## Assignments

**Assignment 1** (Step 1 is Due September 30; Steps 2 and 3 are due October 7) *Give a rigorous proof of Young's inequality. More precisely,*

**Step 1** *First establish, for* $c \in [0, \infty)$*, the formula*

$$\int_0^c \varphi(u)\,du + \int_0^{\varphi(c)} \psi(v)\,dv = c\varphi(c).$$

**Step 2** *Use Step 1 to prove Young's inequality.*

**Step 3** *Prove the "moreover" statement.*

**Assignment 2** (Due September 30)

1. Read sections 1.2,1.3,1.4 in Buck (The lectures will continue with section 1.5). Do not waste your time reading about the concepts *angle, orthogonal, hyperplane, normal vector, line, convexity*, which are discussed in section 1.3 of Buck. We have no immediate use for them. Thus, you may skip pages 15-18 and 21-27 for now.
2. 
   - Buck [§1.2 page 10 #5,10,23]
   - Buck [§1.3 page 18 #2,3,6]
   - Buck [§1.4 page 27 #3,15,16]

**Assignment 3** (Due October 7)

- Buck [§1.5 page 36 #2,5]
- Fix $p \in \mathbf{R}^n$. Show that $\{q \in \mathbf{R}^n : |q - p| > 2\}$ is an open set.

**Assignment 4** (Due October 7) Prove the following assertions:

**(a)** $\operatorname{int} S = \cup\{G : G \text{ is open }, G \subset S\}$

**(b)** $\overline{S} = \cap\{F : F \text{ is closed }, S \subset F\}$

**Assignment 5** (Due October 14) [Buck §1.5 page 36 #6,10,11]

**Assignment 6** (Due October 14) Prove directly the following three assertions. The fourth assertion will be proved in class.

**(a)** If $S$ satisfies BW, then $S$ is a closed set.

**(b)** If $S$ satisfies BW, then $S$ is a bounded set.

**(c)** If $S$ satisfies HB, then $S$ is a bounded set.

**(d)** (This will be done in class, not part of the homework) If $S$ satisfies HB, then $S$ is a closed set.

These assertions are stated in Buck as [§1.8 page 69 #1,2]

**Assignment 7** (Due October 21—no penalty for turning it in on October 24, so you can write it up elegantly)

- Buck page 36, #1,3,4,11 (any two of these four)
- Buck page 36, #7,8,12 (any one of these)
- Buck page 36, #9, Buck page 69, #3 (both of these) (For #3, see the hint at the end of the book)

You will of course be responsible for all of these problems on the midterm.

**Assignment 8** (Due October 28) [Buck, §2.2 page 80 #1 or 2,3 or 4,7 or 8,12 or 13,14 or 17] You are to hand in 5 problems, one from each of these 5 pairs. You will of course be responsible for all of the problems.

**Assignment 9** (Due November 4) [Buck, §2.3 page 88 #1,3 or 4,5 or 6,7]

**Assignment 10** (Due October 28) [Buck, §1.6 page 54 #1 or 2,3 or 4,31 or 33,32 or 35]

**Assignment 11** (Due November 4) In (A) and (B), show that $f$ and $g$ are uniformly continuous on $\mathbf{R}^n$, where

**(A)** $f(p) = |p|$ (Hint: triangle inequality)

**(B)** $g(p) = x_1y_1 + \cdots + x_ny_n$ where $p = (x_1, \ldots, x_n) \in \mathbf{R}^n$ is a variable point and $y_1, \ldots, y_n \in \mathbf{R}$ are fixed.

**(C)** [Buck, p.88#6], namely, a uniformly continuous function preserves Cauchy sequences.

**Assignment 12** (Due November 4) Give three other proofs that the set $\mathrm{cl}\, S$ of cluster points of an arbitrary set $S$ is a closed set, more precisely,

- Show $\mathbf{R}^n - \mathrm{cl}\, S$ is open
- Show $\mathrm{bdy}\,(\mathrm{cl}\, S) \subset \mathrm{cl}\, S$
- Show $\{\lim_k p_k : p_k \in \mathrm{cl}\, S\} \subset \mathrm{cl}\, S$

**Assignment 13** (Due November 14)

**(A)** Let $S \subset \mathbf{R}^n$ be a bounded set and let $f : S \to \mathbf{R}$ be a continuous function. Prove that $f$ has a continuous extension to $\overline{S}$ if and only if $f$ is uniformly continuous on $S$. (You just need to quote two theorems!)

**(B)** Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and suppose that

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Prove that $f$ is uniformly continuous on $\mathbf{R}$.

**Assignment 14** (Due November 14) [Buck, §3.3 page 134 #4,5,11]

**Assignment 15** (Due November 14) Give an example for $n = 1$ where $p^*$ cannot be chosen to be $p_0$. (Hint: almost any example works). What about $n = 2$?

**Assignment 16** (Due November 21) Let $T(p) = (f^1(p), \cdots, f^m(p))$ be a transformation with coordinate functions $f^1, \ldots, f^m$. Prove that $T$ is continuous at $p_0$ if and only if each coordinate function $f^j$, $1 \le j \le m$, is continuous at $p_0$.

**Assignment 17** (Due November 21) Prove Theorem 20.4

**Assignment 18** (Due November 21) Prove Theorem 20.5

**Assignment 19** (Due November 21) State and prove an analog of the Extreme values theorem, Theorem 12.4. (Hint: Since $\mathbf{R}^m$ has no order structure, you have to express the theorem in terms of $|T(p)|$.)

**Assignment 20** (Due November 21) Prove Theorem 20.7

**Assignment 21** (Due November 21) Show that a linear transformation (see [Buck, Section 7.3]) is uniformly continuous. (Hint: Use [Buck, Theorem 8,page 338])

**Assignment 22** (Due November 21) Prove Theorem 20.8

**Assignment 23** (Due November 21) Let $D \subset \mathbf{R}^n$ be a bounded set and let $T : D \to \mathbf{R}^m$ be a continuous transformation. Prove that $T$ has a continuous extension to $\overline{D}$ if and only if $T$ is uniformly continuous on $D$.

**Assignment 24** (Due November 21) Prove that a transformation of class $C^1$ is continuous.

**Assignment 25** (Due November 28) Prove Theorem 23.3.

**Assignment 26** (Due November 28) *Prove that, for a fixed $p_0$, at most one linear transformation $L$ can satisfy* (31). (This is the same as Exercise #10, page 352 in Buck)

**Assignment 27** (Due November 28) *If $T = (f^1, \ldots, f^m)$ is a differentiable transformation at $p_0$, then the partial derivatives $\frac{\partial f^i}{\partial x_j}(p_0)$ exist for all $1 \le j \le n, 1 \le i \le m$. In other words, the Jacobian matrix $J_T(p_0)$ exists.* (Hint: In the definition of partial derivative, let $p = p_0 + te_j$ where $t \in \mathbf{R}$ and $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots,\ e_n = (0, \ldots, 0, 1)$).

**Assignment 28** (Due November 28) Let $T$ be a transformation which is of class $C^1$ on an open set $D$, and let $S$ be a transformation of class $C^1$ on an open set containing $T(D)$. Then $S \circ T$ is of class $C^1$ on $D$.

**Assignment 29** (Due November 28) *Let* $F(x, y) = f(g(x, y), h(x, y))$, *where* $g : \mathbf{R}^2 \to \mathbf{R}$ *and* $h : \mathbf{R}^2 \to \mathbf{R}$. *Use Corollary 24.3 to prove that*

$$F_1(x, y) = f_1(g(x, y), h(x, y))g_1(x, y) + f_2(g(x, y), h(x, y))h_1(x, y)$$

*and*

$$F_2(x, y) = f_1(g(x, y), h(x, y))g_2(x, y) + f_2(g(x, y), h(x, y))h_2(x, y).$$

(Compare using Corollary 24.3 with the method on page 137 of Buck.)

**Assignment 30** (Due November 28) [Buck, §7.4 page 351, #2,5,(7 or 8)](three problems)

**Assignment 31** (Due December 2) *Prove that if* $K$ *is a compact set in* $\mathbf{R}^n$ *and* $q \notin K$, *then*

$$\inf\{|p - q| : p \in K\} > 0.$$

**Assignment 32** (Due December 7—the day of the final exam) Prove Lemma 29.1

**Assignment 33** (Due December 7—the day of the final exam) *If a transformation preserves convergent sequences, then it is continuous.* (Same proof as [Buck, Theorem 2,page 74].)

**Assignment 34** (Due December 7) [Buck, §7.5,page 361,#11,14]

**Assignment 35** (Due December 7) *Show that, for* $F = x^2 + y^2 - 1$, $T = T_F$ *is not one-to-one on* $D = \mathbf{R}^2$ *and* $T(\mathbf{R}^2)$ *is not an open subset of* $\mathbf{R}^2$.

**Assignment 36** (Due December 7) *Let* $F(x, y, z) = x^2 + y^2 + z^2 - 1$ *and take a point* $(x_0, y_0, z_0)$ *on the unit sphere in* $\mathbf{R}^3$: $x_0^2 + y_0^2 + z_0^2 = 1$, *that is,* $F(x_0, y_0, z_0) = 0$. *"Prove" that* $z = \phi(x, y) := \sqrt{1 - x^2 - y^2}$ *satisfies* $F(x, y, \phi(x, y)) = 0$. *According to the implicit function theorem, we need* $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$, *that is,* $2z_0 \neq 0$, *so take, for example* $p_0 = (1/\sqrt{2}, 0, 1/\sqrt{2})$. *Now find* $r > 0$ *such that*

$$(x - \frac{1}{\sqrt{2}})^2 + (y - 0)^2 < r \Rightarrow x^2 + y^2 < 1.$$

**Assignment 37** (Due December 7) *Let* $F(x, y, z) = x^2 + yz^5 - 3xyz + z$, *take the point* $(1, 0, -1)$, *and note that* $F(1, 0, -1) = 0$ *and* $\frac{\partial F}{\partial z}(1, 0, -1) = 1 \neq 0$. *Conclude that there exists* $r > 0$ *and a function* $\phi(x, y)$ *of class* $C^1$ *in the ball*

$$|(x, y) - (1, 0)| < r$$

*such that* $F(x, y, \phi(x, y)) = 0$ *for all* $(x, y)$ *with* $(x - 1)^2 + y^2 < r^2$, *that is*

$$x^2 + y[\phi(x, y)]^5 - (3xy - 1)\phi(x, y) = 0.$$

**Assignment 38** (Due December 7) *Let* $F(x,y,z) = \sin xy + e^z - e$, *take the point* $(x_0, 0, 1)$, *and note that* $F(x_0, 0, 1) = 0$. *Also*

$$\tfrac{\partial F}{\partial x}(x_0,0,1) = 0, \quad \tfrac{\partial F}{\partial y}(x_0,0,1) = x_0, \quad \tfrac{\partial F}{\partial z}(x_0,0,1) = e.$$

*What does the implicit function theorem say in this case? Can you solve for any of the three variables without the help of the implicit function theorem?*

**Assignment 39** (Due December 7) *Let* $F(x,y,z) = (\sin x)e^y + (\cos y)e^{xz} + \sin z$, *take the point* $(0, \pi/2, \pi)$, *and note that* $F(0,\pi/2,\pi) = 0$. *Also*

$$\tfrac{\partial F}{\partial x}(0,\pi/2,\pi) = e^{\pi/2}, \quad \tfrac{\partial F}{\partial y}(0,\pi/2,\pi) = -1, \quad \tfrac{\partial F}{\partial z}(0,\pi/2,\pi) = -1.$$

*By the implicit function theorem, you have*

$$z = \phi(x,y) \text{ for } (x,y) \text{ close to } (0,\pi/2),$$

*as well as*

$$x = \psi(y,z) \text{ for } (y,z) \text{ close to } (\pi/2,\pi),$$

*etc. Now let* $S(x,y) = (x,y,\phi(x,y))$ *and apply the chain rule to* $F \circ S$ *to derive*

$$\tfrac{\partial \phi}{\partial x}(x,y) = \frac{-\tfrac{\partial F}{\partial x}(x,y,\phi(x,y))}{\tfrac{\partial F}{\partial z}(x,y,\phi(x,y))},$$

*and*

$$\tfrac{\partial \phi}{\partial y}(x,y) = \frac{-\tfrac{\partial F}{\partial y}(x,y,\phi(x,y))}{\tfrac{\partial F}{\partial z}(x,y,\phi(x,y))}.$$

**Assignment 40** (Due December 7) [Buck, §7.6,page 366,#1,2,5]