Assignments

Assignment 1 (Step 1 is Due September 30; Steps 2 and 3 are due October 7) *Give a rigorous proof of Young’s inequality. More precisely,*

**Step 1** *First establish, for* \( c \in [0, \infty) \), the formula
\[
\int_0^c \varphi(u) \, du + \int_0^{\varphi(c)} \psi(v) \, dv = c\varphi(c).
\]

**Step 2** *Use Step 1 to prove Young’s inequality.*

**Step 3** *Prove the “moreover” statement.*

Assignment 2 (Due September 30)

1. Read sections 1.2, 1.3, 1.4 in Buck (The lectures will continue with section 1.5). Do not waste your time reading about the concepts *angle, orthogonal, hyperplane, normal vector, line, convexity,* which are discussed in section 1.3 of Buck. We have no immediate use for them. Thus, you may skip pages 15-18 and 21-27 for now.

2. *Buck [§1.2 page 10 #5,10,23]*
   *Buck [§1.3 page 18 #2,3,6]*
   *Buck [§1.4 page 27 #3,15,16]*

Assignment 3 (Due October 7)

*Buck [§1.5 page 36 #2,5]*

*Fix* \( p \in \mathbb{R}^n \). Show that \( \{ q \in \mathbb{R}^n : |q - p| > 2 \} \) is an open set.

Assignment 4 (Due October 7) Prove the following assertions:

(a) \( \text{int} \, S = \bigcup \{ G : G \text{ is open}, G \subset S \} \)

(b) \( \overline{S} = \bigcap \{ F : F \text{ is closed}, S \subset F \} \)

Assignment 5 (Due October 14) [Buck §1.5 page 36 #6,10,11]

Assignment 6 (Due October 14) Prove directly the following three assertions. The fourth assertion will be proved in class.

(a) If \( S \) satisfies BW, then \( S \) is a closed set.

(b) If \( S \) satisfies BW, then \( S \) is a bounded set.

(c) If \( S \) satisfies HB, then \( S \) is a bounded set.
(d) (This will be done in class, not part of the homework) If \( S \) satisfies HB, then \( S \) is a closed set.

These assertions are stated in Buck as [§1.8 page 69 #1,2]

**Assignment 7** (Due October 21—no penalty for turning it in on October 24, so you can write it up elegantly)

- Buck page 36, #1,3,4,11 (any two of these four)
- Buck page 36, #7,8,12 (any one of these)
- Buck page 36, #9, Buck page 69, #3 (both of these) (For #3, see the hint at the end of the book)

You will of course be responsible for all of these problems on the midterm.

**Assignment 8** (Due October 28) [Buck, §2.2 page 80 #1 or 2,3 or 4,7 or 8,12 or 13,14 or 17] You are to hand in 5 problems, one from each of these 5 pairs. You will of course be responsible for all of the problems.

**Assignment 9** (Due November 4) [Buck, §2.3 page 88 #1 or 3 or 4 or 5 or 6,7]

**Assignment 10** (Due October 28) [Buck, §1.6 page 54 #1 or 2,3 or 4,31 or 33,32 or 35]

**Assignment 11** (Due November 4) In (A) and (B), show that \( f \) and \( g \) are uniformly continuous on \( \mathbb{R}^n \), where

(A) \( f(p) = |p| \) (Hint: triangle inequality)

(B) \( g(p) = x_1y_1 + \cdots + x_ny_n \) where \( p = (x_1, \ldots, x_n) \in \mathbb{R}^n \) is a variable point and \( y_1, \ldots, y_n \in \mathbb{R} \) are fixed.

(C) [Buck, p.88#6], namely, a uniformly continuous function preserves Cauchy sequences.

**Assignment 12** (Due November 4) Give three other proofs that the set \( \text{cl} S \) of cluster points of an arbitrary set \( S \) is a closed set, more precisely,

- Show \( \mathbb{R}^n - \text{cl} S \) is open
- Show \( \text{bdy} (\text{cl} S) \subset \text{cl} S \)
- Show \( \{\lim p_k : p_k \in \text{cl} S\} \subset \text{cl} S \)

**Assignment 13** (Due November 14)

(A) Let \( S \subset \mathbb{R}^n \) be a bounded set and let \( f : S \to \mathbb{R} \) be a continuous function. Prove that \( f \) has a continuous extension to \( \overline{S} \) if and only if \( f \) is uniformly continuous on \( S \). (You just need to quote two theorems!)
(B) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be continuous and suppose that
\[
\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0.
\]
Prove that \( f \) is uniformly continuous on \( \mathbb{R} \).

Assignment 14 (Due November 14) [Buck, §3.3 page 134 #4,5,11]

Assignment 15 (Due November 14) Give an example for \( n = 1 \) where \( p^* \) cannot be chosen to be \( p_0 \). (Hint: almost any example works). What about \( n = 2 \)?

Assignment 16 (Due November 21) Let \( T(p) = (f^1(p), \ldots, f^m(p)) \) be a transformation with coordinate functions \( f^1, \ldots, f^m \). Prove that \( T \) is continuous at \( p_0 \) if and only if each coordinate function \( f^j \), \( 1 \leq j \leq m \), is continuous at \( p_0 \).

Assignment 17 (Due November 21) Prove Theorem 20.4

Assignment 18 (Due November 21) Prove Theorem 20.5

Assignment 19 (Due November 21) State and prove an analog of the Extreme values theorem, Theorem 12.4. (Hint: Since \( \mathbb{R}^m \) has no order structure, you have to express the theorem in terms of \( |T(p)| \).)

Assignment 20 (Due November 21) Prove Theorem 20.7

Assignment 21 (Due November 21) Show that a linear transformation (see [Buck, Section 7.3]) is uniformly continuous. (Hint: Use [Buck, Theorem 8, page 338])

Assignment 22 (Due November 21) Prove Theorem 20.8

Assignment 23 (Due November 21) Let \( D \subset \mathbb{R}^n \) be a bounded set and let \( T : D \rightarrow \mathbb{R}^m \) be a continuous transformation. Prove that \( T \) has a continuous extension to \( \overline{D} \) if and only if \( T \) is uniformly continuous on \( D \).

Assignment 24 (Due November 21) Prove that a transformation of class \( \mathcal{C}^1 \) is continuous.

Assignment 25 (Due November 28) Prove Theorem 23.3.

Assignment 26 (Due November 28) Prove that, for a fixed \( p_0 \), at most one linear transformation \( L \) can satisfy (31). (This is the same as Exercise #10, page 352 in Buck)

Assignment 27 (Due November 28) If \( T = (f^1, \ldots, f^m) \) is a differentiable transformation at \( p_0 \), then the partial derivatives \( \frac{\partial f^j}{\partial x_i}(p_0) \) exist for all \( 1 \leq j \leq n, 1 \leq i \leq m \). In other words, the Jacobian matrix \( J_T(p_0) \) exists. (Hint: In the definition of partial derivative, let \( p = p_0 + te_j \) where \( t \in \mathbb{R} \) and \( e_1 = (1,0,\ldots,0), \ e_2 = (0,1,0,\ldots,0), \ldots, \ e_n = (0,\ldots,0,1) \).
Assignment 28 (Due November 28) Let $T$ be a transformation which is of class $C^1$ on an open set $D$, and let $S$ be a transformation of class $C^1$ on an open set containing $T(D)$. Then $S \circ T$ is of class $C^1$ on $D$.

Assignment 29 (Due November 28) Let $F(x, y) = f(g(x, y), h(x, y))$, where $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$. Use Corollary 24.3 to prove that

$$F_1(x, y) = f_1(g(x, y), h(x, y))g_1(x, y) + f_2(g(x, y), h(x, y))h_1(x, y)$$

and

$$F_2(x, y) = f_1(g(x, y), h(x, y))g_2(x, y) + f_2(g(x, y), h(x, y))h_2(x, y).$$

(Compare using Corollary 24.3 with the method on page 137 of Buck.)

Assignment 30 (Due November 28) [Buck, §7.4 page 351, #2,5,(7 or 8)](three problems)

Assignment 31 (Due December 2) Prove that if $K$ is a compact set in $\mathbb{R}^n$ and $q \notin K$, then

$$\inf \{|p - q| : p \in K\} > 0.$$

Assignment 32 (Due December 7—the day of the final exam) Prove Lemma 29.1

Assignment 33 (Due December 7—the day of the final exam) If a transformation preserves convergent sequences, then it is continuous. (Same proof as [Buck, Theorem 2, page 74].)

Assignment 34 (Due December 7) [Buck, §7.5, page 361, #11,14]

Assignment 35 (Due December 7) Show that, for $F = x^2 + y^2 - 1$, $T = T_F$ is not one-to-one on $D = \mathbb{R}^2$ and $T(\mathbb{R}^2)$ is not an open subset of $\mathbb{R}^2$.

Assignment 36 (Due December 7) Let $F(x, y, z) = x^2 + y^2 + z^2 - 1$ and take a point $(x_0, y_0, z_0)$ on the unit sphere in $\mathbb{R}^3$: $x_0^2 + y_0^2 + z_0^2 = 1$, that is, $F(x_0, y_0, z_0) = 0$. “Prove” that $z = \phi(x, y) := \sqrt{1 - x^2 - y^2}$ satisfies $F(x, y, \phi(x, y)) = 0$. According to the implicit function theorem, we need $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$, that is, $2z_0 \neq 0$, so take, for example $p_0 = (1/\sqrt{2}, 0, 1/\sqrt{2})$. Now find $r > 0$ such that

$$(x - \frac{1}{\sqrt{2}})^2 + (y - 0)^2 < r \Rightarrow x^2 + y^2 < 1.$$

Assignment 37 (Due December 7) Let $F(x, y, z) = x^2 + y^2 - 3xyz + z$, take the point $(1, 0, -1)$, and note that $F(1, 0, -1) = 0$ and $\frac{\partial F}{\partial z}(1, 0, -1) = 1 \neq 0$. Conclude that there exists $r > 0$ and a function $\phi(x, y)$ of class $C^1$ in the ball $|(x, y) - (1, 0)| < r$

such that $F(x, y, \phi(x, y)) = 0$ for all $(x, y)$ with $(y - 1)^2 + y^2 < r^2$, that is

$$x^2 + y[\phi(x, y)]^5 - (3xy - 1)\phi(x, y) = 0.$$
Assignment 38 (Due December 7) Let \( F(x, y, z) = \sin xy + e^z - e \), take the point \((x_0, 0, 1)\), and note that \( F(x_0, 0, 1) = 0 \). Also
\[
\frac{\partial F}{\partial x}(x_0, 0, 1) = 0, \quad \frac{\partial F}{\partial y}(x_0, 0, 1) = x_0, \quad \frac{\partial F}{\partial z}(x_0, 0, 1) = e.
\]
What does the implicit function theorem say in this case? Can you solve for any of the three variables without the help of the implicit function theorem?

Assignment 39 (Due December 7) Let \( F(x, y, z) = (\sin x)e^y + (\cos y)e^{xz} + \sin z \), take the point \((0, \pi/2, \pi)\), and note that \( F(0, \pi/2, \pi) = 0 \). Also
\[
\frac{\partial F}{\partial x}(0, \pi/2, \pi) = e^{\pi/2}, \quad \frac{\partial F}{\partial y}(0, \pi/2, \pi) = -1, \quad \frac{\partial F}{\partial z}(0, \pi/2, \pi) = -1.
\]
By the implicit function theorem, you have
\[ z = \phi(x, y) \text{ for } (x, y) \text{ close to } (0, \pi/2), \]
as well as
\[ x = \psi(y, z) \text{ for } (y, z) \text{ close to } (\pi/2, \pi), \]
etc. Now let \( S(x, y) = (x, y, \phi(x, y)) \) and apply the chain rule to \( F \circ S \) to derive
\[
\frac{\partial \phi}{\partial x}(x, y) = -\frac{\frac{\partial F}{\partial x}(x, y, \phi(x, y))}{\frac{\partial F}{\partial z}(x, y, \phi(x, y))},
\]
and
\[
\frac{\partial \phi}{\partial y}(x, y) = -\frac{\frac{\partial F}{\partial y}(x, y, \phi(x, y))}{\frac{\partial F}{\partial z}(x, y, \phi(x, y))}.
\]

Assignment 40 (Due December 7) [Buck, §7.6, page 366, #1,2,5]