Assignments

Assignment 1 (Due September 29)

1. Read sections 1.2,1.3,1.4 in Buck (The lectures will continue with section 1.5). Do not waste your time reading about the concepts *angle, orthogonal, hyperplane, normal vector, line, convexity,* which are discussed in section 1.3 of Buck. We have no immediate use for them. Thus, you may skip pages 15-18 and 21-27 for now.

2.
- Buck [§1.2 page 10 #5,10,23]
- Buck [§1.3 page 18 #1,2,5,6]

Assignment 2 (Due October 6)

- Buck [§1.4 page 27 #3,15,16]
- Buck [§1.5 page 36 #1,5,9,13]

Assignment 3 (Due October 6)
Fix $p \in \mathbf{R}^n$. Show that $\{q \in \mathbf{R}^n : |q - p| > 2\}$ is an open set.

Assignment 4 (Due October 13)
Recall Young's inequality (Theorem 3.1):

> Let φ be differentiable and strictly increasing on $[0, \infty)$, $\varphi(0) = 0$, $\lim_{u\to\infty} \varphi(u) = \infty$, $\psi := \varphi^{-1}$, $\Phi(x) := \int_0^x \varphi(u)\,du$, $\Psi(x) := \int_0^x \psi(u)\,du$. Then for all $a, b \in [0, \infty)$,
>
> $$ab \le \Phi(a) + \Psi(b). \tag{1}$$
>
> Moreover, equality holds in (1) if and only if $b = \varphi(a)$.

Give a rigorous proof of Young's inequality. More precisely,

Step 1 First establish, for $c \in [0, \infty)$, the formula

$$\int_0^c \varphi(u)\,du + \int_0^{\varphi(c)} \psi(v)\,dv = c\varphi(c). \tag{2}$$

Step 2 Use (2) to prove (1).

Step 3 Prove the "moreover" statement.

Assignment 5 (Due October 13)

- Show that every open set in $\mathbf{R}^n$ is the union of a countable collection of open balls. (Hint: The answer is somewhere in the minutes for my 140C class of Fall 2005)

- Show that in $\mathbf{R}^1$, the open balls can be assumed to be disjoint

Assignment 6 (Due October 6) Prove the following assertions:

(a) $\operatorname{int} S = \cup\{G : G \text{ is open }, G \subset S\}$

(b) $\overline{S} = \cap\{F : F \text{ is closed }, S \subset F\}$

Assignment 7 (Due October 6) [Buck §1.5 page 36 #2,6,10,11]

Assignment 8 (Due October 13) Prove directly the following three assertions. The fourth assertion will be proved in class.

(a) If S satisfies BW, then S is a closed set.

(b) If S satisfies BW, then S is a bounded set.

(c) If S satisfies HB, then S is a bounded set.

(d) (This will be done in class, not part of the homework—it is included here for comparison purposes only) If S satisfies HB, then S is a closed set.

Assertions (c) and (d) are stated in Buck as [§1.8 page 69 #1,2]

Assignment 9 (Due October 27) [Buck, §2.2 page 80 #1 or 2, #3 or 4, #7 or 8, #12 or 13, #14 or 17] You are to hand in 5 problems, one from each of these 5 pairs. You will of course be responsible for all of the problems.

Assignment 10 (Due October 20) [Buck, §1.6 page 54 #1, 2, 3, 4, 32, 35]

Assignment 11 (Due October 27) [Buck, §2.3 page 88 #1–7]

Assignment 12 (Due October 27) [Buck, §3.3 page 134 #4,5]

Assignment 13 (Due November 3 Hint: This was done in class)

(A) Give a proof of Problem 3(c) on page 37 of Buck using Corollary 14.2.

(B) Use Corollary 14.2 to prove that if A and B are closed sets in $\mathbf{R}$, then $A \times B$ is a closed set in $\mathbf{R}^2$.

Assignment 14 (Due November 3) Let S be any subset of $\mathbf{R}^n$. Using only the definitions of cluster point and boundary, prove the following statements.

- $\operatorname{cl}(\operatorname{cl} S) \subset \operatorname{cl} S$ (Hint: See the solution to Problem 8(b) on the first midterm for Math 140C, Fall 2005, which is at the top of page 26 of the minutes)
- $\mathbf{R}^n - \operatorname{cl} S$ is open
- $\operatorname{bdy}(\operatorname{cl} S) \subset \operatorname{cl} S$

- $\{\lim_k p_k : p_k \in \mathrm{cl}\, S\} \subset \mathrm{cl}\, S$

Assignment 15 (Due November 3) Prove Theorem 16.4.

Assignment 16 (Due November 3) Show that a linear transformation (see [Buck, Section 7.3]) is uniformly continuous. (Hint: Use [Buck, Theorem 8,page 338])

Assignment 17 (Due November 10) From (19) and (20), show that $D_j f(p_0)$ exists. Hint: Use the property $\mathbf{D}_{-e_j} f = -\mathbf{D}_{e_j} f$ of directional derivatives (see Buck, page 126)

Assignment 18 (Due November 10) [Buck page 351 #1,2,7,8]

Assignment 19 (Due November 17) Show that the function of Problem 4 on page 135 of Buck, namely, $f(x, y) = xy/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is not differentiable at $(0, 0)$.

Assignment 20 (Due November 17)

(a) Use Theorem 20.3 to prove the following theorem from [Buck,section 3.4].

> **Theorem** [Theorem 14, page 136 of Buck] Let $F(t) = f(x, y)$, where $x = g(t)$, $y = h(t)$, the functions g, h are assumed to be of class C^1 on an open interval containing $t_0 \in \mathbf{R}$, and the function f is assumed to be of class C^1 in an open ball with center $p_0 = (x_0, y_0) = (g(t_0), h(t_0))$. Then F is of class C^1 on an open interval containing $t_0 \in \mathbf{R}$, and for t in that interval,
>
> $$F'(t) = \tfrac{\partial f}{\partial x}(g(t), h(t))g'(t) + \tfrac{\partial f}{\partial y}(g(t), h(t))h'(t).$$

(b) Let $F(x, y) = f(g(x, y), h(x, y))$, where $f : \mathbf{R}^2 \to \mathbf{R}$, $g : \mathbf{R}^2 \to \mathbf{R}$, and $h : \mathbf{R}^2 \to \mathbf{R}$ are differentiable. Use Theorem 20.3 to prove that F is differentiable and

$$D_1 F(x, y) = D_1 f(g(x, y), h(x, y)) D_1 g(x, y) + D_2 f(g(x, y), h(x, y)) D_1 h(x, y)$$

and

$$D_2 F(x, y) = D_1 f(g(x, y), h(x, y)) D_2 g(x, y) + D_2 f(g(x, y), h(x, y)) D_2 h(x, y).$$

Assignment 21 (Due November 17) [Buck page 145 #1,2]

Assignment 22 (Due November 22) [Buck page 154 #18] (Look in the index of Buck to find the definitions of *convex* and *Lipschitz condition*)

Assignment 23 (Due November 22) [Buck page 361 #11] (The answer to the question is NO. Look at the hint in Buck to construct a proof)

Assignment 24 (Due December 1) *Show that, for $F = x^2 + y^2 - 1$, $T = T_F$ is not one-to-one on $D = \mathbf{R}^2$ and $T(\mathbf{R}^2)$ is not an open subset of $\mathbf{R}^2$.*

Assignment 25 (Due December 1) [Buck page 366 #2,5,9,11]

Assignment 26 (Due December 1) *Prove that if K is a compact set in $\mathbf{R}^n$ and $q \notin K$, then*

$$\inf\{|p - q| : p \in K\} > 0.$$