

The small Hankel operator in several complex variables

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Abstract. A survey of known results and open problems concerning boundedness, compactness, and trace ideal membership of the small Hankel operator. The setting is either the Bergman or Hardy space over a bounded symmetric domain or a strongly pseudoconvex domain in several complex variables, with special attention to the unit polydisk and multivariable harmonic analysis.

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This expository paper attempts to give the status of research on small Hankel operators in several complex variables insofar as it is concerned with certain basic problems for operators associated with a symbol. Although it is primarily about small Hankel operators on Hardy spaces, the Bergman spaces and big Hankel operators are also mentioned. Hankel operators, Toeplitz operators, and composition operators are at the center of the study of certain aspects of contemporary operator theory in function spaces. For any of these operators, one can consider the following problems: characterize the symbols for which the operator with that symbol is bounded, compact, or in a Schatten p -class. In this note we consider these problems for the small Hankel operator.

Hankel operators are of interest in pure and applied operator theory. They appear in the following contexts, to name a few (for the first three, see [39]):

- H^∞ control theory (engineering)
- interpolation problems (Nevanlinna-Pick, Caratheodory-Fejer)
- approximation theory
- noncommutative geometry (quantum Hall effect, [8])
- $\bar{\partial}_b$ equation ([1])

This paper contains three sections. Section 1 gives the background on the types of function spaces, domains, and operators of interest, and poses the problems to be discussed in later sections. The literature for the Bergman space versions of our problems is discussed here. In Section 2, the state of affairs regarding the Hardy spaces of the unit ball is discussed. Also in that section, the known results for the Hardy spaces of the unit disk are given. Most of these, as well as references to results on more general domains, are given in the monograph [62] so they are only mentioned briefly here without much discussion. Section 3 is an exposition

of the multiparameter harmonic analysis as it applies to the study of the small Hankel operator on the polydisk, which is also presented there.

1. Preliminaries

1.1. Bergman and Hardy spaces

Let Ω be a domain in \mathbf{C}^n . The *Bergman space* is the set of all holomorphic functions on Ω which are p -integrable with respect to Lebesgue volume measure dV on $\mathbf{C}^n = \mathbf{R}^{2n}$:

$$A^p(\Omega) \subset L^p(\Omega, dV) \quad 0 < p < \infty.$$

$A^p(\Omega)$ is a closed subspace of $L^p(\Omega, dV)$. When $n = 1$, we use the notation dA for dV . The Hardy space $\mathcal{H}^p(\Omega)$, $0 < p < \infty$, as well as the embedding $\mathcal{H}^p(\Omega) \subset L^p(\partial\Omega)$ are a little more complicated. We begin with three familiar cases. For any function f and $r > 0$, let $f_r(z) = f(rz)$.

In the following, $d\theta/2\pi$ denotes normalized Lebesgue measure on the unit circle $\mathbf{T} = \partial\Delta$, σ denotes a unique rotation invariant measure on the unit sphere ∂B , where B is the unit ball in \mathbf{C}^n , and in the case of the unit polydisk Δ^n , $r = (r_1, \dots, r_n)$ and $\theta = (\theta_1, \dots, \theta_n)$, with obvious meanings for $0 < r < 1$ and $e^{i\theta}$ in this case.

- $\Omega =$ the unit disk: for a holomorphic function f on the unit disk $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ and $0 < p < \infty$, $f \in \mathcal{H}^p(\Delta)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0 < r < 1} \int_0^{2\pi} |f_r(e^{i\theta})|^p d\theta/2\pi < \infty.$$

- $\Omega =$ the unit ball: for a holomorphic function f on the unit ball $B = \{z = (z_1, \dots, z_n) \in \mathbf{C}^n : \sum |z_j|^2 < 1\}$ and $0 < p < \infty$, $f \in \mathcal{H}^p(B)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0 < r < 1} \int_{\partial B} |f_r(\zeta)|^p d\sigma(\zeta) < \infty.$$

- $\Omega =$ the unit polydisk: for a holomorphic function f on the unit polydisk $\Delta^n \subset \mathbf{C}^n$ and $0 < p < \infty$, $f \in \mathcal{H}^p(\Delta^n)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0 < r < 1} \int_{\mathbf{T}^n} |f_r(e^{i\theta})|^p d\theta_1 \cdots d\theta_n / (2\pi)^n < \infty.$$

In each of the above cases, any \mathcal{H}^p function f has nontangential boundary values f^* almost everywhere, which belong to $L^p(\partial\Omega)$, and the map

$$\mathcal{H}^p \ni f \mapsto f^* \in L^p(\partial\Omega)$$

is norm preserving ([28], [56], [55]). In fact, for any bounded domain in \mathbf{C}^n with C^2 -boundary, f^* exists, see [33, Ch. 8]. Moreover, in the two cases considered below, that is, bounded symmetric domains and strongly pseudoconvex domains, the embedding $\mathcal{H}^p(\Omega) \subset L^p(\partial\Omega)$ is an isometry.

1.2. Domains of Interest

We shall limit our attention in this paper to two types of domains, namely, *bounded symmetric domains* and especially the unit polydisk, and *strongly pseudoconvex domains*. Before discussing the definitions, we show how the Hardy spaces are defined in each case. For a summary of an algebraic approach to bounded symmetric domains, see the survey paper [57].

A bounded symmetric domain can be defined as a domain in \mathbf{C}^n which is the open unit ball of a certain Banach space structure on \mathbf{C}^n . This is because in finite dimensions, the bounded symmetric domains have been classified, first using Lie theory [12], and afterwards using Jordan theory [32], [48]. Namely, the underlying Banach spaces of all finite dimensional bounded symmetric domains are contained in the following list. We shall not specify the norms in the last three cases. For a fuller discussion, see [48] or [57].

- $M_{m,n}(\mathbf{C})$: rectangular m by n complex matrices with the operator norm
- $S_n(\mathbf{C})$: symmetric n by n complex matrices with the operator norm
- $A_n(\mathbf{C})$: anti-symmetric n by n complex matrices with the operator norm
- Spin_n : the complex “spin factor” of dimension n
- I_{16} : the “exceptional” complex Jordan triple system of dimension 16
- I_{27} : the “exceptional” complex Jordan algebra of dimension 27

In particular, we obtain the unit disk, unit ball, and unit polydisk, from $M_{1,1}$, $M_{1,n}$, and $M_{1,1} \times M_{1,1} \times \cdots \times M_{1,1}$, respectively.

For any bounded symmetric domain Ω , there is a unique probability measure σ on the Silov boundary $\partial^* = \partial^*\Omega$, which is invariant under the action of the compact group of linear automorphisms of Ω . Since Ω is the open unit ball for a norm on \mathbf{C}^n , the following definition makes sense for a holomorphic function f on Ω ([27]).

For $0 < p < \infty$, $f \in \mathcal{H}^p(\Omega)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{0 < r < 1} \int_{\partial^*} |f_r(\zeta)|^p d\sigma(\zeta) < \infty.$$

A strongly pseudoconvex domain Ω is given by a defining function $\rho : \mathbf{C}^n \rightarrow (0, \infty)$ with certain properties which will not be mentioned here: $\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\}$. With Ω_ϵ defined by $\{\rho < -\epsilon\}$, the conditions on ρ guarantee the existence of a surface area probability measure σ_ϵ on $\partial\Omega_\epsilon$ so the following definition makes sense ([33, Ch. 8]). For $0 < p < \infty$, $f \in \mathcal{H}^p(\Omega)$ if

$$\|f\|_{\mathcal{H}^p}^p = \sup_{\epsilon > 0} \int_{\partial\Omega_\epsilon} |f(z)|^p d\sigma_\epsilon(z) < \infty.$$

The unit ball is an example of a bounded symmetric domain and of a strongly pseudoconvex domain, the defining function given by $\rho(z) = |z|^2 - 1$.

1.3. Operators of Interest

The Bergman space $A^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\Omega)$ and its orthogonal projection (the Bergman projection) is given as an integral operator with kernel $K(z, w)$ (the Bergman kernel). We shall denote this projection by P ,

$$Pf(z) = \int_{\Omega} f(w)K(z, w) dV(w) \quad , \quad f \in L^2(\Omega), \quad z \in \Omega.$$

Similarly, the Hardy space $\mathcal{H}^2(\Omega)$ is a closed subspace of the Hilbert space $L^2(\partial\Omega)$ and its orthogonal projection (the Szegő projection) is given as an integral operator with kernel $S(z, w)$ (the Szegő kernel). We shall denote this projection by S ,

$$Sf(z) = \int_{\partial\Omega} f(w)S(z, w) d\sigma(w), \quad f \in L^2(\partial\Omega), \quad z \in \Omega.$$

Let $f : \Omega \rightarrow \mathbb{C}$ (say $f \in L^2$) and define formally the following:

Hankel operator $H_f : A^2 \rightarrow A^{2\perp}$; $H_fg = (I - P)(fg)$, $g \in A^2$, $fg \in L^2$

Small Hankel operator $h_f : A^2 \rightarrow A^{2\perp}$; $h_fg = P(f\bar{g})$, $g \in A^2$, $f\bar{g} \in L^2$

We make several remarks in connection with these definitions. The definitions above are for the operators on the Bergman space with $p = 2$. There is a corresponding Hardy space operator in each case; replace Ω by $\partial\Omega$, A^2 by \mathcal{H}^2 and P by S . Although these operators can also be defined on A^p and \mathcal{H}^p for $0 < p \leq \infty$, we shall restrict our attention to the Hilbert space case of $p = 2$. Both of these operators are densely defined, and the small Hankel operators are conjugate linear. The small Hankel operator is essentially the same as the Hankel operator only in the case of $\mathcal{H}^2(\Delta)$, because $\mathcal{H}^2(\Delta)^\perp$ is one dimension away from $\overline{\mathcal{H}^2(\Delta)}$. It is sometimes convenient to consider these operators as acting from L^2 into L^2 .

The Bergman and Szegő projections are important tools in the study of operator theory in function spaces, and indeed are instrumental in the very definition of Hankel operators. Let's give some explicit formulas for the Bergman and Szegő kernels in the cases of interest to us.

unit ball: ([33, p.60,66]),

$$K(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \bar{w})^{n+1}},$$

$$S(z, w) = \frac{(n-1)!}{2\pi^n} \frac{1}{(1 - z \cdot \bar{w})^n}$$

strongly pseudoconvex domain: In this case, there is no explicit formula, but an asymptotic expansion due to C. Fefferman [22] allows one to transfer techniques known for the unit ball to this setting.

unit polydisk: ([33, p.61,67]),

$$K(z, w) = \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{(1 - z_j \bar{w}_j)^2},$$

$$S(z, w) = \frac{1}{(2\pi)^n} \prod_{j=1}^n \frac{1}{(1 - z_j \bar{w}_j)}$$

bounded symmetric domain: In this case, the Bergman kernel can be expressed in terms of the Jordan algebraic structure associated with bounded symmetric domains as follows:

$$K(z, w) = c \det B(z, w)^{-1}$$

where $B(x, y)$ is the “Bergman operator”. This description of the Bergman kernel can be found in [48] and [21], see also [57]. Formulas for these kernels can be found in [29], see [15].

1.4. Problems of Interest

For small Hankel operators on the Hardy space \mathcal{H}^2 or the Bergman space A^2 we shall be interested in the following natural questions.

1. For which symbols f is h_f bounded?
2. For which symbols f is h_f compact?
3. For which symbols f does h_f belong to some Schatten-von Neumann class \mathcal{S}_q , $0 < q < \infty$?

For a given domain, the above list implies that there are six questions of interest, three for the Hardy space and three for the Bergman space. In the case of the Bergman space, all of these problems have been essentially solved except for the third one in the case of a strongly pseudoconvex domain, see Table 2 below. In particular, all three questions have been answered for the Bergman space of a bounded symmetric domain and therefore for the Bergman space of the unit ball and of the unit polydisk.

On the contrary, all three problems are completely open in the case of the Hardy space of a bounded symmetric domain, and one of them (see Table 1) is open in the particular cases of the unit polydisk and a strongly pseudoconvex domain.

The above discussion is summarized in the following two tables, whose entries show the appropriate authors and year of publication for the solution of the problem associated with the entry. We have used the abbreviation SPCD for strongly pseudoconvex domain and BSD for bounded symmetric domain. An entry in parentheses means it was initially proved in a more general context. The author apologizes if there are some other references that should have been included here which have been overlooked.

Table 1: Problems on the Hardy space

	bounded	compact	Schatten
unit disk	Nehari 57	Hartman 58	Peller 80
unit ball	Coifman & Rochberg & Weiss 76	Coifman & Rochberg & Weiss 76	Feldman & Rochberg 90 Zhang 91
unit polydisk	Lin-Russo 95	Lin-Russo 95	OPEN
SPCD	Krantz-Li 95	Krantz-Li 95	OPEN
BSD	OPEN	OPEN	OPEN

Table 2: Problems on the Bergman space

	bounded	compact	Schatten
unit disk	Janson & Rochberg & Peetre 87	Janson & Rochberg & Peetre 87	Janson, Rochberg & Peetre 87; Arazy Fisher & Peetre 88
unit ball	Coifman & Rochberg & Weiss 76	Coifman & Rochberg & Weiss 76	Feldman & Rochberg 90 Burbea 87-unpub.
unit polydisk	(Zhu 95)	(Zhu 95)	(Zhu 95)
SPCD	Coupet 89	Coupet 89	OPEN
BSD	Zhu 95	Zhu 95	Zhu 95

In contrast to the Hardy space case, the theory of the small Hankel operator on the Bergman space is fairly complete. The following is an elaboration of Table 2.

unit disk: Boundedness and compactness have been characterized in terms of Bloch and little Bloch spaces in [31], and trace ideal criteria were worked out in terms of Besov spaces in [3],[31] and in [10].

unit ball: Boundedness and compactness criteria have been established in [17]. Trace ideal criteria are established in the unpublished paper [11], and are obtained as a consequence of the Hardy space theory in [24].

strongly pseudoconvex domain: Boundedness and compactness have been characterized in terms of Bloch and little Bloch spaces in [20]. The trace ideal criteria have not been done up to now, but there are some sufficient conditions in this case, as well as in the case of finite type domains in \mathbf{C}^n (convex if $n > 2$). Since the small Hankel operator is “dominated” by the big Hankel operator (see for example [36]), the work in the 1990s on the latter, for example [7],[40],[42], [38],[50], [51],[43],[44],[58], automatically give sufficient conditions for each of the three problems of interest. Finding conditions which are both necessary and sufficient for the small Hankel operator, and for the big Hankel operator for $p < 2$ has proved difficult to achieve, however see [44]. In this context there is also a useful

relation between Bergman space results and Hardy space results in one higher dimension, see [37] and [58] for example.

bounded symmetric domain: A complete theory of boundedness, compactness, and trace ideal criteria have been established in [65].

The theory of the big Hankel operator differs significantly from that of the small Hankel operator. For example, there are cut-off phenomenon, going back to the setting of \mathbf{R}^n in [30]. In more than one variable, the references below represent work which appeared in print after 1990.

The big Hankel operator on the *Hardy space* has been studied in at least two contexts, the unit ball [24] and the unit polydisk [19]. There does not seem to be any other references which study the big Hankel operator on Hardy spaces over domains more general than the unit ball and unit polydisk.

On the Bergman space there is more activity. The types of problems considered in this paper for the small Hankel operator have been studied for the big Hankel operator in the following works, which however will not be discussed here. In some cases, the operator in question is more general than the Hankel operator defined here. The author apologizes if some relevant references have been overlooked. In addition to the above references for strongly pseudoconvex domains, we also have [5],[3] for the unit disk, [4],[63],[59],[50] for the unit ball, [41],[64] for the unit polydisk, and [6],[61],[2] for bounded symmetric domains.

2. The small Hankel operator on the Hardy space of the unit ball

2.1. The unit disk

Let Δ be the open unit disk in \mathbf{C} with normalized Lebesgue measure dA , and $\mathbf{T} = \partial\Delta$ the unit circle with normalized arc length measure $d\sigma$. Let $\mathcal{H}^p = \mathcal{H}^p(\Delta)$ be the Hardy space for $p \geq 1$, and let $S : L^2(\mathbf{T}, d\sigma) \rightarrow \mathcal{H}^2$ be the orthogonal projection. For holomorphic f , the *small Hankel operator* h_f on \mathcal{H}^2 is defined by

$$h_f g = S(f\bar{g}), \quad g \in \mathcal{H}^2, \quad f\bar{g} \in L^2(\mathbf{T}, d\sigma).$$

We know by the theorems of Nehari and Hartman respectively, that h_f is bounded or compact if and only if $f \in BMOA$ or $f \in VMOA$ (see [62, Chapter 9] for details or [47] for a summary of this). Let \mathcal{S}_p be the Schatten class of operators on \mathcal{H}^2 . The following well known theorem due to Peller [49] characterizes those holomorphic functions f for which $h_f \in \mathcal{S}_p$.

Theorem 2.1. *Let f be a holomorphic function on Δ , and $p \geq 1$. Then $h_f \in \mathcal{S}_p$ if and only if*

$$\int_D |f''(z)|^p (1 - |z|^2)^{2p-2} dA(z) < \infty.$$

For a detailed proof, see [62, Chapter 9].

2.2. Boundedness and Compactness

Let B be the unit ball in \mathbf{C}^n , and let σ denote Lebesgue area measure on ∂B . Recall that the Hardy space $\mathcal{H}^p(B)$ consists of all holomorphic functions $F : B \rightarrow \mathbf{C}$ satisfying

$$\|F\|_p^p = \sup_{0 < r < 1} \int_{\partial B} |F(rz)|^p d\sigma(z) < \infty.$$

The space $BMO(B)$ is defined as the space of functions $b : \partial B \rightarrow \mathbf{C}$ such that

$$\|b\|_{BMO} = \sup_S \frac{1}{|S|} \int_S |b(y) - m_S(b)| d\sigma < \infty$$

where S runs over all spheres in ∂B with respect to the metric $|1 - z \cdot \bar{w}|^{1/2}$, and $m_S(b) = \int_S b d\sigma / |S|$.

The following three theorems form a pattern which can be used in various contexts. For the unit ball, they are contained in [17]

Theorem 2.2 (Factorization). *Every $F \in \mathcal{H}^1(B)$ can be written $F = \sum_i G_i H_i$, where $G_i, H_i \in \mathcal{H}^2(B)$ and $\sum \|G_i\|_2 \|H_i\|_2 \leq c \|F\|_1$.*

Theorem 2.3 (Boundedness). *For $f \in \mathcal{H}^2(B)$, if h_f denotes the small Hankel operator, then*

$$h_f \in B(\mathcal{H}^2(B)) \Leftrightarrow f \in BMOA(B).$$

The space $VMO = VMO(B)$ consists of those functions $b \in BMO(B)$ for which

$$\lim_{|S| \rightarrow 0} \frac{1}{|S|} \int_S |b - m_S(b)| d\sigma = 0,$$

and the space $VMOA(B)$ denotes the BMO -closure of the analytic polynomials. We have the duality relations:

$$VMOA^* = \mathcal{H}^1, \quad \mathcal{H}^{1*} = BMOA.$$

Theorem 2.4 (Compactness). *For $f \in \mathcal{H}^2(B)$, if h_f denotes the small Hankel operator, then*

$$h_f \text{ is compact} \Leftrightarrow f \in VMOA(B).$$

The results of this subsection have been proved in the setting of a bounded strongly pseudoconvex domain in \mathbf{C}^n with smooth boundary, as well as a bounded pseudoconvex domain of finite type in \mathbf{C}^2 in [34]. This work, as well as [35] has established the foundation for harmonic analysis on domains in several complex variables, and the results obtained, about Hardy spaces, BMO , Hankel operators, had been sought for fifteen years or more.

2.3. Trace Ideal Criteria

The problem of Schatten class membership has a history going back to 1980, but when restricted to the small Hankel operator on Hardy space, it has only been solved for the unit ball in one or several complex variables. More precisely, Peller, in [49] proved that the Hankel operator on the Hardy space of the unit disk belongs to the Schatten p -class \mathcal{S}_p if and only if the symbol belongs to the Besov space B^p , $1 \leq p < \infty$ (see 2.4). A similar theorem was obtained for the upper half plane in \mathbb{C} by Coifman and Rochberg [16] for $p = 1$ and by Rochberg [52] for $p > 1$.

In more than one variable, there are two papers which prove the corresponding result for the open unit ball, namely Feldman and Rochberg [24] and Zhang [60]. The former involves the techniques of harmonic analysis on the Heisenberg group as well as the notion of nearly weakly orthonormal sequences [53], [54]. The latter involves duality of Bergman spaces and complex interpolation theory. These two papers show the richness of the problem and provide ideas for generalizations to domains other than the unit ball. For completeness, we state the result here.

Theorem 2.5. *Let f be a holomorphic function on the unit ball B in \mathbb{C}^n and let $p \geq 1$. Then the small Hankel operator h_f belongs to the Schatten class \mathcal{S}_p over the Hardy space $\mathcal{H}^2(B)$ if and only if*

$$\sum_{|\alpha|=n+1} \int_B \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha} \right|^p (1 - |z|^2)^{(p-1)(n+1)} dV(z) < \infty.$$

2.4. Hankel operators in the Dixmier class

Let H be a Hilbert space over \mathbb{C} . For $0 < p < \infty$, recall that $T \in \mathcal{S}_p(H)$ (Schatten-von Neumann p -class) if $\{\mu_n(T)\}_{n=1}^\infty \in \ell^p$, where the $\mu_n(T)$ are the eigenvalues of $|T| = (T^*T)^{1/2}$.

An important class of operators which lies between $\mathcal{S}_1(H)$ and $\mathcal{S}_{1+\epsilon}(H)$ is the Macaev ideal $S_1^+(H)$ (also denoted by $L^{(1,\infty)}(H)$), which we shall call the Dixmier class, cf. [18]. We say that $T \in S_1^+(H)$ if $\{\sigma_n / \log n\}_{n=2}^\infty \in \ell^\infty$ where $\sigma_n = \sum_{j=1}^n \mu_j(T)$. This class was used in 1966 by Dixmier, see [18, p. 303] or [8, p. 5408], to settle in the negative the question of the uniqueness of the trace on $\mathcal{L}(H)$. We mention that S_1^+ is a Banach space under the norm: $\|T\|_{S_1^+} = \sup_{n \geq 2} \{\sigma_n(T) / \log n\}$.

More recently, J. Bellissard and co-workers have connected Hankel operators on the Hardy space of the unit disk with their study of the quantum Hall effect [8], thereby proposing the following question: What is the holomorphic function space which consists of precisely the symbols of Hankel operators belonging to the Dixmier class S_1^+ ? An answer to this question is given in [45] as follows.

Recall that for $1 \leq p < \infty$, $B^p(\Omega)$ denotes the holomorphic Besov space over a domain Ω in \mathbb{C}^n , with the seminorm $\|\cdot\|_{B^p}$ defined as follows:

$$\|f\|_{B^p}^p = \int_\Omega |f^{(n+1)}(z)|^p K(z, z)^{1-p} dV(z), \quad |f^{(n+1)}(z)| = \sum_{|\beta|=n+1} \left| \frac{\partial^{n+1} f}{\partial z^\beta} \right|.$$

For each $\alpha > 0$, we let $dV_\alpha(z) = c_\alpha(1 - |z|^2)^{\alpha-1}dV(z)$ where dV is Lebesgue volume measure, and $\int_\Omega dV_\alpha = 1$. Let $A_\alpha^2(\Omega)$ denote the weighted Bergman space on Ω and $P_\alpha : L^2(\Omega, dV_\alpha) \rightarrow A_\alpha^2(\Omega)$ the Bergman projection with Bergman kernel $K_w^\alpha(z) = K^\alpha(z, w) = c_\alpha(1 - z \cdot \bar{w})^{-n-\alpha}$. Note that, as a limiting case, $\alpha = 0$ gives rise to the Hardy space.

For a domain $\Omega \subset \mathbb{C}^n$, we say that a holomorphic function f over Ω belongs to $B_+^1(\Omega)$ if,

$$\|f\|_{B_+^1(\Omega)} = \int_\Omega \frac{|f^{(n+1)}(z)|}{1 + |\log F(f)|} dV(z) < \infty, \quad F(f) = \frac{1 + |f^{(n+1)}(z)|}{K_z(z)}.$$

Then we have the following theorem.

Theorem 2.6. *Let $\alpha \geq 0$ and let $f \in \mathcal{H}^2(B)$, where B is the unit ball. Then*

- (i) $h_f^\alpha \in S_1^+(A_\alpha^2(B))$ if and only if $\sup_{1 < p \leq 2} \{(p-1)\|f\|_{B^p}^p\} < \infty$.
- (ii) If $f \in B_+^1(B)$, then $h_f^\alpha \in S_1^+(A_\alpha^2(B))$.
- (iii) If $h_f^\alpha \in S_1^+(A_\alpha^2(B))$, then for any $p \in (1, 2)$,

$$\int_B |f^{(n+1)}(z)|(1 + |\log F(f)|)(1 + \log(1 + |\log F(f)|))^{-p} dV(z) < \infty.$$

By using the results on the boundedness and compactness of h_f in [34], [58] and the asymptotic expansion of the Bergman and Szegő kernels given in [22], one can prove Theorem 2.6 in the case of a smoothly bounded strictly pseudoconvex domain in \mathbb{C}^n . This remark can also apply to other domains in \mathbb{C}^n , such as bounded symmetric domains, by using the results proved in [65].

3. The small Hankel operator on the Hardy space of the bidisk

The problem of boundedness and compactness in this setting has been discussed in [47], where sufficient conditions are given, based on the study of multiparameter Fourier analysis done in [13]. A survey of these theorems on the unit disk (Theorems of Nehari and Hartman) as well as on the unit ball in several complex variables (Theorems of Coifman-Rochberg-Weiss) is given in [47, 1.1]. As in [17], a proof of boundedness could be based on a factorization theorem and a proof of compactness could be based on factorization and a duality between H^1 and $VMOA$ on the bidisc. The work in [46, Theorem 2.3] proves a factorization theorem for atoms but the proof for a general H^1 function, as stated there, is incomplete. The author is grateful to Aline Bonami for bringing this to his attention. Thus, the subtle question of whether the sufficient conditions for boundedness and compactness are also necessary is, as with H^1 -factorization, still an open problem.

The unit ball in \mathbb{C}^n is an example of a bounded symmetric domain and of a strongly pseudoconvex domain. For the small Hankel operator on the Hardy space in these contexts (other than for the unit disk or unit ball), the only criteria on boundedness and compactness are those of Krantz-Li for strongly pseudoconvex

domains [34]. It should be noted that some sufficient conditions for the boundedness, compactness, and belonging to a Schatten class are proved in [58] in the case of a bounded pseudoconvex domain of finite type in \mathbf{C}^2 with smooth boundary. Moreover, for a general strongly pseudoconvex domain, necessary and sufficient conditions are proved in [9], which also considers the problem in the setting of complex ellipsoids.

To summarize then, there are three ingredients needed for a theorem of Nehari type on the Hardy space of a domain. Namely,

- duality of \mathcal{H}^1 with BMO
- atomic decomposition of \mathcal{H}^1
- factorization in \mathcal{H}^1

As already noted, these results are known for the unit disk (Fefferman, Coifman), unit ball (Coifman-Rochberg-Weiss), and strongly pseudoconvex domains (Krantz-Li). For the polydisk, the first two are known (Chang-Fefferman). In this section we give an exposition in the case of the polydisk, and in the process correct some inaccuracies in [47] and [46].

3.1. Multiparameter Harmonic Analysis

Hardy spaces of the bidisc

We denote by $\Gamma_j(\theta_j)$ a standard cone in the unit disc Δ with vertex at $e^{i\theta_j} \in \mathbf{T}$, that is, for $j = 1, 2$,

$$\Gamma_j(\theta_j) = \{z_j \in \Delta : |1 - z_j e^{-i\theta_j}| < 1 - |z_j|\},$$

and we set, for $\theta = (\theta_1, \theta_2) \in \mathbf{T}^2$,

$$\Gamma(\theta) = \Gamma_1(\theta_1) \times \Gamma_2(\theta_2)$$

For a measurable function u on Δ^2 , let $N(u)(\theta)$ be the unrestricted nontangential maximal function,

$$N(u)(\theta) = \sup_{z \in \Gamma(\theta)} |u(z)|,$$

and let $A(u)$ denote the area integral of u , that is,

$$A^2(u) = A_{12}^2(u) + A_1^2(u) + A_2^2(u) + |u(0)|^2,$$

where

$$A_{12}^2(u)(\theta) = \int_{\Gamma(\theta)} |\nabla_1 \nabla_2 u(z)|^2 dz,$$

$$A_1^2(u)(\theta) = \int_{\Gamma_1(\theta_1)} |\nabla_1 u(z_1, 0)|^2 dz_1,$$

$$A_2^2(u)(\theta) = \int_{\Gamma_2(\theta_2)} |\nabla_2 u(0, z_2)|^2 dz_2,$$

and dz_j is Lebesgue measure on Δ , $dz = dz_1 dz_2$.

We shall be dealing with functions u which are harmonic in each variable: $\Delta_1 u = \Delta_2 u = 0$. For such a function, it is known ([26, Th.1]) that for $0 < p < \infty$, $N(u) \in L^p(\mathbf{T})$ if and only if $A(u) \in L^p(\mathbf{T})$, and the space $H^p(\Delta^2)$ is defined to be the set of functions u harmonic in each variable, such that this condition is satisfied. The space H^p is normed as follows:

$$\|u\|_{H^p} = \|N(u)\|_{L^p} \text{ or } \|A(u)\|_{L^p},$$

which are equivalent. We let f denote the boundary distribution of u and identify u with f when convenient. It is noteworthy that the usual holomorphic Hardy spaces defined in section 1.2 and denoted here by $H_A^p(\Delta^2)$, are included in these spaces; $H_A^p(\Delta^2) \subset H^p(\Delta^2)$.

There is a companion result which deals with the bi-upper half-plane $D = \mathbf{R}_+^2 \times \mathbf{R}_+^2$. For $x = (x_1, x_2) \in \mathbf{R}^2$, let

$$\Gamma(x) = \Gamma(x_1) \times \Gamma(x_2) = \{(y_1, t_1, y_2, t_2) : |x_1 - y_1| < t_1, |x_2 - y_2| < t_2\}.$$

Let $u(x, t)$ be harmonic in each variable (x_j, t_j) ($j = 1, 2$) and denote by u^* the nontangential maximal function

$$u^*(x_1, x_2) = \sup_{(y, t) \in \Gamma(x)} |u((y_1, t_1, y_2, t_2))|,$$

and Su the square function

$$S^2(u)(x) = \int_{\Gamma(x)} |\nabla_1 \nabla_2 u(y, t)|^2 dy_1 dy_2 dt_1 dt_2.$$

For a function $f : \mathbf{R}^2 \rightarrow \mathbf{C}$, let $u(x, t) = P[f](x, t)$ be its bi-Poisson integral. Then, for $0 < p < \infty$, by definition, f belongs to $H^p(D)$ if $u^* \in L^p(\mathbf{R}^2)$. It is known that this is the case if and only if $S(u) \in L^p(\mathbf{R}^2)$ ([23, pp.103–109]). As above, $H_A^p(D) \subset H^p(D)$ and $\|f\|_{H^p}$ is given by either of the equivalent norms $\|u^*\|_{L^p}$ or $\|Su\|_{L^p}$.

The space $H^1(D)$ will be of special interest to us. It is defined as

$$\begin{aligned} H^1(D) &= \{u : u \text{ harmonic on } D, u^* \in L^1(\mathbf{R}^2)\} \\ &= \{f : f \text{ defined on } \mathbf{R}^2, u = \text{Poisson integral of } f, f^* = u^* \in L^1(\mathbf{R}^2)\}. \end{aligned}$$

It is proved in [13, Th.1] that

$$H^1(D) = \{f = \sum \lambda_k a_k, a_k \text{ atoms}, \sum |\lambda_k| \leq A\|f\|_{H^1}\}. \quad (1)$$

Atoms will be defined below in subsection 3.2. Equation (1) is the atomic decomposition of H^1 and will be discussed further below.

Duality of H^1 with BMO on the bidisc

By the combined efforts of A. Chang and R. Fefferman, the following three conditions serve as criteria for a function φ to belong to BMO in the multiparameter setting. We state only the bidisc version.

Theorem 3.1. Let $\varphi : \mathbf{T}^2 \rightarrow \mathbf{C}$. Then $\varphi \in BMO(\mathbf{T}^2)$ if one of the following equivalent conditions holds:

(i): $\varphi \in (H^1)^*$, that is

$$\left| \int f \varphi \right| \leq \|\varphi\|_{BMO} \|f\|_{H^1}.$$

(ii): $\varphi \in L^\infty + H_1 L^\infty + H_2 L^\infty + H_1 H_2 L^\infty$, where H_j is the Hilbert transform in the variable z_j .

(iii): If $u = P[\varphi]$ and $\Omega \subset \mathbf{T}^2$ is an open set, then

$$\int_{S(\Omega)} |\nabla_1 \nabla_2 u|^2 \log \frac{1}{|z_1|} \log \frac{1}{|z_2|} dA(z_1) dA(z_2) \leq C|\Omega|,$$

where $S(\Omega) = \{(z_1, z_2) \in \Delta^2 : I_{z_1} \times I_{z_2} \subset \Omega\}$.

It is now a simple matter to obtain a holomorphic duality theorem, correcting an omission in [47]. Let $BMOA(\Delta^2) = H_A^2(\Delta^2) \cap BMO(\mathbf{T}^2)$.

Theorem 3.2. $BMOA = (H_A^1)^*$.

Proof. Let $\ell \in (H_A^1)^*$. Then $\ell(f) = \int f \bar{g}$ for some $g \in L^\infty$. For $f \in H_A^2$, $\ell(f) = \int f \bar{g} = \int f \overline{Sg}$, where S is the Szegő projection from L^2 onto H_A^2 . Now $Sg \in S(L^\infty) \cap H_A^2 = BMOA$, since $S(L^\infty) \subset BMO$ by virtue of the relation of the Szegő projection and Hilbert transform in one variable: $Sf = (iHf + f - \hat{f}(0))/2$.

Conversely, if $g \in BMOA$, then for $f \in H_A^2 \subset H_A^1$,

$$\left| \int f \bar{g} \right| \leq C \|f\|_{H^1} \|g\|_{BMO}.$$

Let $\pi : BMOA \rightarrow (H_A^1)^*$ be the map $\pi(g) = \ell_g$. By the above arguments, π is linear and onto. To show that it is one-to-one, suppose $\pi(g_1) = \pi(g_2)$. Then for any $f \in L^2(\mathbf{T}^2)$,

$$\int g \overline{(g_1 - g_2)} = (f, g_1 - g_2)_{L^2} = (Sf, g_1 - g_2)_{L^2} = 0$$

since $Sf \in H_A^2$. Thus $g_1 = g_2$, and since $BMOA$ is complete, by the open mapping theorem and the inequality $\|\pi(g)\| \leq C\|g\|_{BMO}$, the norms $\|\pi(g)\|$ and $\|g\|_{BMO}$ are equivalent. \square

3.2. Factorization of an atom on the bidisc

In this subsection we first elaborate on the atomic decomposition (1).

We work in the context first of the bi-upper half plane D . An *atom* is a function $a = a(x_1, x_2)$ on the Shilov boundary \mathbf{R}^2 , supported in an open set Ω of finite measure, which satisfies the following conditions:

1. $\|a\|_{L^2} \leq |\Omega|^{-1/2}$
2. a has mean zero over every component interval of every x_j -cross section of Ω
3. a is further decomposed into “elementary particles” $a = \sum_R a_R$ where

- (a): Each a_R is supported on a rectangle $R \subset \Omega$ with $R \not\subset 3R'$ for any $R \neq R'$ in the sum
- (b): $\int_{I_j} a_R(x) dx_j = 0$, $R = I_1 \times I_2$
- (c): a_R satisfies
- $\|a_R\|_\infty \leq c_R |R|^{-1/2}$
 - $\|\partial a_R / \partial x_j\|_\infty \leq c_R |I_j|^{-1} |R|^{-1/2}$
 - $\|\partial^2 a_R / \partial x_1 \partial x_2\|_\infty \leq c_R |R|^{-3/2}$
- where $\sum_R c_R^2 \leq A |\Omega|^{-1}$.

With this definition of atom, one has the atomic decomposition (1) of Chang and Fefferman. It is natural to expect that (1) and Theorem 3.3 below would lead to a factorization theorem for an arbitrary element of H^1 , but as of this writing, this has not been proved.

Theorem 3.3 (Theorem 2.3 of [46]). *Let a be an atom. Then for each R in the decomposition $a = \sum_R a_R$, there exist $B_j, C_j \in H^2(D)$ such that*

$$S(a_R) = \sum_1^4 B_j C_j$$

and

$$\sum_1^4 \|B_j\|_2 \|C_j\|_2 \leq c c_R |R|^{1/2}.$$

Because all functions involved are holomorphic, the multivariable Cayley transform can be used to transfer Theorem 3.3 to the setting of the bidisc. Moreover, the factorization can be done for any p -atom for $0 < p \leq 1$, as defined in [14], and in this case, the holomorphic component functions B_j, C_j belong to H^{2p} . The corresponding factorization theorem for arbitrary elements of H^p for this range of p , for the unit ball and for strongly pseudoconvex domains, were proved in [25] and [34] respectively.

References

- [1] E. Amar, *Big Hankel operator and $\bar{\partial}_b$ -equation*, J. Operator Theory 33 (1995), 223–233.
- [2] J. Arazy, *Boundedness and compactness of generalized Hankel operators on bounded symmetric domains*, J. Funct. Anal. 137 (1996), 97–151.
- [3] J. Arazy, S.D. Fisher and J. Peetre *Hankel operators on weighted Bergman spaces*, Amer. J. Math. 110 (1988), 989–1053.
- [4] J. Arazy, S.D. Fisher, S. Janson and J. Peetre, *Membership of Hankel operators on the ball in unitary ideals*, J. London Math. Soc 43 (1991), 485–508.
- [5] S. Axler, *The Bergman space, the Bloch space, and commutators of multiplication operators*, Duke Math. J. 53 (1986), 315–332.

- [6] D. Bekolle, C. Berger, L. Coburn, and K. Zhu, *BMO in the Bergman metric of bounded symmetric domains*, J. Funct. Anal. 93 (1990), 310–350.
- [7] F. Beatrous and S.Y. Li, *Trace ideal criteria for operators of Hankel type*, Ill. J. Math. 39 (1995), 723–754.
- [8] J. Bellissard, A. van Elst, H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect*, J. Math. Phys. 35 (1994), 5373–5451.
- [9] A. Bonami, M. Peloso, and F. Symesak, *Powers of the Szegő kernel and Hankel operators on Hardy spaces*, preprint 1998.
- [10] F. F. Bonsall, *Hankel operators on the Bergman space for the disc*, J. London Math. Soc. 33 (1986), 355–364.
- [11] J. Burbea, *Trace ideal criteria for Hankel operators in the ball*, preprint 1987.
- [12] E. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg 11 (1935), 116–162.
- [13] S.-Y. A. Chang and R. Fefferman, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. 112 (1980), 179–201.
- [14] S.-Y. A. Chang and R. Fefferman, *The Calderon-Zygmund decomposition on product domains*, Amer. J. Math. 104 (1982), 455–468.
- [15] J. M. Cohen and F. Colonna, *Bounded holomorphic functions on bounded symmetric domains*, Trans. Amer. Math. Soc. 343 (1994), 135–156.
- [16] R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in L^p* , Astérisque 77 (1980), 11–66.
- [17] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 611–635.
- [18] A. Connes, *Noncommutative geometry*, Academic Press, 1994.
- [19] M. Cotlar and C. Sadosky, *Nehari and Nevanlinna-Pick problems and holomorphic extensions in terms of restricted BMO*, J. Funct. Anal. 124 (1994), 205–210.
- [20] B. Coupet, *Décomposition atomique des espaces de Bergman*, Indiana Univ. Math. J. 38 (1989), 917–941.
- [21] J. Faraut and A. Koranyi, *Analysis on symmetric cones*, Oxford University Press, 1994.
- [22] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. 26 (1974), 1–65.
- [23] R. Fefferman, *Multiparameter Fourier Analysis*, In: Beijing Lectures, Annals of Mathematics Studies 112, Princeton Univ. Press (1986), 47–130.
- [24] M. Feldman and R. Rochberg, *Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group*, 121–160, Analysis and PDE (C. Sadosky, ed.), A collection of papers dedicated to M. Cotlar, Dekker, New York 1990.
- [25] J. Garnett and R. Latter, *The atomic decomposition for Hardy spaces in several complex variables*, Duke Math. J. 45 (1978), 815–845.
- [26] R. F. Gundy and E. M. Stein, *H^p theory for the poly-disc*, Proc. Natl. Acad. Sci. USA 76 (1979), 1026–1029.

- [27] K. T. Hahn and J. Mitchell, *Representation of linear functionals in H^p spaces over bounded symmetric domains in \mathbb{C}^n* , J. Math. Anal. Appl. 56 (1976), 379–396.
- [28] Hoffman, K., *Banach spaces of analytic functions*, Prentice-Hall 1962
- [29] L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Translation of Mathematical Monographs, 6. Amer. Math. Soc. 1979.
- [30] S. Janson and T. Wolfe, *Schatten classes and commutators of singular integral operators*, Ark. Mat. 20 (1982), 301–310.
- [31] S. Janson, J. Peetre, R. Rochberg, *Hankel forms and the Fock space*, Revista Math. Iberoamericano 3 (1987), 61–138.
- [32] M. Koecher, *An elementary approach to bounded symmetric domains*, Lecture Notes, Rice University, 1969.
- [33] S. Krantz, *Function theory of several complex variables*, Wadsworth and Brooks/Cole, 1992.
- [34] S. Krantz and S.-Y. Li, *On decomposition theorems for Hardy spaces on domains in \mathbb{C}^n and applications*, J. Fourier Anal. Appl. 2 (1995), 65–107.
- [35] S. G. Krantz and S.-Y. Li, *Duality theorems for Hardy and Bergman spaces on convex domains of finite type in \mathbb{C}^n* , Ann. Inst. Fourier (Grenoble) 45 (1995), 1305–1327.
- [36] S. G. Krantz, S.-Y. Li, P. Lin and R. Rochberg, *The effect of boundary regularity on the singular numbers of Friedrichs operators on Bergman spaces*, Michigan Math. J. 43 (1996), 337–348.
- [37] S. G. Krantz, S.-Y. Li and R. Rochberg, *Analysis of some function spaces associated to Hankel operators*, Ill. J. Math. 41 (1997), 398–411.
- [38] S. G. Krantz, S.-Y. Li and R. Rochberg, *The effect of boundary geometry on Hankel operators belonging to the trace ideals of Bergman spaces*, Int. Eq. Operator Theory 28 (1997), 196–213.
- [39] S. V. Khrushchev and V. V. Peller, *Hankel operators, best approximations and stationary Gaussian processes*, Uspekhi Mat. Nauk 37 (1982), 53–124. English translation: Russian Math. Surveys 37 (1982), 61–144.
- [40] H. Li, *Schatten class Hankel operators on the Bergman spaces of strongly pseudoconvex domains*, Proc. Amer. Math. Soc. 119 (1993), 1211–1221.
- [41] H. Li, *Hankel operators on the Bergman space of the unit polydisc*, Proc. Amer. Math. Soc. 120 (1994), 1113–1121.
- [42] H. Li, *Hankel operators on the Bergman space of strongly pseudoconvex domains*, Int. Eq. Oper. Theory 19 (1994), 458–476.
- [43] H. Li and D. Luecking, *BMO on strongly pseudoconvex domains: Hankel operators, duality, and $\bar{\partial}$ -estimates*, Trans. Amer. Math. Soc. 346 (1994), 661–691.
- [44] H. Li and D. Luecking, *Schatten class of Hankel and Toeplitz operators on the Bergman space of strongly pseudoconvex domains*, Contemporary Math. 185 (1995), 237–257.

- [45] S.Y. Li and B. Russo *Hankel operators in the Dixmier class*, C. R. Acad. Sci. Paris, 325 (1997), 21–26.
- [46] I.-J. Lin, *Factorization theorems for Hardy spaces of the bidisc*, $0 < p \leq 1$, Proc. Amer. Math. Soc. 124 (1996), 549–560.
- [47] I.-J. Lin and B. Russo, *Applications of factorization in the Hardy spaces of the polydisk*, In: Interactions between functional analysis, harmonic analysis, and probability (Columbia Missouri), edited by N. Kalton, E. Saab, S. Montgomery-Smith, Marcel Dekker 1995, pp. 131–146.
- [48] O. Loos, *Bounded symmetric domains and Jordan pairs*, University of California, Irvine, 1977.
- [49] V. V. Peller, *Hankel operators of class S_p and applications*, Mat. Sb. 113 (1980), 538–581; translation in Math. USSR Sbornik 4 (1982), 443–479.
- [50] M. Peloso, *Besov spaces, mean oscillation, and generalized Hankel operators*, Pacific J. Math. 161 (1993), 155–184.
- [51] M. Peloso, *Hankel operators on weighted Bergman spaces on strongly pseudoconvex domains* III. J. Math. 38 (1994), 223–249.
- [52] R. Rochberg, *Trace ideal criteria for Hankel operators and commutators*, Indiana U. Math. J. 31 (1982), 913–925.
- [53] R. Rochberg and S. Semmes, *A decomposition theorem for BMO and applications*, J. Funct. Anal. 67 (1986), 228–263.
- [54] R. Rochberg and S. Semmes, *Nearly weakly orthonormal sequences, singular value estimates, and Calderon-Zygmund operators*, J. Funct. Anal. 86 (1989), 237–306.
- [55] W. Rudin, *Function Theory in Polydisks*, W. A. Benjamin 1969
- [56] W. Rudin, *Function Theory in the unit ball of \mathbb{C}^n* , Springer-Verlag 1980
- [57] B. Russo, *Composition operators in several complex variables*, In: Composition Operators, Proceedings of the Rocky Mountain Mathematics Consortium Conference 1996, Eds: C. Cowen, F. Jafari, B. MacCluer, D. Porter, American Mathematical Society, Contemporary Mathematics, to appear.
- [58] F. Symesak, *Hankel operators on pseudoconvex domains of finite type in \mathbb{C}^2* , Can. J. Math. 50 (1998), 658–672.
- [59] R. Wallsten, *Hankel operators between weighted Bergman spaces in the ball*, Ark. Mat. 28 (1990), 183–192.
- [60] G. Zhang, *Hankel operators on Hardy spaces and Schatten classes*, Chinese Ann. of Math. 3 (1991), 282–293.
- [61] D. Zheng, *Schatten class Hankel operators on the Bergman space*, Int. Eq. Operator Theory 13 (1990), 442–459.
- [62] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, Inc., 1990.
- [63] K. Zhu, *Schatten class Hankel operators on the Bergman space of the unit ball*, Amer. J. Math. 113 (1991), 147–167.
- [64] K. Zhu, *Hankel operators on the Bergman space of bounded symmetric domains*, Trans. Amer. Math. Soc. 324 (1991), 707–730.
- [65] K. Zhu, *Holomorphic Besov spaces on bounded symmetric domains. II*, Indiana U. Math. J. 44 (1995), 1017–1031.

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