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Projections on von Neumann algebras as limits of elementary operators

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ABSTRACT

If B is a subalgebra of a von Neumann algebra $A \subset \mathcal{B}(H)$ and B contains the rank one projections corresponding to an orthonormal basis of H, then a linear B-bimodule projection P on A with range B is of the form

$$P(x) = \sum_j p_j x p_j \quad x \in \mathcal{B}(H)$$

for orthogonal projections p_j in A which are diagonal with respect to the basis. An analogous result holds if $A = \mathcal{B}(H)$ and B is a weakly closed ternary ring of operators.

1. Introduction

In ring theory and in the context of algebras, idempotents have many well-established uses. In particular, if $e \in R$ is an idempotent of a ring R, then the subring eRe has unit e and there is an eRe-bimodule projection $x \mapsto exe$ from R onto eRe. The kernel

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$eR(1-e)+(1-e)Re+(1-e)R(1-e)$ of the projection is a complementary eRe-submodule of R.

In probability theory, and in the theory of von Neumann algebras, the notion of conditional expectation (as a completely positive map $E : M \to M$ on a von Neumann algebra M, with M commutative in the case of probability theory) satisfies similar algebraic properties as the Peirce projections on a ring R or on an algebra A. A result of J. Tomiyama states that a unital and bounded projection $E : A \to A$ with range $S = E(A)$ a C*-subalgebra of A must have norm one, must be positive, must satisfy the conditional expectation property $E(s_1xs_2) = s_1E(x)s_2$ (for $s_1, s_2 \in S, x \in A$) and also the Schwarz type inequality $E(x)^2 \leq E(x^2)$ for self-adjoint x (see [1, II.6.10.2]). In one of the themes of recent research, the notion of injective operator space, a similar algebraic 'conditional expectation' property plays a significant role, interacting with the notion of a ternary ring of operators (TRO, see [11]).

In [7], T. Y. Lam proposed abstracting the algebraic properties of the Peirce projection $E_e : R \to R$ associated with an idempotent e in a ring R, which is given by $E_e(x) = exe$, ($x \in R$), and investigating algebraic properties that hold in this more general context. His proposal is to consider (additive) maps $E : R \to R$ with $E \circ E = E$, $S = E(R)$ a subring of R under the assumption that E is an S-bimodule map (which means that it satisfies the conditional expectation property $E(s_1xs_2) = s_1E(x)s_2$ for $s_1, s_2 \in S, x \in R$). Lam refers to such subrings S as 'corners'.

We consider this notion principally in the context of a (complex) C*-algebra A in place of a ring R and with the assumption that the corner $S = E(A)$ is a complex subalgebra. Our aim is to characterize such corners as fully as we can, ideally by establishing that they are related to the ranges of the more well-known completely positive (unital) conditional expectations.

In the general approach of Lam (in the context of rings), although a ring-theoretic Lam corner S of a unital algebra A need not be a subalgebra, if S is a subalgebra then the corresponding projection E must be linear (that is, homogeneous), which justifies the definition of corner algebra we use (Definition 2.1). Thus we adopt a definition modified from the ring-theoretic one (which insists that we deal with corners that are subalgebras and have vector space complements, or equivalently we deal only with linear projections E).

While simple examples show that Lam corners S in C*-algebras need not be self-adjoint subalgebras, Peirce corners in C*-algebras and certain 'generalized' Peirce corners behave like self-adjoint corners (see [10, section 3.6]). In Proposition 2.5, we characterize corners in finite dimensional C*-algebras that contain the diagonal and use that in Theorem 1 to characterize corners of von Neumann algebras that contain the diagonal in some basis for H. A consequence of this result is a version where the range of the projection on $\mathcal{B}(H)$ is a weakly closed ternary ring of operators (Theorem 2).

2. Main result

Let $\mathcal{B}(H)$ be the algebra of bounded linear operators on a Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$, and with an orthonormal basis $(\xi_i)_{i \in I}$ (which may be countable or uncountable). For any $i \in I$, consider the *diagonal operator* $\xi_i \otimes \xi_i^* \in \mathcal{B}(H)$ defined by $(\xi_i \otimes \xi_i^*)(\xi) = \langle \xi, \xi_i \rangle \xi_i$ for $\xi \in H$, which is the orthogonal self-adjoint projection of H onto the one-dimensional subspace of H spanned by ξ_i. This terminology for such operators $\xi_i \otimes \xi_i^*$ recalls the notion of 'diagonal matrices' $e_{ii} \in M_I(\mathbb{C})$ with 1 on the (i, i) position and 0 elsewhere. We shall call a (self-adjoint) projection $p \in \mathcal{B}(H)$ a *diagonal projection* if $p\xi_i \in \mathbb{C}\xi_i$ for each $i \in I$.

The objects of study in this section are corner algebras S of C*-subalgebras of $\mathcal{B}(H)$, with S containing the diagonal operators $\xi_i \otimes \xi_i^*$. We need the following definitions.

Definition 2.1. Let A be an algebra. A subalgebra S of A is called a *corner algebra* (or simply a *corner*) of A if there exists a vector subspace M of A such that

$$A = S \oplus M, \quad SM \subset M, \quad MS \subset M.$$

M is called a *complement* of S.

Corners of concrete C*-algebras need not be closed in any of the operator topologies (see [10, section 3.2]), but our main examples of corners will be closed subalgebras of $\mathcal{B}(H)$.

Definition 2.2. Corners of the form pAp, where p is an idempotent in A are called *Peirce corners*. If $e_1, \ldots, e_n$ are idempotents in an algebra A with $e_ie_j = 0$ for $i \neq j$, then the corner $\oplus_{i=1}^n e_iAe_i$ is called a *generalized Peirce corner*.

It is shown in [7, Proposition 2.1] that S is a corner of A if and only if there exists a linear S-bimodule map $\mathcal{E} : A \to A$ with $\mathcal{E}(A) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$.

Proposition 2.3. *If R is a ring and $e_1, \ldots, e_n$ are idempotents in R with $e_ie_j = 0$ for $i \neq j$, then the generalized Peirce corner $S = \oplus_{i=1}^n e_iRe_i$ has a unique complement and the unique idempotent mapping on R with range S is given by $\mathcal{E}(x) = \sum_{i=1}^n e_ixe_i$.*

Proof. Let M be a complement for S and let $\mathcal{E}_0$ be a corresponding idempotent S-bimodule map with range S. The idempotent $e = \sum_{i=1}^n e_i$ is the identity element for S and it follows from $R \ni x = s + m$ ($s \in S$, $m \in M$) that $exe = s + eme$ with $eme \in M$. So $\mathcal{E}_0(x) = s = \mathcal{E}_0(exe)$.

Note that for $z \in S$ we have $z = \sum_{k=1}^n e_kze_k$.

For $y \in eRe$ we have $y = eye = \sum_{i,j=1}^n e_iye_j = \sum_{i=1}^n e_iye_i + \sum_{i \neq j} e_iye_j$. For $i \neq j$ we have $\mathcal{E}_0(e_iye_j) = \sum_{k=1}^n e_k\mathcal{E}_0(e_iye_j)e_k = \sum_{k=1}^n \mathcal{E}_0(e_ke_ixe_je_k) = 0$. Hence $e_iye_j \in M$ for $i \neq j$ and $\mathcal{E}_0(y) = \sum_{i=1}^n e_iye_i$.

It follows that for $x \in R$,

$$\mathcal{E}_0(x) = \mathcal{E}_0(exe) = \sum_{i=1}^{n} e_i exee_i = \sum_{i=1}^{n} e_i xe_i.$$

Furthermore, any complement M of S must be equal to the kernel of $\mathcal{E}$. Indeed, if $x = s + m$, then $\mathcal{E}(x) = s + \sum_i e_i me_i$ and $\sum_i e_i me_i \in M \cap S = \{0\}$ so that if $x \in \ker \mathcal{E}$, $s = 0$ and $x \in M$. Similarly, $M \subset \ker \mathcal{E}$. □

We will need the following result, which follows from Wedderburn's theorem.

Proposition 2.4 *([2, Proposition 5.2.6]). Any semisimple finite-dimensional algebra R over an algebraically closed field k is a direct product of full matrix rings over k.*

Proposition 2.5. *Let A be a C^*-subalgebra of $B(H)$, where H is finite dimensional with orthonormal basis $\xi_1, \dots, \xi_n$. Let $\mathcal{E}\colon A \to A$, have range S which is a subalgebra of A containing the rank 1 projections $e_{ii} = \xi_i \otimes \xi_i^*$. Suppose $\mathcal{E}$ is an idempotent S-bimodule linear map. Then S is a self-adjoint generalized Peirce corner and $\|\mathcal{E}\| = 1$. Moreover there are orthogonal diagonal projections $p_1, \dots, p_k$ in A such that $\mathcal{E}(x) = \sum_{j=1}^k p_j x p_j$. (Each p_j is a sum of some of the e_{ii}.)*

Proof. We identify $\mathcal{B}(H)$ with $M_n(\mathbb{C})$. Since $S = \mathcal{E}(A)$ is a finite dimensional algebra over $\mathbb{C}$ and by [6, Theorem 1], semisimple, it must be isomorphic to a finite direct sum of full matrix algebras over $\mathbb{C}$ by Proposition 2.4. Let e_{ii} denote the n-by-n matrix with entry 1 in the (i, i) position and 0 elsewhere, and let

$$\phi\colon S \to M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

be an isomorphism. Since $e_{ii} \in S$ $(1 \le i \le n)$, it follows that $\phi(e_{ii})$ is an idempotent in $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, and so $\phi(e_{ii}) = f_{i1} \oplus \cdots \oplus f_{ik}$ with $f_{ij} \in M_{n_j}(\mathbb{C})$ $(1 \le j \le k)$ an idempotent. Since e_{ii} is minimal in S, we must have $f_{ij} \ne 0$ for just one j; $\phi(e_{ii}) = f_{ij}$ for some j. Moreover $\phi(e_{ii}) = f_{ij}$ must be a rank one idempotent in $M_{n_j}(\mathbb{C})$, and we can partition $\{1, \dots, n\}$ into k classes where the j^{th} class is $C_j = \{i : \phi(e_{ii}) \in M_{n_j}(\mathbb{C})\}$. Put $p_j = \sum_{i \in C_j} e_{ii}$, and let 1_n denote the n-by-n identity matrix. Then $\phi(1_n) = \sum_{i=1}^n \phi(e_{ii}) = \sum_{j=1}^k \phi(p_j)$ is the identity of $\phi(S)$. Hence $\phi(p_j) \in M_{n_j}(\mathbb{C})$ is the identity. It follows that C_j must have n_j members and $\sum_{j=1}^k n_j = n$. Therefore, if $x \in S$, then $\phi(x) \in M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ and $\phi(x) = \sum_{j=1}^k \phi(p_j)\phi(x)\phi(p_j) = \phi(\sum_{j=1}^k p_j x p_j)$, so $S \subseteq \bigoplus_{j=1}^k p_j A p_j$ and

$$\dim S \le \dim \bigoplus_{j=1}^{k} p_j A p_j = \sum_j \dim p_j A p_j$$

$$\leq \sum_j (\text{rank } p_j)^2 = \sum_j (\text{rank } \phi(p_j))^2$$
$$= \sum_j n_j^2 = \dim \phi(S) = \dim S.$$

Therefore $S = \bigoplus_{j=1}^k p_j A p_j$, and since $\bigoplus_{j=1}^k p_j A p_j$ is a generalized Peirce corner of A, $\mathcal{E}(x) = \sum_{j=1}^k p_j x p_j$ by Proposition 2.3. Thus $\mathcal{E}$ is a positive unital map and so $\|\mathcal{E}\| = 1$. (That $\|\mathcal{E}\| = 1$ follows also from [1, II.6.9.4].) $\square$

Remark 2.6. If $R = M_n(\mathbb{C})$ and $E(x) = \text{tr}\,(x)1$ then $\mathbb{C}1$ is a corner of R but E is not of the form $\sum_{j=1}^k p_j x p_j$ for orthogonal projections $p_1, \ldots, p_k$, since then $p_i = E(p_i) = \text{tr}\,(p_i)1$ implies E is the identity map. Thus the assumption on the rank one projections in Proposition 2.5, and in Theorem 1, is essential.

Our main result, Theorem 1 below, is an infinite dimensional version of Proposition 2.5. For motivation purposes, we shall give a constructive proof in the case that $A = \mathcal{B}(H)$, with H a separable Hilbert space with orthonormal basis $\xi_1, \ldots, \xi_n, \ldots$. Let $\mathcal{E}\colon A \to A$ be an idempotent S-bimodule linear map with range S which is a subalgebra of A containing the rank 1 projections $\xi_i \otimes \xi_i^*$.

If $\alpha \subset I$ is a finite set, we write $\pi = \pi_\alpha$ for the orthogonal projection of H onto the span$\{\xi_i : i \in \alpha\}$, which is $\pi_\alpha = \sum_{i\in\alpha} \xi_i \otimes \xi_i^*$ and is in the range of $\mathcal{E}$. Let $A_\alpha = \{x \in A : x = \pi x \pi\} = \pi A \pi$, a C^*-subalgebra (in fact a self-adjoint Peirce corner) of A. Note that if $x \in A_\alpha$ then $\mathcal{E}(x) = \mathcal{E}(\pi x \pi) = \pi \mathcal{E}(x) \pi \in A_\alpha$.

We now define $\mathcal{E}_\alpha\colon A_\alpha \to A_\alpha$ by $\mathcal{E}_\alpha = \mathcal{E}|_{A_\alpha}$ (restriction of $\mathcal{E}$) and we can check easily that $\mathcal{E}_\alpha$ is an idempotent $\mathcal{E}_\alpha(A_\alpha)$-bimodule map on A_α. Moreover the range of $\mathcal{E}_\alpha$ contains the diagonal and so Proposition 2.5 applies. (Of course, $A_\alpha \simeq \mathcal{B}(\pi_\alpha H)$.)

With $\alpha = \{1, \ldots, n\}$, denote $\pi_n = \pi_\alpha$, and $A_n = A_\alpha$. By Proposition 2.5 we can write

$$\mathcal{E}_n(x) = \sum_{j=1}^{k_n} p_{nj} x p_{nj} \text{ for } x \in A_n \tag{2.1}$$

where the p_{nj} are orthogonal diagonal projections in A_n for $j = 1, \ldots, k_n$.

We know that $\mathcal{E}_n = \mathcal{E}_{n+1}|A_n$. We now define by induction a family of projections in A. First, $\mathcal{P}_1 = \{e_{11}\}$ where $e_{11} = \xi_1 \otimes \xi_1^*$ and $\mathcal{E}_1(x) = e_{11} x e_{11}$ for $x \in A_1$. More generally, we define $e_{i_1 i_2} = \xi_{i_1} \otimes \xi_{i_2}^*$, for $i_1, i_2 \in I$. The projection $\mathcal{E}_2$ is either the identity on $A_2 \simeq M_2(\mathbb{C})$ or $\mathcal{E}_2(x) = e_{11} x e_{11} + e_{22} x e_{22}$ for $x \in A_2$. In the first case, we define $\mathcal{P}_2 = \{e_{11} + e_{22}\}$ and in the second case $\mathcal{P}_2 = \{e_{11}, e_{22}\}$. Each of these cases gives rise to two possible choices for $\mathcal{P}_3$, namely if $\mathcal{P}_2 = \{e_{11}, e_{12}\}$, then $\mathcal{P}_3$ is either $\{e_{11}, e_{22} + e_{33}\}$ or $\{e_{11}, e_{22}, e_{33}\}$; and if $\mathcal{P}_2 = \{e_{11} + e_{22}\}$, then $\mathcal{P}_3$ is either $\{e_{11} + e_{22}, e_{33}\}$ or $\{e_{11} + e_{22} + e_{33}\}$; and so forth.

By (2.1),

$$\mathcal{E}_{n+1}(x) = \sum_{j=1}^{k_{n+1}} p_{n+1,j} x p_{n+1,j} \text{ for } x \in A_{n+1}.$$

Since $\mathcal{E}_n = \mathcal{E}_{n+1}|A_n$, we have $p_{nj} = p_{n+1,j}$ for $j = 1, \dots, k_n - 1$. As above, there are two possibilities. Either $k_{n+1} = k_n$ and $p_{n+1,k_{n+1}} = p_{n,k_n} + e_{n+1,n+1}$; or $k_{n+1} = k_n + 1$ and $p_{n+1,k_n} = p_{n,k_n}$ and $p_{n+1,k_{n+1}} = e_{n+1,n+1}$. Depending on which possibility holds, we define

$$\mathcal{P}_{n+1} = \{p_{nj} : j = 1, \dots k_n - 1\} \cup \{p_{n,k_n} + e_{n+1,n+1}\}$$

or

$$\mathcal{P}_{n+1} = \mathcal{P}_n \cup \{e_{n+1,n+1}\}.$$

Finally we define

$$\mathcal{P} = \cup_{n=1}^{\infty} \mathcal{P}_n,$$

and to avoid overlap we define

$$\mathcal{Q} = \mathcal{P} - \{p \in \mathcal{P} : p \le q \text{ for some } q \in \mathcal{P}, q \ne p\}.$$

Note that $\mathcal{Q}$ consists of orthogonal diagonal projections, and that for each finite subset $\alpha \subset I$, there is a finite subset $Q_\alpha \subset \mathcal{Q}$ such that $\mathcal{E}_\alpha(x) = \sum_{p \in Q_\alpha} pxp$ for $x \in A$. It follows that if $\sigma, \tau \in H$ are finite linear combinations of the basis vectors, say $\sigma = \sum_{i \in \alpha} \sigma_i \xi_i$ and $\tau = \sum_{i \in \alpha} \tau_i \xi_i$, and $x \in A$ then

$$\begin{aligned}
\langle \mathcal{E}(x)\sigma, \tau \rangle &= \langle \mathcal{E}(x)\pi_\alpha \sigma, \pi_\alpha \tau \rangle = \langle \pi_\alpha \mathcal{E}(x)\pi_\alpha \sigma, \pi_\alpha \tau \rangle \\
&= \langle \mathcal{E}(\pi_\alpha x \pi_\alpha)\sigma, \pi_\alpha \tau \rangle = \langle \mathcal{E}_\alpha(x)\sigma, \tau \rangle \\
&= \left\langle \left(\sum_{p \in Q_\alpha} pxp \right) \sigma, \tau \right\rangle,
\end{aligned}$$

and therefore, since $p\sigma = 0 = p\tau$ if $p \in \mathcal{Q} - \mathcal{Q}_\alpha$, we may take $\tau = \tau_\beta = \sum_{i \in \beta} \tau_i \xi_i$ with β a finite subset of I containing α and then in the limit as τ_β approaches an arbitrary vector τ' in H, we have

$$\langle \mathcal{E}(x)\sigma, \tau' \rangle = \left\langle \left(\sum_{p \in \mathcal{Q}} pxp \right) \sigma, \tau' \right\rangle,$$

so that

$$\mathcal{E}(x)\sigma = \left(\sum_{p\in\mathcal{Q}} pxp\right)\sigma.$$

We conclude

$$\mathcal{E}(x) = \text{S-lim}(\sum_{p\in\mathcal{Q}} pxp) \quad (\text{for } x \in A).$$

This completes the proof of the special case of Theorem 1 below in which $A = \mathcal{B}(H)$ with H separable. This argument does not seem to work if A is not equal to $\mathcal{B}(H)$, but some of its notation will be useful in the proof below of Theorem 1, which is valid for arbitrary A and H, and which is adapted from [10, Theorems 3.12.5 and 3.13.4] (however, see Remark 2.9(ii)). We first need a couple of Lemmas.

Lemma 2.7. *A von Neumann algebra $A \subset B(H)$ which contains all the rank one projections $e_{ii} = \xi_i \otimes \xi_i^*$ corresponding to an orthonormal basis of H is necessarily atomic, that is, generated by its minimal projections, and is therefore a direct sum of factors of type I (see [4, Remark 1.10]).*

Proof. If p is a non-zero projection in A, then $q := \sum_{pe_{ii}\neq o} e_{ii}$ is not zero and $p(1-q) = 0$, so $p = pq = pqp = \sum_{pe_{ii}\neq 0} pe_{ii}p$ and $pe_{ii}p = p(\xi_i \otimes \xi_i^*)p = p\xi_i \otimes (p\xi_i)^* \in A$ so p dominates each minimal projection $q_i = (\|p\xi_i\|^2)^{-1}p\xi_i \otimes (p\xi_i)^*$. Indeed, with $\lambda_i = \|p\xi_i\|^2$, $p \geq \lambda_i q_i \Rightarrow \operatorname{ran}(1-p) = \ker p \subset \ker q_i = \operatorname{ran}(1-q_i), 1-p \leq 1-q_i, p \geq q_i$. It follows that every projection in A is the sum of an orthogonal family of minimal projections, so A is generated as a von Neumann algebra by its minimal projections. □

The following lemma is well-known, so we omit its proof, which can be found in [10, Lemma 3.12.3].

Lemma 2.8. *If $(p_i)_{i\in I}$ are orthogonal projections in $\mathcal{B}(H)$, then we can define an idempotent S-bimodule map, $\mathcal{E}\colon \mathcal{B}(H) \to \mathcal{B}(H)$, S being the range of $\mathcal{E}$, by*

$$\mathcal{E}(x) = \sum_{i\in I} p_i x p_i = \lim_{\alpha\in\mathcal{F}(I)} \sum_{i\in\alpha} p_i x p_i$$

where the limit is taken in the strong operator topology of $\mathcal{B}(H)$ and $\mathcal{F}(I)$ denotes the collection of finite subsets $\alpha \subseteq I$ (ordered by inclusion).

Theorem 1. *Let $A \subset \mathcal{B}(H)$ be a von Neumann algebra. Let $\mathcal{E}\colon A \to A$ be an idempotent S-bimodule map, where $S = \mathcal{E}(A)$ is a subalgebra (not necessarily self-adjoint or norm closed) such that $\xi_i \otimes \xi_i^* \in S$ for all $i \in I$, $\{\xi_i : i \in I\}$ being an orthonormal basis of H. Then A is atomic, and $\mathcal{E}$ has the form*

$$\mathcal{E}(x) = \sum_{j \in J} p_j x p_j$$

for 'diagonal' orthogonal projections $\{p_j : j \in J\} \subset A$ *(that is projections with* $p_j \xi_i \in \mathbb{C}\xi_i$ *for each* $i \in I$*).*

Proof. We adopt the notation of the discussion preceding Lemma 2.7, namely if $\alpha \subset I$ is a finite set, we write $\pi = \pi_\alpha$ for the orthogonal projection of H onto the span$\{\xi_i : i \in \alpha\}$, $A_\alpha = \{x \in A : x = \pi x \pi\} = \pi A \pi$, and define $\mathcal{E}_\alpha \colon A_\alpha \to A_\alpha$ by $\mathcal{E}_\alpha = \mathcal{E}|_{A_\alpha}$. $\mathcal{E}_\alpha$ is an idempotent $\mathcal{E}_\alpha(A_\alpha)$-bimodule map on A_α whose range contains the diagonal.

We begin, as above, by assuming that $A = \mathcal{B}(H)$. Define a relation on I by $i_1 \sim i_2$ if $\mathcal{E}(\xi_{i_1} \otimes \xi_{i_2}^*) = \xi_{i_1} \otimes \xi_{i_2}^*$. Since the range of $\mathcal{E}$ contains the diagonal, $i \sim i$ for all $i \in I$. As seen above, the projection $\mathcal{E}_{\{i_1,i_2\}}$ is either the identity on $A_{\{i_1,i_2\}} \simeq M_2(\mathbb{C})$ if $e_{i_1 i_2} \in \mathcal{E}(A)$, in which case $e_{i_2,i_1} \in \mathcal{E}(A)$; or $\mathcal{E}_{\{i_1,i_2\}}(x) = e_{i_1 i_1} x e_{i_1 i_1} + e_{i_2 i_2} x e_{i_2 i_2}$, so $\sim$ is symmetric. Moreover, if $i_1 \nsim i_2$, then

$$\mathcal{E}_{\{i_1,i_2\}}(e_{i_1 i_2}) = e_{i_1 i_1} e_{i_1 i_2} e_{i_1 i_1} + e_{i_2 i_2} e_{i_1 i_2} e_{i_2 i_2} = 0. \tag{2.2}$$

To show transitivity of $\sim$, assuming $i_1 \sim i_2$ and $i_2 \sim i_3$, we have $\xi_{i_1} \otimes \xi_{i_3}^* = (\xi_{i_1} \otimes \xi_{i_2}^*)(\xi_{i_2} \otimes \xi_{i_3}^*) \in \mathcal{E}(A)$, so we have an equivalence relation $\sim$ on I.

Take J to be the set of equivalence classes and for $j \in J$ define

$$p_j = \sum_{i \in j} \xi_i \otimes \xi_i^*$$

(sum converging in strong operator topology).

Observe that

$$\mathcal{E}(\xi_{i_1} \otimes \xi_{i_2}^*) = \sum_{j \in J} p_j (\xi_{i_1} \otimes \xi_{i_2}^*) p_j \tag{2.3}$$

for all $i_1, i_2 \in I$ because if $i_1 \sim i_2$ then $p_j(\xi_{i_1} \otimes \xi_{i_2}^*)p_j$ is zero for all equivalence classes j other than the one containing i_1, while $p_j(\xi_{i_1} \otimes \xi_{i_2}^*)p_j = \xi_{i_1} \otimes \xi_{i_2}^*$ when j is the equivalence class of i_1. On the other hand, if $i_1 \nsim i_2$, then both sides of (2.3) are zero, by (2.2)

Also observe that for $x \in A$, α a finite subset of I, and $j \in J$,

$$p_j \pi_\alpha x \pi_\alpha p_j = p_j x p_j,$$

since both sides are equal to $\sum_{k,\ell \in j} \langle x\xi_k, \xi_\ell \rangle \xi_\ell \otimes \xi_k^*$.

It follows, as above, that if $\sigma, \tau \in H$ are finite linear combinations of the basis vectors, say $\sigma = \sum_{i \in \alpha} \sigma_i \xi_i$ and $\tau = \sum_{i \in \alpha} \tau_i \xi_i$, and $x \in A$ then

$$\langle \mathcal{E}(x)\sigma, \tau \rangle = \langle \mathcal{E}(\pi_\alpha x \pi_\alpha)\sigma, \pi_\alpha \tau \rangle$$

$$= \left\langle \left(\sum_{j \in J} p_j \pi_\alpha x \pi_\alpha p_j \right) \sigma, \tau \right\rangle$$

$$= \left\langle \left(\sum_{j \in J} p_j x p_j \right) \sigma, \tau \right\rangle,$$

and thus $\mathcal{E}(x) = \text{S-}\lim(\sum_{j \in J} p_j x p_j)$ for $x \in A$. This completes the proof in case $A = \mathcal{B}(H)$.

We now consider the general case. By Lemma 2.7, A is atomic, so that $A = \oplus_{k \in K} B_k$ where $B_k \simeq \mathcal{B}(H_k)$ with $H \simeq \oplus_{k \in K} H_k$. Each minimal projection of A belongs to one of the summands B_k as a minimal projection and the orthonormal basis $\{\xi_i : i \in I\}$ consists of the union of orthonormal bases in each H_k.

Denote by $\{e_i : i \in I\}$ the orthogonal minimal projections in the range of $\mathcal{E}$ which sum to 1. Define a relation on I by $i_1 \sim i_2$ if $0 \neq e_{i_1} A e_{i_2} \subset \mathcal{E}(A)$. Clearly $i \sim i$ for every $i \in I$ since $e_i A e_i = \mathbb{C} e_i$.

If $i_1 \neq i_2$ and $i_1 \sim i_2$ then the minimal projections e_{i_1}, e_{i_2} belong to the same summand B_k, and $e_{i_1} A e_{i_2} = \mathbb{C} u_{21}$ where u_{21} is the partial isometry in $B_k \simeq \mathcal{B}(H_k)$ with initial projection e_{i_2} and final projection e_{i_1}. Moreover, with $\alpha = \{i_1, i_2\}$, since $\mathcal{E}_\alpha$ is either the identity on A_α or $\mathcal{E}_\alpha(x) = e_{i_1} x e_{i_1} + e_{i_2} x e_{i_2}$ and $\mathcal{E}_\alpha(u_{21}) = u_{21}$, it follows that $\mathcal{E}_\alpha$ is the identity so that $0 \neq e_{i_2} B_k e_{i_1} \subset \mathcal{E}(A)$ and $\sim$ is symmetric.

Finally, if $i_1 \sim i_2$ and $i_2 \sim i_3$, with $A_\alpha = \{i_1, i_2, i_3\}$, then $e_{i_1} A e_{i_3} = \mathbb{C} u_{21} u_{32} \subset \mathcal{E}(A)$, where u_{32} is the partial isometry in $\mathcal{B}(H)$ with initial projection e_{i_3} and final projection e_{i_2}.

Take J to be the set of equivalence classes and for $j \in J$ define

$$p_j = \sum_{i \in j} \xi_i \otimes \xi_i^*.$$

Then as in earlier parts of the proof

$$\mathcal{E}(\xi_{i_1} \otimes \xi_{i_2}^*) = \sum_{j \in J} p_j (\xi_{i_1} \otimes \xi_{i_2}^*) p_j$$

for all $i_1, i_2 \in I$ and therefore $\mathcal{E}(x) = \sum_{j \in J} p_j x p_j$ for all $x \in A$. □

Remark 2.9.

- **(i)** Since $\mathcal{E}(1) = 1$ and $\mathcal{E}$ is positive, $\|\mathcal{E}\| = 1$ and $\mathcal{E}(A)$ is a C*-subalgebra of A.
- **(ii)** Theorem 1 is an improvement of [10, Theorem 3.12.5] which had the additional assumption that $\mathcal{E}$ is a self-adjoint map.
- **(iii)** The maximal abelian $*$-subalgebra associated with the orthonormal basis is more than just the linear span of the diagonal rank one operators $\xi_i \otimes \xi_i^*$. The maximal abelian $*$-subalgebra would be the weak*-closure of that span.

(iv) There is a significant literature concerning idempotent $\mathcal{D}$-module maps on von Neumann algebras, where $\mathcal{D}$ is a maximal abelian self adjoint subalgebra, for example [5], [12], [9]. These papers focus on proving algebraic properties of the range.

The referee has suggested an alternate approach to Theorem 1, under the additional assumption that the range $S = \mathcal{E}(A)$ contains the maximal abelian $*$-subalgebra $\mathcal{D}$ associated with the orthonormal basis. In that case there is the additional conclusion that S is a von Neumann subalgebra of A. The proof proceeds along the following lines.

Assume that $A = B(H)$. Since S contains $\mathcal{D}$, and $\mathcal{E}$ is an S-bimodule map, $\mathcal{E}$ is given by a Schur multiplier, that is, in the orthonormal basis $\{\xi_i\}$, $\mathcal{E}$ is given by $[x_{ij}] \mapsto [a_{ij}x_{ij}]$ for a fixed infinite matrix $[a_{ij}]$ (for the finite dimensional case, see [8, Exercise 4.4, p. 56], which could be used to shorten the proof of Proposition 2.5, and more generally see [13]). Using that $\mathcal{E}$ is idempotent, it follows that each entry a_{ij} is either 0 or 1, thus each matrix unit e_{ij} (corresponding to the given basis) is either in the image of $\mathcal{E}$ (that is, in S) or in the kernel $\ker \mathcal{E}$. It follows that S must in fact be self-adjoint. Namely, if for a fixed i and j, we have that $e_{ij} \in S$, then $\mathcal{E}(e_{ji}) = \mathcal{E}(e_{ji}e_{ij}) = \mathcal{E}(e_{jj}) = e_{jj}$ so $e_{ji} \notin \ker \mathcal{E}$ and therefore $e_{ji} \in S$. Further, $S = \ker(1 - \mathcal{E})$ is weak* closed since Schur multipliers are known to be weak* continuous. In the general (atomic) case, because of the form of $\mathcal{E}$, it follows that $\mathcal{E}$ is a direct sum of $\mathcal{E}_k : B_k \to B_k$ which are each weak*-continuous, so that $\mathcal{E}(A)$ is a von Neumann subalgebra.

Example 2.10. For a positive integer n, let $M_n(\mathbb{C})$ denote the algebra of all n-by-n-matrices over the field of complex numbers $\mathbb{C}$. Then $\bigoplus_{n=1}^{\infty} M_n(\mathbb{C})$ is a Type I finite von Neumann algebra with center isomorphic to ℓ^∞. There are many ways to write the identity as a sum of projections $\sum_{i\in I} p_i$ such that each p_i is of the form $p_i = (p_{i,n})_{n=1}^{\infty}$ with $p_{i,n} \in M_n(\mathbb{C})$ diagonal. Such a sum $\sum_{i\in I} p_i$ gives rise to projection on $\bigoplus_{n=1}^{\infty} M_n(\mathbb{C})$ as in Theorem 1.

3. Ternary rings of operators

We shall use the main result from [11], stated in Lemmas 3.1 and 3.2 below.

For Hilbert spaces H and K, $B(H, K)$ denotes the set of all bounded operators from H to K. A *ternary ring of operators*, TRO for short, is a norm closed subspace $T \subset B(H, K)$ such that $TT^*T \subset T$. For such T, the norm closed linear spans $C =: \langle TT^* \rangle$ and $D = \langle T^*T \rangle$ are C*-subalgebras of $B(K)$ and $\mathcal{B}(H)$ respectively. The *linking algebra* of T is the C*-algebra

$$A_T = \begin{bmatrix} C & T \\ T^* & D \end{bmatrix} \subset B(K \oplus H).$$

If T is a TRO and $P : T \to T$ is a completely contractive projection onto a sub-TRO X, then P is a TRO conditional expectation in the sense that for $a \in T, x, y \in X$,

$$P(ax^*y) = P(a)x^*y$$
$$P(xa^*y) = xP(a)^*y$$
$$P(xy^*a) = xy^*P(a).$$

If T is a C*-algebra, the result was proved in [14, Corollary 3]. If T is a TRO, it was proved with the weaker assumption that P is a contractive projection in [3, Theorem 2.5].

A sub-TRO X of T is *non degenerate* if $\langle XT^*T\rangle = T$ and $\langle TT^*X\rangle = T$.

Lemma 3.1. ([11, Theorem 2.1]) *Let T be a TRO and let $P : T \to T$ be a contractive projection with range X a non degenerate sub-TRO of T. Then there is a (C^*-algebra) conditional expectation from the linking algebra A_T onto the linking algebra A_X,*

$$E = \begin{bmatrix} E_{11} & P \\ P^\dagger & E_{22} \end{bmatrix} : A_T \to A_X,$$

where $P^\dagger(t) = P(t^*)^*$ for $t \in T$,

$$E_{11}\left(\sum_{i=1}^n a_i x_i^*\right) = \sum_{i=1}^n P(a_i)x_i^* \text{ and } E_{22}\left(\sum_{i=1}^n x_i^* a_i\right) = \sum_{i=1}^n x_i^* P(a_i),$$

for $a_i \in T$ and $x_i \in X$.

A W*-TRO is a TRO $T \subset B(H, K)$ that is closed in the weak operator topology.

Lemma 3.2. ([11, Theorem 3.3]) *Let T be a W*-TRO and let $P : T \to T$ be a normal contractive projection with range X a non degenerate sub-W*-TRO of T. Then P extends to a normal conditional expectation from the linking von Neumann algebra A''_T of T, onto the linking von Neumann algebra*

$$A''_X = \begin{bmatrix} \langle XX^*\rangle'' & X \\ X^* & \langle X^*X\rangle'' \end{bmatrix}$$

of X.

Theorem 2. *Let $P : \mathcal{B}(H) \to \mathcal{B}(H)$ be a normal contractive projection onto a non degenerate sub-W*-TRO X of $\mathcal{B}(H)$. Suppose that there is an orthonormal basis $\{\xi_i : i \in I\}$ of H such that for all $i \in I$, $\xi_i \otimes \xi_i^* \in \langle XX^*\rangle'' \cap \langle X^*X\rangle''$. Then there are pairwise* *orthogonal diagonal projections* $\{p_j : j \in J\} \in \mathcal{B}(H)$ *and pairwise orthogonal diagonal projections* $\{q_j : j \in J\} \in \mathcal{B}(H)$ *such that*

$$P(x) = \sum_{j \in J} p_j x q_j \text{ for } x \in \mathcal{B}(H).$$

Proof. Let $\xi_i' = (\xi_i, 0)$ and $\xi_i'' = (0, \xi_i)$, so that $\{\xi_i', \xi_i'' : i \in I\}$ is an orthonormal basis for $H \oplus H$. Identifying $A_{\mathcal{B}(H)}$ with $M_2(\mathcal{B}(H)) = \mathcal{B}(H \oplus H)$ shows that

$$\xi_i' \otimes (\xi_i')^* = \begin{bmatrix} \xi_i \otimes \xi_i^* & 0 \\ 0 & 0 \end{bmatrix} \in A_X''$$

and

$$\xi_i'' \otimes (\xi_i'')^* = \begin{bmatrix} 0 & 0 \\ 0 & \xi_i \otimes \xi_i^* \end{bmatrix} \in A_X''.$$

If $E : \mathcal{B}(H \oplus H) \to \mathcal{B}(H \oplus H)$ is the extension of P given by Lemma 3.1, then by Lemma 3.2 and Theorem 1, there are pairwise orthogonal diagonal projections $r_j \in \mathcal{B}(H \oplus H)$ with

$$E\left(\begin{bmatrix} a & x \\ y^* & b \end{bmatrix}\right) = \sum_j r_j \begin{bmatrix} a & x \\ y^* & b \end{bmatrix} r_j.$$

It follows by diagonality that $r_j = \begin{bmatrix} p_j & 0 \\ 0 & q_j \end{bmatrix}$ and therefore

$$\begin{bmatrix} 0 & P(x) \\ 0 & 0 \end{bmatrix} = E\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \sum_j \begin{bmatrix} 0 & p_j x q_j \\ 0 & 0 \end{bmatrix},$$

where $\{p_j : j \in J\}$ and $\{q_j : j \in J\}$ are each a family of orthogonal and diagonal projections in $\mathcal{B}(H)$. □

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