ON THE STRUCTURE OF WEAKLY CLOSED TERNARY RINGS OF OPERATORS

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Dedicated to the memory of Richard Timoney

Abstract

Calculation of the universal enveloping TROs of continuous JBW*-triples, and application of the techniques used to supplement some structural results of Ruan for W*-TROs.

1. Introduction

In 2004, Zhong-Jin Ruan [20] presented a classification scheme and proved various structure theorems for weakly closed ternary rings of operators (W*-TROs) of particular types. A W*-TRO of type I, II or III was defined according to the Murray-von Neumann type of its linking von Neumann algebra. W*-TROs of type II were further designated as either of type $II_{1,1}, II_{1,\infty}, II_{\infty 1}$ or $II_{\infty,\infty}$. Representation theorems for W*-TROs of various types were given in Ruan's paper (see Theorem 2.1 below), with the possible exception of type $II_{1,1}$ (however, see the remarks at the end of subsection 2.1 of this paper).

The purpose of this paper is to shed some light on the structure of W*-TROs (Theorem 4.1), in particular, those of type $II_{1,1}$ (Corollary 4.4), using ideas from [6] together with the well-established structure theory of JBW*-triples (cf. [15, 16]). A W*-TRO is an example of a JBW*-triple.

Let us recall the structure of all JBW*-triples U from [15, 16]: there is a surjective linear isometric triple isomorphism

$$U \mapsto \oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha) \oplus pM \oplus H(N,\beta), \tag{1.1}$$

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where for each α, Ω_α is an appropriate measure space, C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, and β is a *-antiautomorphism of N of order 2 with fixed points $H(N,\beta)$.

A basic tool in our approach is the universal enveloping TRO $T^*(X)$ of a JC*-triple X as developed in [4] and its sequels [5, 6]. The TROs $T^*(C)$ where C is a Cartan factor have been determined in [4], and independently and simultaneously in the finite dimensional cases in [3]. Both [4] and [3] make very strong use of [19].

Our main new results are the determination of the universal enveloping TROs $T^*(pM)$ and $T^*(H(N,\beta))$ of the continuous JBW*-triples occurring in (1.1). In Theorem 3.2 it is shown that $T^*(pM) = pM \oplus M^t p^t$, and in Theorem 3.4 it is shown that $T^*(H(N,\beta)) = N$.

Only one of these results is needed in the proof of Theorem 4.1, but each is of interest in its own right. Moreover, the other one is used in an alternate proof of a portion of Theorem 4.1 as an illustration of the power of universal enveloping TROs. It is planned to use these techniques in future research (see Section 5).

A decomposition result, obtained simultaneously and independently by different methods in 2013, and stated in the following theorem, plays a key role in some of our proofs. Although we only use the existence of the decomposition, we note that the uniqueness has now been proved in [7].

Theorem 1.1. *Let X be a W*-TRO.*

(a) *(Bunce-Timoney [6, lemma 5.17]) X is TRO-isomorphic to the direct sum $eW \oplus Wf$, where W is a von Neumann algebra and e, f are centrally orthogonal projections in W.*

(b) *(Kaneda [18]) X can be decomposed into the direct sum of TROs X_L, X_R, X_T, and there is a complete isometry of X into a von Neumann algebra M that maps X_L (resp. X_R, X_T) into a weak*-closed left ideal (resp. right ideal, two-sided ideal).*

2. Preliminaries

2.1. Ruan classification scheme

A ternary ring of operators (hereafter TRO) is a norm closed complex subspace of $B(K,H)$ that contains xy^*z whenever it contains x, y, z, where K and H are complex Hilbert spaces. A TRO that is closed in the weak operator topology is called a W*-TRO. A TRO-homomorphism is a linear map φ between two TROs respecting the ternary product: $\varphi(xy^*z) = \varphi(x)\varphi(y)^*\varphi(z)$.

If R is a von Neumann algebra and e is a projection in R, then $V := eR(1-e)$ is a W*-TRO. Conversely, if $V \subset B(K,H)$ is a W*-TRO, then, with $V^* = \{x^* : x \in V\} \subset B(H,K)$, $M(V) = \overline{XX^*}^{sot} \subset B(H)$, $N(V) = \overline{X^*X}^{sot} \subset B(K)$, let

$$R_V = \begin{bmatrix} M(V) & V \\ V^* & N(V) \end{bmatrix} \subset B(H \oplus K)$$

denote the linking von Neumann algebra of V. Then there is a SOT-continuous TRO-isomorphism $V \simeq eRe^\perp$, where $e = \left[\begin{smallmatrix} 1_H & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e^\perp = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_K \end{smallmatrix}\right]$.

In particular, if $V = pM$ where p is a projection in a von Neumann algebra M, then

$$R_V = \begin{bmatrix} pMp & pM \\ Mp & c(p)M \end{bmatrix} \subset B(H \oplus H),$$

where $c(p)$ denotes the central support of p (see [4, p. 965]).

A W*-TRO V is of type I, II or III according to whether R_V is a von Neumann algebra of the corresponding type. A W*-TRO of type II is said to be of type $II_{\epsilon,\delta}$, where $\epsilon, \delta \in \{1, \infty\}$, if $M(V)$ is of type II_ϵ and $N(V)$ is of type II_δ.

Ruan's main representation theorems from [20] are summarised in the following theorem, in which $S\bar{\otimes}T$ denotes the weak operator closure of the algebraic tensor product of subspaces S, T of operators on Hilbert spaces.

Theorem 2.1. *(Ruan [20]) Let V be a W*-TRO.*

(i) *If V is a W*-TRO of type I, then V is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha, B(K_\alpha, H_\alpha))$ ([20, theorem 4.1]).*

(ii) *If V is a W*-TRO of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or III, acting on a separable Hilbert space, then V is a stable W*-TRO, and hence TRO-isomorphic to a von Neumann algebra ([20, corollary 4.3]).*

(iii) *If V is a W*-TRO of type $II_{1,\infty}$ (resp. $II_{\infty,1}$), then V is TRO-isomorphic to $B(H, \mathbb{C})\bar{\otimes}M$ (resp. $B(\mathbb{C}, H)\bar{\otimes}N$), where M (resp. N) is a von Neumann algebra of type II_1 ([20, theorem 4.4]).*

According to Ruan [20, p. 862], 'The structure of a type $II_{1,1}$ W*-TRO is a little bit more complicated.' Nevertheless, using techniques developed for approximately finite dimensional (AFD) von Neumann algebras of type II_1, he is able to prove that every injective W*-TRO of type $II_{1,1}$ acting on a separable Hilbert space is *r*ectangularly AFD ([20, theorem 5.4]). Together with other results from [20, sections 3 and 4], he proves that any W*-TRO acting on a separable Hilbert space is injective if and only if it is rectangularly AFD ([20, theorem 5.5]).

2.2. Horn-Neher classification scheme

A complex *JBW*-triple* is a complex JB*-triple that is also a dual Banach space. The definition of a JB*-triple will not be given here (see, for example, [8, 4, 15, 16]), since only its concrete realisations, which are called JC*-triples, will be involved: namely, norm closed complex subspaces of $B(K, H)$ that contain $xy^*z + zy^*x$ whenever they contain x, y, z. A JC*-homomorphism is a linear map φ between two JC*-triples respecting the triple product: $\{x, y, z\} := (xy^*z + zy^*x)/2$, that is, $\varphi\{x, y, z\} = \{\varphi(x), \varphi(y), \varphi(z)\}$. Such maps are called triple homomorphisms to distinguish them from TRO-homomorphisms.

The structure of JBW*-triples is fairly well understood. Every JBW*-triple is a direct sum of a JBW*-triple of type I and a continuous JBW*-triple (defined below). JBW*-triples of type I have been defined and classified in [15], and continuous JBW*-triples have been classified in [16]. JBW*-triples of type I will not be defined here. Their classification theorem from [15] states: a JBW*-triple of type I is an ℓ^∞-direct sum of JBW*-triples of the form $A\bar{\otimes}C$, where A is a commutative von Neumann algebra and C is a Cartan factor. For Cartan factors of types 1–6, see [8, theorem 2.5.9 and p. 168]. A Cartan factor of type 1 is by definition $B(H, K)$, where H and K are complex Hilbert spaces. No other information about Cartan factors is needed in this paper.

A JBW^*-triple A is said to be *continuous* if it has no type I direct summand. In this case it is known that, up to isometry, A is a JW^*-triple, that is, a JC*-triple that is closed in the weak operator topology. More importantly, it has a decomposition into weak*-closed triple ideals, $A = H(W, \alpha) \oplus pV$, where W and V are continuous von Neumann algebras, p is a projection in V, α is a *-antiautomorphism of W order 2, and $H(W, \alpha) = \{x \in W : \alpha(x) = x\}$ (see [16, (1.20)]).

Notice that the triple product in pV is given by $(xy^*z + zy^*x)/2$, and that $H(W, \alpha)$ is a JW*-algebra (= weakly closed JC*-algebra) with the Jordan product $x \circ y = (xy + yx)/2$. A JC-algebra is a norm closed real subspace of $B(H)$ that is stable for the Jordan product $x \circ y = (xy + yx)/2$. A JC*-algebra is a norm closed complex Jordan *-subalgebra of $B(H)$.

The following theorem can be derived from [16, (2.1), (2.4) and (3.1)], results that led to a uniqueness theorem ([16, (4.8)]). We are indebted to Les Bunce for pointing this out to us, as we use this theorem twice in section 4.

Theorem 2.2. *Up to linear isometry, a continuous JW*-triple A has a unique decomposition*

$$M \oplus_\infty pV,$$

where M is a continuous JW-triple of the form $H(W, \alpha)$ with no non-zero weak* closed ideals Jordan *-isomorphic to a von Neumann algebra, and p is a projection in a continuous von Neumann algebra V.*

2.3. Universal enveloping TROs

If E is a JC*-triple, denote by $C^*(E)$ and $T^*(E)$ the universal C*-algebra and the universal TRO of E resp. (see [4, theorem 3.1, corollary 3.2, definition 3.3]). Recall that the former means that: $C^*(E)$ is a C*-algebra; there is an injective JC*-homomorphism $\alpha_E : E \to C^*(E)$ with the properties that $\alpha_E(E)$ generates $C^*(E)$ as a C*-algebra; and for each JC*-homomorphism $\pi : E \to A$, where A is a C*-algebra, there is a unique *-homomorphism $\tilde{\pi} : C^*(E) \to A$ such that $\tilde{\pi} \circ \alpha_E = \pi$. The latter means that: $T^*(E)$ is a TRO; there is an injective TRO-homomorphism $\alpha_E : E \to T^*(E)$ with the properties that $\alpha_E(E)$ generates $T^*(E)$ as a TRO; and for each JC*-homomorphism $\pi : E \to T$, where T is a TRO, there is a unique TRO-homomorphism $\tilde{\pi} : T^*(E) \to T$ such that $\tilde{\pi} \circ \alpha_E = \pi$.

The property of being universally reversible (cf. [6]) will be important for some of the proofs to follow. A JC-algebra $A \subset B(H)_{sa}$ is called *reversible* if

$$a_1, \ldots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A.$$

A is *universally reversible* if $\pi(A)$ is reversible for each representation (= Jordan homomorphism) $\pi : A \to B(K)_{sa}$. A JC*-algebra $A \subset B(H)$ is called *reversible* if

$$a_1, \dots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A.$$

A is *universally reversible* if $\pi(A)$ is reversible for each representation (= Jordan *-homomorphism) $\pi : A \to B(K)$. Since JC-algebras are exactly the self-adjoint parts of JC*-algebras, a JC*-algebra A is reversible (resp. universally reversible) if and only if the JC-algebra A_{sa} is reversible (resp. universally reversible).

A JC*-triple $A \subset B(H, K)$ is called *reversible* if $a_1, \dots, a_{2n+1} \in A \Rightarrow$

$$a_1 a_2^* a_3 \cdots a_{2n-1} a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* a_{2n-1} \cdots a_3 a_2^* a_1 \in A,$$

and A is *universally reversible* if $\pi(A)$ is reversible for each representation (= triple homomorphism) $\pi : A \to B(H', K')$.

It is easy to check that if a JC*-algebra is universally reversible as a JC*-triple, then it is universally reversible as a JC*-algebra.

For the convenience of the reader, we state two theorems from [6].

Theorem 2.3. *([6, theorem 4.11]) Let T be a TRO. Then the following are equivalent:*

(a) *T is universally reversible (as a JC*-triple);*

(b) *T has no triple homomorphisms onto a Hilbert space of dimension at least 3;*

(c) *T has no TRO homomorphisms onto a Hilbert space of dimension at least 3.*

Theorem 2.4. *([6, theorem 5.4(a)]) Let $V \subset B(H)$ be a TRO with no non-zero representations onto a Hilbert space of any dimension other than (possibly) 2. Suppose that $x \mapsto x^t$ is a transposition of $B(H)$. Then, $T^*(V) = V \oplus V^t$ with $\alpha_V(x) = x \oplus x^t$ (for $x, y \in V$).*

Given a JC-algebra A, there is a universal C*-algebra $C_u^*(A)$ of A, analogous to the definition of $C^*(E)$ given above for JC*-triples E, with the following properties: there is a Jordan homomorphism ψ_A from A into $(C_u^*(A))_{sa}$ such that $C_u^*(A)$ is the C*-algebra generated by $\psi_A(A)$; and for every Jordan homomorphism π from A into C_{sa} for some C*-algebra C, there is a *-homomorphism $\widehat{\pi} : C_u^*(A) \to C$ such that $\pi = \widehat{\pi} \circ \psi_A$ (see [12, section 4], where $C_u^*(A)$ is denoted by $C^*(A)$, or [13, 7.1.8]). It is clear that $C_u^*(A) = C^*(E)$, where E is the complexification of A.

For the convenience of the reader, the following theorem is stated.

Theorem 2.5. *([12, theorem 4.4]) Let A be a universally reversible JC-algebra, B a C*-algebra, and $\theta : A \to B_{sa}$ an injective homomorphism such that B is the C*-algebra generated by $\theta(A)$. If B admits an antiautomorphism φ such that $\varphi \circ \theta = \theta$, then θ extends to a *-isomorphism of $C_u^*(A)$ onto B.*

3. The universal enveloping TROs of continuous JBW*-triples

The proofs of the theorems in this section are very short, since several results from [6] are used, as well as one each from [4] and [11].

3.1. The universal enveloping TRO of pM

A continuous von Neumann algebra W does not admit a non-zero *-homomorphism into $\mathbb{C}$; for by [9, corollary 3, p. 247] there is a projection $e \in W$ with $e \sim 1-e$, so if φ were such a *-homomorphism, we would have $\varphi(e) = 0 \Leftrightarrow \varphi(1-e) = 0 \Leftrightarrow \varphi(e) = 1$.

Although it will not be used, the above statement also holds for Jordan *-homorphisms onto $\mathbb{C}$, since a Jordan *-homomorphism of any C*-algebra onto $\mathbb{C}$ is already a *-homomorphism due to commutativity of $\mathbb{C}$. (See the argument in the first paragraph of the proof of Lemma 3.1.)

Lemma 3.1. *Let W be a continuous von Neumann algebra, and let e be a projection in W. Then the TRO eW does not admit a non-zero triple homomorphism onto $\mathbb{C}$.*

Proof. Suppose, by way of contradiction, that f is a non-zero triple homomorphism of eW onto $\mathbb{C}$. The map f is in fact a TRO-homomorphism, since $f = \tilde{f} \circ q$, where q is the canonical TRO-homomorphism of eW onto $eW/\ker f$, and $\tilde{f}$ is the induced triple isomorphism of $eW/\ker f$ onto $\mathbb{C}$, which is a TRO-isomorphism by commutativity.

Since $f(e) = f(ee^*e) = f(e)|f(e)|^2$, either $f(e) = 0$ or $|f(e)| = 1$. The former case can be ruled out since for $x \in W$, $f(ex) = f((e1)(e1)^*(ex)) = |f(e)|^2 f(ex)$ and f would be zero. If then $f(e) = \lambda$ with $|\lambda| = 1$, then, replacing f by $\overline{\lambda} f$, it can be assumed that $f(e) = 1$.

For $x, y \in W$, $f((exe)(eye)) = f(exee^*eye) = f(exe)\overline{f(e)}f(eye) = f(exe)f(eye)$ and $f((exe)^*) = f(ex^*e) = f(e(exe)^*e) = \overline{f(exe)}$ so that $f|eWe$ is a *-homomorphism of a continuous von Neumann algebra onto $\mathbb{C}$, a contradiction. ∎

Corresponding to an orthonormal basis of a complex Hilbert space H, let J be the unique conjugate linear isometry that fixes that basis elementwise. The transpose $x^t \in B(H)$ of an element $x \in B(H)$ is then defined by $x^t = Jx^*J$.

Theorem 3.2. *Let $W \subset B(H)$ be a continuous von Neumann algebra, and let e be a projection in W. Then $T^*(eW) = eW \oplus W^t e^t$, where x^t is any transposition on $B(H)$.*

Proof. By [6, proposition 3.9], eW is universally reversible, and so, by Theorem 2.3, it does not admit a TRO homomorphism onto a Hilbert space of dimension greater than 2. The proof is completed by applying Lemma 3.1 and Theorem 2.4. ∎

3.2. The universal enveloping TRO of $H(N,\beta)$

Let E be a JC*-algebra. Similar to the construction of $C^*(E)$ when E is considered as a JC*-triple, there is a C*-algebra $C^*_J(E)$ and a Jordan *-homomorphism $\beta_E : E \to C^*_J(E)$ such that $C^*_J(E)$ is the C*-algebra generated by $\beta_E(E)$, and every Jordan *-homomorphism $\pi : E \to B$, where B is a C*-algebra, extends to a *-homomorphism of $C^*_J(E)$ into B (see [4, remark 3.4]). By [4, proposition 3.7], $T^*(E) = C^*_J(E)$. This fact is used in the following theorem.

Remark 3.3. If E is a JC*-algebra, then $C^*_J(E)$ is *-isomorphic to a quotient of $C^*(E)$. Indeed, by definition of $C^*(E)$, there exists a surjective *-homomorphism $\widetilde{\beta}_E : C^*(E) \to C^*_J(E)$ such that $\widetilde{\beta}_E \circ \alpha_E = \beta_E$. In particular, if $C^*(E)$ is simple, then $C^*_J(E)$ is *-isomorphic to $C^*(E)$.

Theorem 3.4. *If N is a continuous von Neumann algebra, then*

$$T^*(H(N,\beta)) = N.$$

Proof. Let $E = H(N,\beta)$. By [6, proposition 2.2], E is universally reversible. In Theorem 2.5, let $A = E_{sa}$, $B = N$, $\varphi = \beta$, and $\theta(x) = x$ for $x \in A$. By [11, corollary 2.9], N is the C*-algebra generated by $\theta(A)$, so that Theorem 2.5 applies to conclude that $N = C^*_J(E) = T^*(E)$. ■

Remark 3.5. [11, corollary 2.9], which was used in the proof of Theorem 3.4, is a corollary to [11, theorem 2.8], which states that if N is a von Neumann algebra admitting a *-antiautomorphism α, and if $H(N,\alpha)_{sa}$ has no type I_1 part, then N is generated as a von Neumann algebra by $H(N,\alpha)_{sa}$. The author of [11] was apparently unaware that [11, corollary 2.9] was proved in the case of a continuous factor by Ayupov in 1985 [1], and the result in this case appeared as Theorem 1.5.2 in [2] in 1997. It is interesting to note that in that case, N is generated algebraically by $H(N,\alpha)_{sa}$.

4. Structure of W*-TROs via JC*-triples

Suppose that V is a W*-TRO, and consider the space V with the JC*-triple structure given by $\{xyz\} = (xy^*z+zy^*x)/2$, so that V becomes a JBW*-triple. As noted in (1.1), $V = V_1 \oplus V_2 \oplus V_3$, where V_i are weak*-closed orthogonal triple ideals of V with V_1 triple isomorphic to a JBW*-triple $\oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha)$ of type I, V_2 triple isomorphic to a right ideal pM in a continuous von Neumann algebra M, and V_3 triple isomorphic to $H(N,\beta)$ for some continuous von Neumann algebra N admitting a *-antiautomorphism β of order 2.

The separability assumption in Theorem 4.1 is due to the fact that we use direct integral theory in the analysis of V_1.

Theorem 4.1. *Let V be a W*-TRO acting on a separable Hilbert space. Then V*

is TRO-isomorphic to

$$\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha)) \oplus eA \oplus Af,$$

where A is a continuous von Neumann algebra.

PROOF. Write $V = V_1 \oplus V_2 \oplus V_3$ as above. Since the triple ideals coincide with the TRO ideals ([14, proposition 5.8]), in particular each V_i is a sub-W*-TRO of V.

Consider first V_2, which is triple isomorphic to a right ideal pM in a continuous von Neumann algebra M. For notation's sake, denote V_2 by V and pM by W. By Theorem 3.2, $T^*(W) = W \oplus W^t$ and $\alpha_W(x) = x \oplus x^t$. By [6, proposition 3.9] and Theorem 2.3, W does not admit a TRO-homomorphism onto a Hilbert space of dimension greater than 2, and therefore V does not admit a triple homomorphism onto a Hilbert space of dimension greater than 2. By Theorem 2.3, V does not admit a TRO-homomorphism onto a Hilbert space of dimension greater than 2. By Lemma 3.1, $W = pM$ does not admit a TRO-homomorphism onto $\mathbb{C}$, so by the same argument, $V = V_2$ does not admit a TRO-homomorphism onto $\mathbb{C}$. Then, by Theorem 2.4, we have that $T^*(V) = V \oplus V^t$ and $\alpha_V(x) = x \oplus x^t$.

By [10, proposition 2.4], the TRO-isomorphism of $T^*(V)$ onto $T^*(W)$ is weak*-continuous. Thus, V is TRO-isomorphic to a weak*-closed ideal I in $W \oplus W^t$. Writing $I = (I \cap W) \oplus (I \cap W^t)$, then $I \cap W$ is a weak*-closed ideal in W, let's call it I_1, and $I \cap W^t$ is a weak*-closed ideal in W^t, let's call it I_2. As noted in [16, p. 574], there are projections $p_1 \leq p, p_2 \leq p^t$ such that $I_1 = p_1 M$ and $I_2 = M^t p_2$.

More precisely,

$$I = I_1 \oplus I_2 = (p_1 \oplus 0)(M \oplus M^t) \oplus (M \oplus M^t)(0 \oplus p_2) = eA \oplus Af,$$

where $A = M \oplus M^t$ is a continuous von Neumann algebra, $e = p_1 \oplus 0$ and $f = 0 \oplus p_2$.

Next, it is shown that $V_3 = 0$. V_3 is triple isomorphic to $H(N, \beta)$ and TRO-isomorphic to $eA \oplus Af$, for a von Neumann algebra A. By Theorem 2.2, the continuous JBW*-triple $H(N, \beta)$ can be chosen so that it has no weak*-closed ideals Jordan *-isomorphic to a von Neumann algebra; but it is triple isomorphic to the continuous JBW*-triple $(e \oplus f^t)(A \oplus A^t)$. By Theorem 2.2, $H(N, \beta) = 0$.

Finally, consider V_1. There are weak*-closed TRO ideals V_α such that $V_1 = \oplus_\alpha V_\alpha$ with V_α triple isomorphic to $L^\infty(\Omega_\alpha, C_\alpha)$ provided that $V_\alpha \neq 0$. It is shown in [17, lemma 2.4 and proof of theorem 1.1] that no Cartan factor of type 2, 3, 4, 5 or 6 can be isometric to a TRO. It follows easily that $L^\infty(\Omega_\alpha, C_\alpha)$ cannot be isometric to a TRO unless C_α is a Cartan factor of type 1. Therefore, each C_α is a Cartan factor of type 1, and V_α is triple isomorphic to $L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$ for suitable Hilbert spaces H_α and K_α.

Next, consider V_α for a fixed α. To simplify notation, let U denote V_α and W denote $L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$. By Theorem 1.1(a), U is TRO-isomorphic to $eA \oplus Af$, for some von Neumann algebra A, and therefore

$$eA \oplus Af \overset{TRO}{\simeq} U \overset{triple}{\simeq} W = L^\infty(\Omega, B(H, K)). \tag{4.1}$$

The right side of (4.1) is a JBW*-triple of type I and thus eA is a JBW*-triple of type I, which implies that A is a von Neumann algebra of type I.

Summarising up to this point, V acts on a separable Hilbert space but is otherwise arbitrary, and $V = V_1 \oplus V_2 \oplus V_3$, where

$$V_1 \overset{triple}{\simeq} \oplus_\alpha e_\alpha A_\alpha \oplus A_\alpha f_\alpha, \tag{4.2}$$

$$V_2 \overset{TRO}{\simeq} eA \oplus Af, \quad V_3 = 0,$$

where each A_α is a von Neumann algebra of type I, and A is a continuous von Neumann algebra. It is required to show that

$$V_1 \overset{TRO}{\simeq} \oplus_\alpha e_\alpha A_\alpha \oplus A_\alpha f_\alpha. \tag{4.3}$$

We return to V_1, and focus on a component on the right side of (4.2) for a fixed α, which is denoted, again for notation's sake, by $eB \oplus Bf$, where B is a von Neumann algebra of type I. Write $B = \oplus_{\gamma\in\Gamma} L^\infty(\Sigma_\gamma, B(H_\gamma))$, $e = \oplus_\gamma e_\gamma$, and $f = \oplus_\gamma f_\gamma$, so that

$$eB = \oplus_{\gamma\in\Gamma} e_\gamma L^\infty(\Sigma_\gamma, B(H_\gamma)),$$

$$Bf = \oplus_{\gamma\in\Gamma} L^\infty(\Sigma_\gamma, B(H_\gamma)) f_\gamma.$$

Since B acts on a separable Hilbert space, the reduction theory of von Neumann algebras ([9, part II]) can now be used to conclude this proof. For a fixed $\gamma \in \Gamma$,

$$L^\infty(\Sigma_\gamma, B(H_\gamma)) = \int_{\Sigma_\gamma}^{\oplus} B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

$$L^2(\Sigma_\gamma, H_\gamma) = \int_{\Sigma_\gamma}^{\oplus} H_\gamma\, d\mu_\gamma(\sigma_\gamma),$$

$$B = \sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

$$e_\gamma = \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

and

$$eB = \sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma) B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma).$$

For notation's sake, for a fixed $\gamma \in \Gamma$, let $\sigma = \sigma_\gamma$, $\mu = \mu_\gamma$, $e = e_\gamma$, $\Sigma = \Sigma_\gamma$,

$H = H_\gamma$, and suppose H is a separable Hilbert space. For each $n \leq \aleph_0$, let $\Sigma_n = \{\sigma \in \Sigma : e(\sigma)$ has rank $n\}$, $e_n = e|_{\Sigma_n}$, and let K_n be a Hilbert space of dimension n. Then,

$$\int_\Sigma^\oplus e(\sigma)B(H)\,d\mu(\sigma) = \sum_{n\leq\aleph_0}^\oplus \int_{\Sigma_n}^\oplus e_n(\sigma)B(H)\,d\mu(\sigma).$$

For each $\sigma \in \Sigma_n$, let $G_\sigma = \{$all unitaries $U : e_n(\sigma)H \to K_n\}$, let $G = \cup_{\sigma\in\Sigma_n} G_\sigma$, and then set

$$E = \{(\sigma, U) \in \Sigma_n \times G : U \in G_\sigma\}.$$

By the measurable selection theorem [9, appendix V], there exists a μ-measurable subset $\Sigma_n' \subset \Sigma_n$ of full measure, and a μ-measurable mapping η of Σ_n' into G, such that $\eta(\sigma) \in G_\sigma$ for every $\sigma \in \Sigma_n'$.

It is easy to verify that for each $\sigma \in \Sigma_n'$, $T_{n,\sigma} : e_n(\sigma)x \mapsto \eta(\sigma)e_n(\sigma)x$ is a TRO-isomorphism of $e_n(\sigma)B(H)$ onto $B(H, K_n)$, and that $\{T_{n,\sigma} : \sigma \in \Sigma_n'\}$ is a μ-measurable field of TRO-isomorphisms.

Hence, $\int_{\Sigma_n}^\oplus T_{n,\sigma}\,d\mu(\sigma)$ is a TRO-isomorphism of

$$\int_{\Sigma_n}^\oplus e_n(\sigma)B(H)\,d\mu(\sigma) \text{ onto } \int_{\Sigma_n}^\oplus B(H, K_n)\,d\mu(\sigma),$$

that is,

$$\int_{\Sigma_n}^\oplus e_n(\sigma)B(H)\,d\mu(\sigma) \overset{TRO}{\simeq} L^\infty(\Sigma_n, B(H, K_n)).$$

Going back to the earlier notation, since

$$eB = \sum_{\gamma\in\Gamma}^\oplus \int_{\Sigma_\gamma}^\oplus e_\gamma(\sigma_\gamma)B(H_\gamma)\,d\mu_\gamma(\sigma_\gamma),$$

it follows that

$$eB \overset{TRO}{\simeq} \sum_{\gamma\in\Gamma}^\oplus \sum_{n\leq\aleph_0} L^\infty(\Sigma_{\gamma,n}, B(H_\gamma, K_n)).$$

By the same arguments, it is clear that also

$$Bf \overset{TRO}{\simeq} \sum_{\gamma\in\Gamma'}^\oplus \sum_{n\leq\aleph_0} L^\infty(\Sigma_{\gamma,n}', B(K_n, H_\gamma')).$$

Since B was one of the A_α in (4.2), this completes the proof of (4.3). ■

In the following three corollaries, the type of a W*-TRO refers to Ruan's classification.

Corollary 4.2. *Let V be a W^*-TRO. If V has no type I part, then it is TRO-isomorphic to $eA \oplus Af$, where A is a continuous von Neumann algebra.*

PROOF. Suppose that V has no type I part. Then $M(V)$ has no type I part and the same holds for $M(V_\alpha)$, by (4.3). But $M(V_\alpha)$ is *-isomorphic to $e_\alpha A_\alpha e_\alpha \oplus c(f_\alpha)A_\alpha$, which is a von Neumann algebra of type I, hence $V_\alpha = 0$. ■

The following Corollary was proved by Ruan without the separability assumption (see Theorem 2.1(i)).

Corollary 4.3. *A W^*-TRO of type I, acting on a separable Hilbert space, is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$.*

PROOF. If $V_2 = pM$ is not zero,

$$R_{V_2} = R_{pM} = \begin{bmatrix} pMp & pM \\ Mp & c(p)M \end{bmatrix}$$

is a von Neumann algebra of type I, so pMp is of type I and continuous, $pMp = 0$, $p = 0$, a contradiction. ■

Corollary 4.4. *A W^*-TRO of type $II_{1,1}$ is TRO-isomorphic to $eA \oplus Af$, where e, f are centrally orthogonal projections in a von Neumann algebra A of type II_1.*

PROOF. Suppose that the W*-TRO $V = V_2 = eA \oplus Af$ is of type $II_{1,1}$. It will be shown that A can be chosen to be of type II_1. Since

$$R_V \overset{*-iso.}{\simeq} \begin{bmatrix} eAe & eA \\ Ae & c(e)A \end{bmatrix} \oplus \begin{bmatrix} c(f)A & Af \\ fA & fAf \end{bmatrix},$$

it follows that $c(f)A$ and $c(e)A$ are each of type II_1. Then, with $\tilde{A} = c(e)A \oplus Ac(f)$, $\tilde{e} = e \oplus 0$, and $\tilde{f} = 0 \oplus f$, we have $V_2 = \tilde{e}\tilde{A} + \tilde{A}\tilde{f}$. ■

Presented now is an alternate approach to the proof of the assertion concerning V_3 in the proof of Theorem 4.1, along the lines of the proof of the assertion concerning V_2. The purpose for doing this is that this proof, despite being longer, further illustrates the power of the techniques used from [4] and [6].

Recall that for any W*-TRO, by (1.1), we can write $V = V_1 \oplus V_2 \oplus V_3$, where V_i are weak*-closed orthogonal triple ideals of V with V_1 triple isomorphic to a JBW*-triple $\oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha)$ of type I, V_2 triple isomorphic to a right ideal pM in a continuous von Neumann algebra M, and V_3 triple isomorphic to $H(N, \beta)$ for some continuous von Neumann algebra N admitting a *-antiautomorphism β of order 2. Since we will only be dealing with V_3, we do not have to assume separability (cf. Corollary 4.2).

We shall show that $V_3 = 0$. V_3 is triple isomorphic to $H(N, \beta)$ for some continuous von Neumann algebra N that admits a *-anti-automorphism β of order 2. We may assume, by Theorem 2.2, that it has no weak*-closed ideals isomorphic to a von Neumann algebra. For notation's sake, denote V_3 by V and $H(N, \beta)$ by W.

Note first that V is a universally reversible TRO. Indeed, by [6, proposition 2.2] and the paragraph preceding it, W is a universally reversible JC*-triple, and therefore so is V. As before, by Theorem 2.3, V does not admit a TRO-homomorphism onto a Hilbert space of dimension greater than 2.

On the other hand, V has no non-zero TRO-homomorphism onto $\mathbb{C}$, since such a homomorphism would extend to a *-homomorphism of the linking von Neumann algebra R_V of V onto $M_2(\mathbb{C})$, whose restriction ρ to the upper left corner of R_V would be zero by the remarks in the paragraph preceding Lemma 3.1.

As before, by Theorem 2.4, $T^*(V) = V \oplus V^t$, $\alpha_V(x) = x \oplus x^t$; and $V \oplus V^t$ is TRO-isomorphic to $T^*(W) = N$, by Theorem 3.4. By [10, proposition 2.4], the TRO-isomorphism is weak*-continuous. Hence, the weak*-closed TRO ideal V in $V \oplus V^t$ is mapped onto a weak*-closed TRO ideal in N, which is necessarily a two-sided ideal in N ([14, proposition 5.8]), say zN for some central projection z in N, forcing $V = V_3 = 0$, by Theorem 2.2.

5. Universal enveloping W*-TROs – prospectus

The work in this paper suggests that it should be fruitful to have a theory of universal enveloping W^*-TROs of JW^*-triples. In this section we propose a definition and suggest four test questions, which can be pursued in a subsequent paper. Our approach is modeled closely after [13, 7.1.9].

Let E be a JW*-triple and consider

$$E \xrightarrow{\alpha_E} C^*(E) \xrightarrow{i} C^*(E)^{**},$$

where i is the canonical embedding of $C^*(E)$ in its enveloping von Neumann algebra $C^*(E)^{**}$. Let $\{e_\alpha\}$ be a maximal family of orthogonal central projections in $C^*(E)^{**}$ such that

$$E \to e_\alpha C^*(E)^{**}, \quad a \mapsto e_\alpha(i \circ \alpha_E(a))$$

is normal, and let $e = \sum_\alpha e_\alpha$. Then e is the maximal central projection with this property. Define

$$W^*(E) = eC^*(E)^{**},$$

and let $\psi_E : E \to W^*(E)$ be the map $a \mapsto e(i \circ \alpha_E(a))$.

For each normal triple homomorphism $\phi : E \to N$, where N is a von Neumann algebra, there is a *-homomorphism $\phi_1 : C^*(E) \to N$ such that $\phi_1 \circ \alpha_E = \phi$. The map ϕ_1 extends to a normal *-homomorphism

$$\overline{\phi} : C^*(E)^{**} \to N.$$

Let f be the support projection of $\overline{\phi}$ in $C^*(E)^{**}$, so that $\overline{\phi}(1 - f) = 0$ and $\overline{\phi}|_{fC^*(E)^{**}}$ is injective. The map

$$E \to fC^*(E)^{**}, \quad a \mapsto f(i \circ \alpha_E(a))$$

decomposes as

$$a \mapsto \phi(a) \to (\overline{\phi}|_{fC^*(E)^{**}})^{-1}(\phi(a)),$$

which is a composition of normal maps, hence is normal, and thus $f \leq e$. Moreover, $W^*(E)$ is the von Neumann algebra generated by $\psi_E(E)$. This proves the following theorem, which is the analog of [4, theorem 3.1].

Theorem 5.1. *If E is a JW^*-triple, then there exists a unique pair $(W^*(E), \psi_E)$, where $W^*(E)$ is a von Neumann algebra, and ψ_E is an injective normal triple homomorphism from E into $W^*(E)$ such that*

(a) $\psi_E(E)$ generates $W^*(E)$ as a von Neumann algebra, and

(b) if $\pi : E \to A$ is a normal triple homomorphism into a von Neumann algebra A, then there exists a unique normal *-homomorphism $\tilde{\pi} : W^*(E) \to A$ such that $\tilde{\pi} \circ \psi_E = \pi$.

By letting $WT^*(E)$ denote the W^*-TRO generated in $W^*(E)$, we obtain the following, which is the analog of [4, corollary 3.2].

Corollary 5.2. *If E is a JW^*-triple, then there exists a unique pair $(WT^*(E), \psi_E)$, where $WT^*(E)$ is W^*-TRO, and ψ_E is an injective normal triple homomorphism from E into $WT^*(E)$ such that*

(a) $\psi_E(E)$ generates $WT^*(E)$ as a W^*-TRO, and

(b) if $\pi : E \to T$ is a normal triple homomorphism into a W^*-TRO T, then there exists a unique normal TRO-homomorphism $\tilde{\pi} : WT^*(E) \to T$ such that $\tilde{\pi} \circ \psi_E = \pi$.

We now state our four test questions.

In [4, theorem 5.4], it is proved that if $B(H, K)$ is of rank at least 2, then $T^*(B(H, K)) = B(H, K) \oplus B(K, H)$.

Question 1. *What is $WT^*(B(H, K))$ in this case?*

In [4, proposition 3.6], it is proved that if the JC*-triple E is the sum of orthogonal ideals I, J, then $T^*(E) = T^*(I) \oplus T^*(J)$.

Question 2. *If the W^*-TRO E is the sum of weak*-closed orthogonal ideals I, J, does it follow that $WT^*(E) = WT^*(I) \oplus WT^*(J)$?*

In [4, theorem 5.1], it is proved that if H is a Hilbert space, then $T^*(H)$ is the TRO generated by the annihilation operators in the CAR algebra of H.

Question 3. *What is $WT^*(H)$?*

In [5, theorem 4.9], it is proved that if A is an Abelian von Neumann algebra and E is a JC*-triple, then $T^*(A\widehat{\otimes}_\epsilon E) = A\widehat{\otimes}_\epsilon T^*(E)$, where $\widehat{\otimes}_\epsilon$ is the injective tensor product of Banach spaces.

If a JC*-triple E is contained in a C*-algebra B, then $A\widehat{\otimes}_\epsilon E$ is a JC*-subtriple of $A\widehat{\otimes}_\epsilon B = A \otimes_{\min} B$. Similarly, if A is an Abelian von Neumann algebra and $E \subset B(H)$ is a JW^*-triple, then the spatial tensor product $A \otimes E$ is a subtriple of $A \otimes B(H)$, and $A\overline{\otimes}E$ denotes the weak closure of $A \otimes E$.

Question 4. *Do we have $WT^*(A\overline{\otimes}E) = A\overline{\otimes}WT^*(E)$, at least for $E =$ Hilbert space and $E = B(H,K)$ of rank at least 2?*

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References

[1] S. Ayupov, JW-factors and anti-automorphisms of von Neumann algebras, *Mathematics of the USSR-Izvestiya* **26** (1986), 201–9.

[2] S. Ayupov, A. Rakhimov and S. Usmanov, *Jordan, real and Lie structures in operator algebras*, Mathematics and its Applications vol. 418, Kluwer Academic Publishers Group, Dordrecht, 1997.

[3] D. Bohle and W. Werner, The universal enveloping ternary ring of operators of a JB*-triple system, *Proceedings of the Edinburgh Mathematical Society* **57** (2) (2014), 347–66.

[4] L.J. Bunce, B. Feely and R.M. Timoney, Operator space structure of JC*-triples and TROs, I, *Mathematische Zeitschrift* **270** (3-4) (2012), 961–82,

[5] L.J. Bunce and R.M. Timoney, On the universal TRO of a JC*-triple, ideals and tensor products, *The Quarterly Journal of Mathematics* **64** (2) (2013), 327–40.

[6] L.J. Bunce and R.M. Timoney, Universally reversible JC*-triples and operator spaces, *Journal of the London Mathematical Society* **88** (2) (2013), 271–93.

[7] L.J. Bunce and R.M. Timoney, On decompositions of TROs and their morphisms, *Mathematical Proceedings of the Royal Irish Academy*, (to appear).

[8] Cho-Ho Chu, *Jordan structures in geometry and analysis*, Cambridge Tracts in Mathematics vol. 190, Cambridge University Press, Cambridge, 2012.

[9] J. Dixmier, *Von Neumann Algebras*, Elsevier, North-Holland, 1981.

[10] E.G. Effros, N. Ozawa and Zhong-Jin Ruan, On injectivity and nuclearity for operator spaces, *Duke Mathematical Journal* **110** (3) (2001), 489–522.

[11] J. Gasemyr, Involutory antiautomorphisms of von Neumann and C*-algebras, *Mathematica Scandinavica* **67** (1990), 87–96.

[12] H. Hanche-Olsen, On the structure and tensor products of JC-algebras, *Canadian Journal of Mathematics* **35** (6) (1983), 1059–74.

[13] H. Hanche-Olsen and E. Stormer, *Jordan operator algebras*, Monographs and Studies in Mathematics vol. 21, Pitman (Advanced Publishing Program), Boston, MA, (1984).

[14] L.A. Harris, A generalization of C*-algebras, *Proceedings of the London Mathematical Society* **42** (3) (1981), 331–61.

[15] G. Horn, Classification of JBW*-triples of type I, *Mathematische Zeitschrift* **196** (2) (1987), 271–91.

[16] G. Horn and E. Neher, Classification of continuous JBW*-triples, *Transactions of the American Mathematical Society* **306** (1988), 553–78.

[17] J.M. Isidro and L.L. Stachó, On the Jordan structure of ternary rings of operators, *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica* **46**, (2003–4), 149–56.

[18] M. Kaneda, Ideal decompositions of a ternary ring of operators with predual, *Pacific Journal of Mathematics* **266** (2) (2013), 297–303.

[19] M. Neal and B. Russo, Contractive projections and operator spaces, *Transactions of the American Mathematical Society* **355** (6) (2003), 2223–62.

[20] Zhong-Jin Ruan, Type decomposition and the rectangular AFD property for W*-TROs, *Canadian Journal of Mathematics* **36** (4) (2004), 843–70.