\operatorname{Ann}_E\big(\sigma_X(\delta)\big), \sigma_X(\delta), \operatorname{Ann}_E\big(\sigma_X(\delta)\big)\big\} = 0. \tag{9}$$

Considering $L(a,b)$ and $Q(a,b)$ as linear mappings from X to X defined by $L(a,b)(x) = \{a,b,x\}$ and $Q(a,b)(x) = \{a,x,b\}$ $(x \in X)$, we deduce from (8) and (9) that $\sigma_X(\delta) \subset \ker L(a,b) \cap \ker Q(a,b)$ and therefore that $L(a,b)\delta$, $Q(a,b)\delta : E \to X$ are continuous operators for every $a, b \in \operatorname{Ann}_E(\sigma_X(\delta))$.

When $L(a,b)$ and $Q(a,b)$ as considered as (real) linear operators from E to E, the compositions $\delta L(a,b)$ and $\delta Q(a,b)$ satisfy the identities

$$\begin{aligned}\delta L(a,b)(c) &= \big\{\delta(a), b, c\big\} + \big\{a, \delta(b), c\big\} + \big\{a, b, \delta(c)\big\}\\ &= \big\{\delta(a), b, c\big\} + \big\{a, \delta(b), c\big\} + L(a,b)\delta(c)\end{aligned}$$

and

$$\begin{aligned}\delta Q(a,b)(c) &= \big\{\delta(a), c, b\big\} + \big\{a, \delta(c), b\big\} + \big\{a, c, \delta(b)\big\}\\ &= \big\{\delta(a), c, b\big\} + Q(a,b)\delta(c) + \big\{a, c, \delta(b)\big\},\end{aligned}$$

for an arbitrary $c \in E$, where the terms in the right-hand sides are defined in terms of the module triple product. Since X is a Banach triple E-module, the continuity of $L(a,b)\delta$ and $Q(a,b)\delta$ as operators from E to X implies that the assignments $c \mapsto \delta(\{a,b,c\})$ and $c \mapsto \delta(\{a,c,b\})$ define continuous mappings from E to X. In summary, we have proved that given a, b in E, the mappings $E \to X$, $c \mapsto \delta(\{a,b,c\})$ and $c \mapsto \delta(\{a,c,b\})$ are continuous.

Let $W : \operatorname{Ann}_E(\sigma_X(\delta)) \times \operatorname{Ann}_E(\sigma_X(\delta)) \times \operatorname{Ann}_E(\sigma_X(\delta)) \to X$ be the real trilinear mapping defined by $W(a,b,c) := \delta(\{a,b,c\})$. We have already seen that W is separately continuous whenever we fix two of the variables in $(a,b,c) \in \operatorname{Ann}_E(\sigma_X(\delta)) \times \operatorname{Ann}_E(\sigma_X(\delta)) \times \operatorname{Ann}_E(\sigma_X(\delta))$. By repeated applications of the uniform boundedness principle, W is (jointly) continuous. Therefore, there exists a positive constant M such that $\|\delta\{a,b,c\}\| \leqslant M\ \|a\|\|b\|\|c\|$, for every $a, b, c \in \operatorname{Ann}_E(\sigma_X(\delta))$.

Finally, since $\operatorname{Ann}_E(\sigma_X(\delta))$ is a JB*-subtriple of E, for each a in $\operatorname{Ann}_E(\sigma_X(\delta))$, there exists b in $\operatorname{Ann}_E(\sigma_X(\delta))$ satisfying that $b^{[3]} = a$. In this case

$$\big\|\delta(a)\big\| = \big\|\delta\{b,b,b\}\big\| \leqslant M\|b\|^3 = M\big\|\{b,b,b\}\big\| = M\|a\|,$$

which shows that the restriction of δ to $\operatorname{Ann}_E(\sigma_X(\delta))$ is continuous. □

We can state now the main result of the paper.

Theorem 8. *Let E be a complex JB*-triple, X a Banach triple E-module, and let $\delta : E \to X$ be a triple derivation. Then δ is continuous if and only if $\operatorname{Ann}_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of E and*

$$\big\{\operatorname{Ann}_E\big(\sigma_X(\delta)\big), \operatorname{Ann}_E\big(\sigma_X(\delta)\big), \sigma_X(\delta)\big\} = 0,$$

in the triple split null extension $E \oplus X$.

Proof. If δ is continuous $\mathrm{Ann}_E(\sigma_X(\delta)) = \mathrm{Ann}_E(\{0\}) = E$ is a linear subspace of E and $\{E, E, 0\} = 0$.

Conversely, let us suppose that E is a complex JB*-triple and that $\mathrm{Ann}_E(\sigma_X(\delta))$ is a norm-closed subspace of E and hence a norm-closed triple ideal of E.

In order to simplify notation, we denote $J = \mathrm{Ann}_E(\sigma_X(\delta))$, while the projection of E onto E/J will be denoted by $a \mapsto \pi(a) = \bar{a}$.

By Corollary 6, E/J is algebraic of bounded degree m. Thus, for each element $\bar{a}$ in E/J there exist mutually orthogonal minimal tripotents $\bar{e}_1, \dots, \bar{e}_k$ in E/J and $0 < \lambda_1 \leqslant \dots \leqslant \lambda_k$ with $k \leqslant m$ such that $\bar{a} = \sum_{j=1}^{k} \lambda_j \bar{e}_j$. We shall show in the next two paragraphs that $e_1, \dots, e_k \in J$, and hence, $a \in J$. This will show that $E = J$ and application of Proposition 7 will complete the proof.

Suppose that $\bar{e}$ is a minimal tripotent in E/J, where $e \in E$ is a representative in the class $\bar{e}$. In this case $(E/J)_2(\bar{e}) = \mathbb{C}\bar{e}$. Take an arbitrary sequence (a_n) converging to 0 in E. For each natural n, there exists a scalar $\mu_n \in \mathbb{C}$ such that

$$\pi\big(Q(e)(a_n)\big) = Q(\bar{e})\big(\pi(a_n)\big) = Q(\bar{e})(\bar{a}_n) = \mu_n \bar{e} = \pi(\mu_n e).$$

The continuity of π and the Peirce-2 projection $P_2(\bar{e})$ assure that $\mu_n \to 0$. Thus, the sequence $Q(e)(a_n) - \mu_n e$ lies in J and tends to zero in norm.

By Proposition 7, $\delta|_J$ is continuous. Therefore,

$$\delta\big(Q(e)(a_n)\big) = \delta\big(Q(e)(a_n) - \mu_n e\big) + \mu_n \delta(e) \to 0.$$

Since (a_n) is an arbitrary norm null sequence in E, the linear mapping $\delta Q(e) : E \to X$ is continuous, and hence $e \in \mathrm{Ann}_E(\sigma_X(\delta)) = J$, or equivalently, $\bar{e} = 0$. □

4. Applications

4.1. Applications to JB*-triples

Let E be a real JB*-triple. By [27, Proposition 2.2], there exists a unique complex JB*-triple structure on the complexification $\widehat{E} = E \oplus iE$, and a unique conjugation (i.e., conjugate-linear isometry of period 2) τ on $\widehat{E}$ such that $E = \widehat{E}^{\tau} := \{x \in \widehat{E}\colon \tau(x) = x\}$, that is, E is a real form of a complex JB*-triple. Let us consider

$$\tau^{\sharp} : \widehat{E}^* \to \widehat{E}^*$$

defined by

$$\tau^{\sharp}(f)(z) = \overline{f\big(\tau(z)\big)}.$$

The mapping $\tau^{\sharp}$ is a conjugation on $\widehat{E}^*$. Furthermore the map

$$\big(\widehat{E}^*\big)^{\tau^{\sharp}} \to \big(\widehat{E}^{\tau}\big)^* \ (= E^*),$$
$$f \mapsto f|_E$$

is an isometric bijection, where $(\widehat{E}^*)^{\tau^{\sharp}} := \{f \in \widehat{E}^*\colon \tau^{\sharp}(f) = f\}$ (compare [27, p. 316]).

Remark 9. Let $\delta : E \to E^*$ be a triple derivation from a real JB*-triple to its dual. It is not hard (but tedious) to see that, under the identifications given in the above paragraph, the mapping $\widehat{\delta} : \widehat{E} \to \widehat{E}^*$, $\widehat{\delta}(x + iy) := \delta(x) - i\delta(y)$ is conjugate linear and a triple derivation from $\widehat{E}$ to $\widehat{E}^*$, when the latter is seen as a triple E-module.

Actually, although the calculations are tedious, the triple products of every real triple E-module, X, can be appropriately extended to its algebraic complexification $\widehat{X} = X \oplus iX$ to make the latter a complex triple $\widehat{E}$-module. Further, every (real linear) triple derivation $\delta : E \to X$ can be extended to a (conjugate linear) triple derivation $\widehat{\delta} : \widehat{E} \to \widehat{X}$.

The first statement of the following corollary was already established in [3, Corollary 2.2] and [24, Remark 1]. The proof given below is completely independent. The second statement is new and has significance in the recent study of weak amenability for JB*-triples [25].

Corollary 10. *Let E be a real or complex JB*-triple.*

(*a*) *Every derivation $\delta : E \to E$ is continuous.*
(*b*) *Every derivation $\delta : E \to E^*$ is continuous.*

The proof of this important corollary requires some preparation.

Let E be a JB*-triple (resp., a real JB*-triple). For each x in E, $E(x)$ will denote the norm-closure of $\{x, E, x\}$ in E. It is known that $E(x)$ coincides with the norm-closed inner ideal of E generated by x and $E_x \subseteq E(x)$ (see [7]). By [7, Proposition 2.1], $E(x)$ is a JB*-subalgebra of the JBW*-algebra $E(x)^{**} = \overline{E(x)}^{w^*} = E_2^{**}(r(x))$, where $r(x)$ is the (so-called) range tripotent of x in E^{**}. It is also known that $x \in E(x)_+$.

For each functional $\varphi \in E^*$, there exists a unique tripotent $s = s(\varphi)$ in E^{**} satisfying that $\varphi = \varphi P_2(s)$ and $\varphi|_{E_2^{**}(s)}$ is a faithful normal positive functional on $E_2^{**}(s)$ (compare [17, Proposition 2] and [35, Lemma 2.9] and [36, Lemma 2.7], respectively). The tripotent $s(\varphi)$ is called the *support tripotent* of φ in E^{**}.

Proposition 11. *Let E be a JB*-triple (resp., a real JB*-triple). For each triple submodule $S \subset E^*$,*

(*a*) *the quadratic annihilator* $\mathrm{Ann}_E(S)$ *is a norm-closed triple ideal of E,*
(*b*) $\mathrm{Ann}_E(S) = E \cap (\bigcap_{\varphi \in S} E_0^{**}(s(\varphi)))$*,*
(*c*) $\{\mathrm{Ann}_E(S), \mathrm{Ann}_E(S), S\} = 0$ *in the triple split null extension $E \oplus E^*$.*

Proof. We prove (b) first. For each $a \in \mathrm{Ann}_E(S)$ and each $\varphi \in S$, we have by definition, $\{a, \varphi, a\} = 0$ and hence $\varphi Q(a)(E) = 0$. It follows that $E(a) \subseteq \ker(\varphi)$ for every $\varphi \in S$, $a \in \mathrm{Ann}_E(S)$. In particular, $\varphi(a) = 0$. Since S is a triple submodule, for every $b \in E$, $\{\varphi, b, a\} \in S$, so $\{\varphi, b, a\}(a) = 0$, that is, $\varphi\{a, a, b\} = 0$.

Fix $\varphi \in S$. We have already seen that $\varphi\{a, a, b\} = 0$ for every $b \in E$. Since E is weak*-dense in E^{**} and $\varphi\{a, a, .\}$ is weak*-continuous on E^{**}, we deduce that $\varphi\{a, a, b\} = 0$, for every $b \in E^{**}$. Thus,

$$\varphi\big\{a, a, s(\varphi)\big\} = 0, \tag{10}$$

where $s = s(\varphi) \in E^{**}$ denotes the support tripotent of φ in E^{**}.

By [17, Proposition 2, Lemma 1.5] together with Peirce arithmetic, the mapping

$$(x, y) \mapsto \varphi\{x, y, s\} = \varphi\big\{P_2(s)x, P_2(s)y, s\big\} + \varphi\big\{P_1(s)x, P_1(s)y, s\big\}$$

is faithful and positive on $E_2^{**}(s) \oplus E_1^{**}(s)$, that is, $\varphi\{x, x, s\} \geqslant 0$ for every $x \in E_2^{**}(s) \oplus E_1^{**}(s)$ and $\varphi\{x, x, s\} = 0$ if and only if $x = 0$. By (10),

$$0 = \varphi\big\{a, a, s(\varphi)\big\} = \varphi\big\{P_2(s)a + P_1(s)a, P_2(s)a + P_1(s)a, s\big\},$$

which implies that $P_2(s)a = P_1(s)a = 0$.

We have shown that $\mathrm{Ann}_E(S) \subseteq E \cap E_0^{**}(s(\varphi))$, for every $\varphi \in S$. This assures that

$$\mathrm{Ann}_E(S) \subseteq E \cap \left(\bigcap_{\varphi \in S} E_0^{**}\big(s(\varphi)\big) \right). \tag{11}$$

To prove the reverse inclusion, let b belong to the right side of (11), let $\varphi \in S$ and let $c \in E$ have Peirce decomposition $c = c_2 + c_1 + c_0$ with respect to $s(\varphi)$. From Peirce arithmetic, $\{b, \varphi, b\}(c) = \varphi\{b, c, b\} = \varphi\{b, c_0, b\} = 0$, proving equality in (11) and establishing (b).

To prove (c), let $b, c \in \mathrm{Ann}_E(S)$ and $\varphi \in S$. Then for $x = x_2 + x_1 + x_0 \in E$ (where $x_i = P_i(s(\varphi))(x)$), by Peirce rules and properties of the support tripotent, $\{b, c, \varphi\}(x) = \varphi\{c, b, x\} = \varphi\{c, b, x_2\} + \varphi\{c, b, x_1\} + \varphi\{c, b, x_0\} = 0$, which proves (c).

Because of (4) and (5), to prove (a) it remains to show that $\mathrm{Ann}_E(S)$ is a linear subspace of E. Take $a, b \in \mathrm{Ann}_E(S)$. Since, by Peirce arithmetic, with $2Q(a,b) = Q(a+b) - Q(a) - Q(b)$, $Q(a,b)(E) \subseteq E \cap E_0^{**}(s(\varphi))$, and $L(a,b)(E) \subseteq E \cap (E_0^{**}(s(\varphi)) \oplus E_1^{**}(s(\varphi)))$, for every $\varphi \in S$, it follows that $\{a, \varphi, b\} = 0$, and $\{a, b, \varphi\} = 0$, for every $\varphi \in S$. Therefore (using only the first of these two facts),

$$Q(a+b)\varphi = Q(a)\varphi + Q(b)\varphi + 2Q(a,b)\varphi = 0,$$

for every $a, b \in \mathrm{Ann}_E(S)$ and $\varphi \in S$, which implies that $\mathrm{Ann}_E(S)$ is a linear subspace of E and completes the proof. □

We can now prove Corollary 10. The proof in the complex case follows now from Proposition 11 and Theorem 8. (In Theorem 8, we consider E as a real triple and as a real triple E-module, and δ as a real-linear map.) The statements in the real setting are, by Remark 9, direct consequences of the corresponding results in the complex case.

Recall that every (associative binary) derivation of a complex C*-algebra A into a Banach A-bimodule is automatically continuous [39]. The class of Banach triple modules over real or complex JB*-triples is strictly wider than the class of Banach bimodules over C*-algebras. Our next remark shows that, in the more general setting of triple derivations from real or complex JB*-triples to Banach triple modules the continuity is not, in general, automatic.

Remark 12. Let H be a real Hilbert space with inner product denoted by $(.,.)$. Suppose that $\dim(H) \geqslant 2$. Let J denote the Banach space $\mathbb{C}1 \oplus^{\ell_1} H$. It is known that J is a JB-algebra with respect to the product

$$(\lambda_1 1 + a_1) \circ (\lambda_2 1 + a_2) := \lambda_1 a_2 + \lambda_2 a_1 + \big(\lambda_1\lambda_2 + (a_1, a_2)\big)1.$$

The JB-algebra $(J, \circ)$ is called a *spin factor* (see [20]). It follows that J is a real JB*-triple via $\{a, b, c\} := (a \circ b) \circ c + (c \circ b) \circ a - (a \circ c) \circ b$, $(a, b, c \in J)$.

It was already noticed by Hejazian and Niknam (see [22, Definition 3.2]) that every Banach space X can be considered as a (degenerate) Jordan J-module with respect to the products

$$(\lambda_1 1 + a_1) \circ x = x \circ (\lambda_1 1 + a_1) = \lambda_1 x \quad (x \in X, \lambda_1 \in \mathbb{R}, a_1 \in H).$$

Since every linear mapping $D : J \to X$ with $D(1) = 0$ is a Jordan derivation (i.e. $D(a \circ b) = D(a) \circ b + a \circ D(b)$, $\forall a, b \in J$), for every infinite dimensional spin factor J, there exists a discontinuous derivation from J to a degenerate Jordan J-module.

Each degenerate Banach Jordan J-module X is a Banach triple J-module with respect to $\{a, b, x\} := (a \circ b) \circ x + (x \circ b) \circ a - (a \circ x) \circ b$ and $\{a, x, b\} = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x$ $(a, b \in J, x \in X)$, and each linear mapping $\delta : J \to X$ with $\delta(1) = 0$ is a triple derivation. Thus, for each infinite dimensional spin factor J there exists a discontinuous triple derivation from J to a Banach triple J-module.

4.2. Applications to C*-algebras

In view of the intense interest in automatic continuity problems in the past half century, it was natural to ask if the assumption of boundedness can be removed in Johnson's result, stated earlier, affirming that every bounded Jordan derivation from a C*-algebra A into a Banach A-bimodule is a derivation (compare [47, Question 14.i.] and [1]). In this section, we give a positive answer to this question, as well as a Jordan module version of it, as consequences of a more general theorem on triple derivations (Theorem 16 below). We shall also note how the answer also follows from results already in the literature.

The above Theorem 8 has been established only for complex JB*-triples and we shall need a technical reformulation of it. Actually, the proof given in Section 3 is not valid for real JB*-triples. The obstacles appearing in the real setting concern the structure of the Peirce-2 subspace associated with a minimal tripotent. We have already commented that, in case of E being a complex JB*-triple, the identity $E^{-1}(e) = iE^1(e)$ holds for every tripotent e in E, whereas in the real situation the dimensions of $E^1(e)$ and $E^{-1}(e)$ are not, in general, correlated. For example, every infinite dimensional rank-one real Cartan factor C contains a minimal tripotent e satisfying that $C^1(e) = \mathbb{R}e$ and $\dim(C^{-1}(e)) = +\infty$ (compare [15, Remark 2.6]).

Following [34, 11.9], we shall say that a real JB*-triple E is *reduced* whenever $E_2(e) = \mathbb{R}e$ (equivalently, $E^{-1}(e) = 0$) for every minimal tripotent $e \in E$. Reduced real Cartan factors were studied and classified in [34, 11.9] and in [32, Table 1]. Reduced real JB*-triples played an important role in the study of the surjective isometries between real JB*-triples developed in [15].

Having the above comments in mind, it is not hard to check that, in the particular subclass of reduced real JB*-triples the proof of Theorem 8 remains valid line by line. We therefore have:

Proposition 13. *Let E be a reduced real JB*-triple, X a Banach triple E-module, and let $\delta : E \to X$ be a triple derivation. Then δ is continuous if, and only if, $\operatorname{Ann}_E(\sigma_X(\delta))$ is a (norm-closed) linear subspace of E and*

$$\{\operatorname{Ann}_E(\sigma_X(\delta)), \operatorname{Ann}_E(\sigma_X(\delta)), \sigma_X(\delta)\} = 0,$$

in the triple split null extension $E \oplus X$. □

Every closed triple ideal of a reduced real JB*-triple is a reduced real JB*-triple. It is also true that the self-adjoint part, A_{sa}, of a C*-algebra, A, is a reduced real JB*-triple with respect to the product

$$\{a, b, c\} := \frac{abc + cba}{2} \quad (a, b, c \in A_{sa}). \tag{12}$$

Indeed, writing $e = p - q$ for a minimal partial isometry $e \in A_{sa}$ with p and q orthogonal projections, it is easy to check that $e = p$ or $e = -q$ and it follows that if $exe = -x$, then $x = 0$. In particular, for each closed triple ideal J of A_{sa}, the quotient A_{sa}/J is a reduced real JB*-triple.

Our next result is a consequence of the previous proposition. Note that the fact that A_{sa} is a reduced JB*-triple is only needed in the case that A is an abelian C*-algebra.

Proposition 14. *Let A be an abelian C*-algebra whose self-adjoint part is denoted by A_{sa}. Then, every triple derivation from A_{sa} to a real Jordan–Banach triple A_{sa}-module is continuous. Hence, every triple derivation from A into a real Jordan–Banach triple A-module is continuous.*

Proof. Let $\delta : A_{sa} \to X$ be a triple derivation from A_{sa} into a real Jordan triple A_{sa}-module. The statement of the proposition will follow from Proposition 13 as soon as we prove that $\operatorname{Ann}(\sigma_X(\delta)) = \operatorname{Ann}_{A_{sa}}(\sigma_X(\delta))$ is a (norm-closed) linear subspace of A_{sa} and

$$\{\operatorname{Ann}(\sigma_X(\delta)), \operatorname{Ann}(\sigma_X(\delta)), \sigma_X(\delta)\} = 0.$$

Let us take $a \in \mathrm{Ann}(\sigma_X(\delta))$. Having in mind that $a \in \mathrm{Ann}(\sigma_X(\delta))$ if, and only if, $Q(a)\delta$ (or equivalently, $\delta Q(a)$) is a continuous operator from A_{sa} to X (see the comments after Lemma 2), we observe that $\delta Q(a)$ is a continuous mapping from A_{sa} to X. Obviously, for each b in A_{sa}, the operator $L_b : A_{sa} \to A_{sa}$, $c \mapsto cb = bc$ is continuous. Since A is abelian we have $L(a^2, b) = Q(a)L_b = L_b Q(a)$. Therefore $\delta L(a^2, b) = \delta Q(a)L_b$ is a continuous operator from A_{sa} to X. The identity

$$\delta L\big(a^2, b\big) = L\big(\delta\big(a^2\big), b\big) + L\big(a^2, \delta(b)\big) + L\big(a^2, b\big)\delta$$

shows that $L(a^2, b)\delta$ is continuous. It is easy to check, from the definition of $\sigma_X(\delta)$, that $\{a^2, b, x\} = 0$, for every $x \in \sigma_X(\delta)$. It follows that

$$\big\{a^2, b, x\big\} = 0, \quad \text{for every } a \in \mathrm{Ann}\big(\sigma_X(\delta)\big),\ b \in A_{sa} \text{ and } x \in \sigma_X(\delta). \tag{13}$$

Let us write a in the form $a = a_1 - a_2$, where a_1 and a_2 are two orthogonal positive elements in A_{sa}. It is also known that $Q(a)(A_{sa}) \in \mathrm{Ann}(\sigma_X(\delta))$. Therefore, $a_1^3 = Q(a)(a_1) \in \mathrm{Ann}(\sigma_X(\delta))$ and hence $a_1^6 A_{sa} = Q(a_1^3)(A_{sa}) \subseteq \mathrm{Ann}(\sigma_X(\delta))$. This implies that the ideal of A_{sa} generated by a_1^6 lies in $\mathrm{Ann}(\sigma_X(\delta))$, which guarantees that $a_1 \in \mathrm{Ann}(\sigma_X(\delta))$. We can similarly show that a_2 belongs to $\mathrm{Ann}(\sigma_X(\delta))$. A similar argument shows that $a_1^{\frac{1}{2}}, a_2^{\frac{1}{2}} \in \mathrm{Ann}(\sigma_X(\delta))$. Now, we deduce from (13) that

$$\{a, b, x\} = \{a_1, b, x\} - \{a_2, b, x\} = 0, \tag{14}$$

for every $a \in \mathrm{Ann}(\sigma_X(\delta))$, $b \in A_{sa}$ and $x \in \sigma_X(\delta)$, or equivalently, $\delta L(a, b)$ and $L(a, b)\delta$ are continuous operators for every $a \in \mathrm{Ann}(\sigma_X(\delta))$ and $b \in A_{sa}$.

Since A is abelian, $L(a, b) = Q(a, b)$ in A_{sa}, and it follows from (14), that $\delta Q(a, b)$ and $Q(a, b)\delta$ are continuous operators from A_{sa} to X for every $a \in \mathrm{Ann}(\sigma_X(\delta))$ and $b \in A_{sa}$. This implies that

$$\{a, x, b\} = 0, \quad \text{for every } a \in \mathrm{Ann}\big(\sigma_X(\delta)\big),\ b \in A_{sa} \text{ and } x \in \sigma_X(\delta). \tag{15}$$

Finally, given a, c in $\mathrm{Ann}(\sigma_X(\delta))$, we deduce from (15) that

$$Q(a+c)\big(\sigma_X(\delta)\big) = Q(a)\big(\sigma_X(\delta)\big) + Q(c)\big(\sigma_X(\delta)\big) + 2Q(a, c)\big(\sigma_X(\delta)\big) = 0,$$

which shows that $a + c \in \mathrm{Ann}(\sigma_X(\delta))$, and hence the latter is a linear subspace of A_{sa}. □

Given any element x in a C*-algebra A, we shall denote by $C(x)$ the C*-subalgebra of A generated by x.

The following theorem, due to J. Cuntz (see [10]) is instrumental to our proof of Theorem 16.

Theorem 15. *(See [10, Theorem 1.3].) Let A be a C*-algebra and f a linear functional on A. If f is continuous on $C(h)$ for all $h = h^*$ in A, then f is continuous on A. Hence, by the uniform boundedness theorem, a linear mapping T from A to a normed space X is continuous if, and only if, it restriction to $C(h)$ is continuous for all $h = h^*$ in A.* □

Let $\delta : A \to X$ be a triple derivation from a C*-algebra to a Banach triple A-module. For each self-adjoint element h in A, the Banach space X can be regarded as a Jordan Banach $C(h)$-module by restricting the module operation from A to $C(h)$. Since $\delta|_{C(h)} : C(h) \to X$ is a triple derivation from an abelian C*-algebra into a Banach triple $C(h)$-module, Proposition 14 assures that $\delta|_{C(h)}$ is continuous. Combining this argument with the above Cuntz's theorem we have:

Theorem 16. *Let A be a C*-algebra. Then every triple derivation from A (resp., from A_{sa}) into a complex (resp., real) Jordan Banach triple A-module is continuous.* □

It is due to B.E. Johnson that every continuous Jordan derivation from a C*-algebra A to a Banach A-bimodule is a derivation (cf. [29, Theorem 6.2]). Since every Banach A-bimodule over a C*-algebra A, is a real Jordan triple A-module, and the restriction of a Jordan derivation to the self-adjoint part of a C*-algebra is a triple derivation, we have:

Corollary 17. *Let A be a C*-algebra. Then every Jordan derivation from A into a Banach A-bimodule X is continuous. In particular, every Jordan derivation from A to X is a derivation, by Johnson's theorem.* $\square$

Corollary 17 was already conjectured in [47, Question 14.i]. According to [5, §5], it "is an intriguing open question". In 2004, J. Alaminos, M. Brešar and A.R. Villena gave a positive answer to the above problem for some classes of C*-algebras including the class of von Neumann algebras and the class of abelian C*-algebras (cf. [1]). In the setting of general C*-algebras the question had remained open and never explicitly solved.

Corollary 17 has a natural generalization to the setting of Banach Jordan algebras. We recall that a linear mapping D from a JB*-algebra J to a Jordan Banach J-bimodule is said to be a *Jordan derivation* if $D(a \circ b) = D(a) \circ b + a \circ D(b)$, for every a, b in J, where $\circ$ denotes the Jordan product in J and the action of J on the Jordan J-module. Since every Jordan derivation is a triple derivation, and every Jordan module is a Jordan triple module, we have:

Corollary 18. *Let A be a C*-algebra. Then every Jordan derivation from A into a Jordan–Banach A-module X is continuous.* $\square$

In the category of JB*-algebras, S. Hejazian and A. Niknam established in [22] that every Jordan derivation from a JB*-algebra J into J or into J^* is automatically continuous. They also proved a theorem which provides necessary and sufficient conditions to guarantee that a Jordan derivation from a JB*-algebra J into a Jordan Banach J-module is continuous (cf. [22, Theorem 2.2]). When the domain JB*-algebra is a commutative or a compact C*-algebra A, the same authors proved that every Jordan derivation from A into a Jordan Banach A-module is continuous (cf. [22, Theorem 2.4 and Corollary 2.7]). In the setting of general C*-algebras, however, the question had remained open.

Prior to the writing of this paper, it apparently had escaped the attention of functional analysts that combining a theorem of Cuntz ([10], see Lemma 15 above) with the theorems just quoted from [1] and [22] concerning commutative C*-algebras yielded proofs of Corollaries 17 and 18.

In [19], U. Haagerup and N.J. Laustsen presented a new proof of Johnson's Theorem. Applying a result of automatic continuity in [22, Corollary 2.3], the just quoted authors proved that every Jordan derivation from a C*-algebra A to A^* is bounded and hence an inner derivation (cf. [19, Corollary 2.5]). This result can be improved now replacing A^* with a Banach A-bimodule or with a Jordan–Banach A-module.

Let $D : A \to X$ be an associative (resp., Jordan) derivation from a C*-algebra to a Banach A-bimodule. The space X, regarded as a real Banach space, is a real Banach triple A_{sa}-module with respect to the product defined in (12), where, in this case, one element in (a, b, c) is taken in X and the other two in A_{sa}. The restriction of D to A_{sa}, $\delta = D|_{A_{sa}} : A_{sa} \to X$ is a (real linear) triple derivation. Hence, Theorem 16 implies that δ (and hence D) is continuous. We thus have a new proof of a celebrated result of Ringrose.

Corollary 19 *(Ringrose). Let A be a C*-algebra. Then every derivation from A into a Banach A-bimodule X is continuous.* $\square$

In [5], M. Brešar studied a more general class of Jordan derivations from a C*-algebra A to an A-bimodule X. An additive mapping $d : A \to X$ satisfying $d(a \circ b) = d(a) \circ b + a \circ d(b)$, for every $a, b \in A$, is called an *additive Jordan derivation*. An additive Jordan derivation is said to be *proper* when it is not an associative derivation. Every (linear) Jordan derivation $D : A \to X$ is an additive Jordan derivation. However, the reciprocal implication is, in general, false. Actually, from [5, Theorem 5.1], for each unital C*-algebra A, there exists a proper additive Jordan derivation from A into some unital A-bimodule if, and only if, A contains an ideal of codimension one.

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